On the expressiveness of the interval logic of Allen's relations over finite and discrete linear orders

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Abstract. Interval temporal logics take time intervals, instead of time instants, as their primitive temporal entities. One of the most studied interval temporal logics is Halpern and Shoham's modal logic of time intervals HS, which associates a modal operator with each binary relation between intervals over a linear order (the so-called Allen's interval relations). A complete classification of all HS fragments with respect to their relative expressive power has been recently given for the classes of all linear orders and of all dense linear orders. The cases of discrete and finite linear orders turn out to be much more involved. In this paper, we make a significant step towards solving the classification problem over those classes of linear orders. First, we illustrate various non-trivial temporal properties that can be expressed by HS fragments when interpreted over finite and discrete linear orders; then, we provide a complete set of definabilities for the HS modalities corresponding to the Allen's relations *meets*, *later*, *begins*, *finishes*, and *during*, as well as the ones corresponding to their inverse relations. Given the results presented here, the only missing piece of the expressiveness puzzle is that of the definabilities for the modality corresponding to the Allen relation *overlaps* (those for the inverse relation *overlapped by* would immediately follow by symmetry).

1 Introduction

Interval reasoning naturally arises in various fields of computer science and artificial intelligence, ranging from hardware and real-time system verification to natural language processing, from constraint satisfaction to planning [4,5,14,22,23,25]. Interval temporal logics make it possible to reason about interval structures over linearly ordered domains, where time intervals, rather than time instants, are the primitive ontological entities. The distinctive features of interval temporal logics turn out to be useful in various application domains [8,11,21,22,25]. For instance, they allow one to model *telic statements*, that is, statements that express goals or accomplishments, e.g., the statement: 'The airplane flew from Venice to Toronto' [21]. Moreover, when we restrict ourselves to discrete linear orders, such as, for instance, $\mathbb{N}$ or $\mathbb{Z}$, some interval temporal logics are expressive enough to constrain the length of intervals, thus allowing one to specify safety properties involving quantitative conditions [21]. This is the case, for instance, with the well-known 'gas-burner' example [25]. Temporal logics with interval-based semantics have also been proposed as suitable formalisms for the specification and verification of hardware [22] and of real-time systems [25].

The variety of binary relations between intervals in a linear order was first studied by Allen [4], who investigated their use in systems for time management and planning. In [16], Halpern and Shoham introduced and systematically analyzed the (full) logic of Allen's relations, called HS in this paper, that features one modality for each Allen relation. In particular, they showed that HS is highly undecidable over most classes of linear orders. This result motivated the search for (syntactic) HS fragments offering a good balance between expressiveness and decidability/complexity [6,7,9,10,12,18,20,21]. A comparative analysis of the expressive power of HS fragments is far from being trivial, because some HS modalities are definable in terms of others, and thus syntactically different fragments may turn out to be equally expressive. Moreover, the definability of a specific modality in terms of other ones depends, in general, on the class of linear orders over which the logic is interpreted, and the classification of the relative expressive power of HS fragments with respect to a given class of linear orders cannot be directly transferred to another class. More precisely, while definabilities do transfer from a class $\mathcal{C}$ to all its proper sub-classes, there might be new definability relations that hold in some sub-class of $\mathcal{C}$, but not in $\mathcal{C}$ itself. Conversely, undefinabilities do transfer from a class to all its proper super-classes, but not vice versa. Proving a specific undefinability result amounts to providing a counterexample based on concrete linear orders from the considered class. As a matter of fact, different assumptions on the underlying linear orders give rise, in general, to different sets of definabilities [2,13].

Contribution. Many classes of linear orders are of practical interest, including the class of all (resp., dense, discrete, finite) linear orders, as well as the particular linear order on $\mathbb{R}$ (resp., $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$). A precise characterization of the expressive power of all HS fragments with respect to the class of all linear orders and that of all dense linear orders has been given in [13] and [2], respectively. The classification of HS fragments over the classes of discrete and finite linear orders presents a number of convoluted technical difficulties. In [12], the authors focus on strongly discrete linear orders, by characterizing and classifying all *decidable* fragments of HS with respect to both complexity of the satisfiability problem and relative expressive power. In this paper, we make a significant step towards a complete classification of the expressiveness of all (*decidable* and *undecidable*) fragments of HS over finite and discrete linear orders, and in doing so we considerably extend the expressiveness results presented in [12]. As a matter of fact, given the present contributions, the only missing piece of the expressiveness puzzle is that of the definabilities for the modality corresponding to the Allen relation *overlaps* (those for the inverse relation *overlapped by* would immediately follow by symmetry).

Structure of the paper. In the next section, we introduce the logic HS. Then, in Section 3, we introduce the notion of definability of a modality in an HS fragment, and we present the main tool we use to prove our results. In order to provide the reader with an idea of the expressive power of HS modalities, we also illustrate some meaningful temporal properties, like counting and boundedness properties, which can be expressed

HS **modalities**	**Allen's relations**
$\langle A \rangle$	$[x,y]R_A[x',y'] \Leftrightarrow y = x'$
$\langle L \rangle$	$[x,y]R_L[x',y'] \Leftrightarrow y < x'$
$\langle B \rangle$	$[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$
$\langle E \rangle$	$[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$
$\langle D \rangle$	$[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$
$\langle O \rangle$	$[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$

Graphical representation

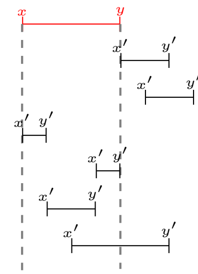

Fig. 1. Allen's interval relations and the corresponding HS modalities.

in HS fragments when interpreted over discrete linear orders. Then, as a warm-up, in Section 4 we present a first, simple expressiveness result, by providing the complete set of definabilities for the HS modalities $\langle A \rangle$, $\langle L \rangle$, $\langle \overline{A} \rangle$, and $\langle \overline{L} \rangle$, corresponding to Allen's relations *meets* and *later*, and their inverses *met by* and *before*, respectively. Section 5 contains our main technical result, that is, a complete set of definabilities for the HS modalities $\langle D \rangle$, $\langle E \rangle$, $\langle B \rangle$, $\langle \overline{D} \rangle$, $\langle \overline{E} \rangle$, and $\langle \overline{B} \rangle$, corresponding to Allen's relations *during*, *finishes*, and *begins*, and their inverses *contains*, *finished by*, and *begun by*, respectively. The proofs of the results in this section are rather difficult and much more technically involved than the ones in Section 4. Therefore, we limit ourselves to giving an overview of the proofs, and we refer the interested reader to [3] for the details. We conclude the paper with some final remarks.

2 Preliminaries

Let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[a, b]$, where $a, b \in D$ and $a \leq b$. An interval is called a *point interval* if $a = b$ and a *strict interval* if $a < b$. In this paper, we assume the *strict semantics*, that is, we exclude point intervals and only consider strict intervals. The adoption of the strict semantics, excluding point intervals, instead of the *non-strict semantics*, which includes them, conforms to the definition of interval adopted by Allen in [4], but differs from the one given by Halpern and Shoham in [16]. It has at least two strong motivations: first, a number of representation paradoxes arise when the non-strict semantics is adopted, due to the presence of point intervals, as pointed out in [4]; second, when point intervals are included, there seems to be no intuitive semantics for interval relations that makes them both pairwise disjoint and jointly exhaustive. If we exclude the identity relation, there are 12 different relations between two strict intervals in a linear order, often called *Allen's relations* [4]: the six relations R_A (adjacent to), R_L (later than), R_B (begins), R_E (ends), R_D (during), and R_O (overlaps), depicted in Fig. 1, and their inverses, that is, $R_{\overline{X}} = (R_X)^{-1}$, for each $X \in \{A, L, B, E, D, O\}$.

We interpret interval structures as Kripke structures, with Allen's relations playing the role of the accessibility relations. Thus, we associate a modality $\langle X \rangle$ with each Allen relation R_X. For each $X \in \{A, L, B, E, D, O\}$, the *transpose* of modality $\langle X\rangle$ is modality $\langle\overline{X}\rangle$, corresponding to the inverse relation $R_{\overline{X}}$ of R_X. Halpern and Shoham's logic HS [16] is a multi-modal logic with formulae built from a finite, non-empty set $\mathcal{AP}$ of atomic propositions (also referred to as proposition letters), the propositional connectives $\vee$ and $\neg$, and a modality for each Allen relation. With every subset $\{R_{X_1}, \ldots, R_{X_k}\}$ of these relations, we associate the fragment $\mathsf{X_1X_2\ldots X_k}$ of HS, whose formulae are defined by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1\rangle\varphi \mid \ldots \mid \langle X_k\rangle\varphi,$$

where $p \in \mathcal{AP}$. The other propositional connectives and constants (e.g., $\wedge$, $\rightarrow$, and $\top$), as well as the dual modalities (e.g., $[A]\varphi \equiv \neg\langle A\rangle\neg\varphi$), can be derived in the standard way. We define the *modal depth* of a formula as the largest nesting of modal operators in it. For a fragment $\mathcal{F} = \mathsf{X_1X_2\ldots X_k}$ and a modality $\langle X\rangle$, we write $\langle X\rangle \in \mathcal{F}$ if $X \in \{X_1, \ldots, X_k\}$. Given two fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X\rangle \in \mathcal{F}_1$ implies $\langle X\rangle \in \mathcal{F}_2$, for every modality $\langle X\rangle$. Finally, for a fragment $\mathcal{F} = \mathsf{X_1X_2\ldots X_k}$ and a formula φ, we write $\varphi \in \mathcal{F}$ or, equivalently, we say that φ is an $\mathcal{F}$-formula, meaning that φ belongs to the language of $\mathcal{F}$.

The (strict) semantics of HS is given in terms of *interval models* $M = \langle\mathbb{I}(\mathbb{D}), V\rangle$, where $\mathbb{D}$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all (strict) intervals over $\mathbb{D}$, and V is a *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$, which assigns to each atomic proposition $p \in \mathcal{AP}$ the set of intervals $V(p)$ on which p holds. The *truth* of a formula on a given interval $[x, y]$ in an interval model M is defined by structural induction on formulae as follows:

- $M, [x, y] \Vdash p$ if and only if $[x, y] \in V(p)$, for each $p \in \mathcal{AP}$;
- $M, [x, y] \Vdash \neg\psi$ if and only if it is not the case that $M, [x, y] \Vdash \psi$;
- $M, [x, y] \Vdash \varphi \vee \psi$ if and only if $M, [x, y] \Vdash \varphi$ or $M, [x, y] \Vdash \psi$;
- $M, [x, y] \Vdash \langle X\rangle\psi$ if and only if there exists $[x', y']$ such that $[x, y]R_X[x', y']$ and $M, [x', y'] \Vdash \psi$, for each modality $\langle X\rangle$.

Formulae of HS can be interpreted over a class of interval models (built on a given class of linear orders). Among others, we mention the following classes of (interval models built on important classes of) linear orders: *(i)* the class of *all* linear orders Lin; *(ii)* the class of (all) *dense* linear orders Den, that is, those in which for every pair of distinct points there exists at least one point in between them (e.g., $\mathbb{Q}$ and $\mathbb{R}$); *(iii)* the class of (all) *discrete* linear orders Dis, that is, those in which every element, apart from the greatest element, if it exists, has an immediate successor, and every element, other than the least element, if it exists, has an immediate predecessor (e.g., $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}+\mathbb{Z}$); *(iv)* the class of (all) *finite* linear orders Fin, that is, those having only finitely many points. A formula ϕ of HS is *valid* over a class $\mathcal{C}$ of linear orders, denoted by $\Vdash_{\mathcal{C}} \phi$, if it is true on every interval in every interval model belonging to $\mathcal{C}$. Two formulae ϕ and ψ are *equivalent* relative to the class $\mathcal{C}$ of linear orders, denoted by $\phi \equiv_{\mathcal{C}} \psi$, if $\Vdash_{\mathcal{C}} \phi \leftrightarrow \psi$.

3 Definability and expressivenesss

Definition 1 (Definability). *A modality $\langle X\rangle$ of* HS *is* definable *in an* HS *fragment $\mathcal{F}$ relative to a class $\mathcal{C}$ of linear orders, denoted $\langle X\rangle \lhd_{\mathcal{C}} \mathcal{F}$, if $\langle X\rangle p \equiv_{\mathcal{C}} \psi$ for some $\mathcal{F}$-formula ψ over the atomic proposition p, for any $p \in \mathcal{AP}$. Then, the equivalence $\langle X\rangle p \equiv_{\mathcal{C}} \psi$ is called a* definability equation *for $\langle X\rangle$ in $\mathcal{F}$ relative to $\mathcal{C}$. We write $\langle X\rangle \ntriangleleft_{\mathcal{C}} \mathcal{F}$ if it is not the case that $\langle X\rangle \lhd_{\mathcal{C}} \mathcal{F}$.*

As we have already noted, smaller classes of linear orders inherit the definabilities holding for larger classes: if $\mathcal{C}_1$ and $\mathcal{C}_2$ are classes of linear orders such that $\mathcal{C}_1 \subset \mathcal{C}_2$, then all definabilities holding for $\mathcal{C}_2$ are also valid for $\mathcal{C}_1$. However, more definabilities can possibly hold for $\mathcal{C}_1$. On the other hand, undefinability results for $\mathcal{C}_1$ hold also for $\mathcal{C}_2$. In the rest of the paper, we omit the class of linear orders when it is clear from the context (e.g., we will simply write $\langle X\rangle p \equiv \psi$ and $\langle X\rangle \lhd \mathcal{F}$ for $\langle X\rangle p \equiv_{\mathcal{C}} \psi$ and $\langle X\rangle \lhd_{\mathcal{C}} \mathcal{F}$, respectively).

It is known from [16] that, when the strict semantics is assumed, all HS modalities are definable in the fragment containing modalities $\langle A\rangle$, $\langle B\rangle$, and $\langle E\rangle$, and their transposes $\langle \overline{A}\rangle$, $\langle \overline{B}\rangle$, and $\langle \overline{E}\rangle$, while in the non-strict semantics, the four modalities $\langle B\rangle$, $\langle E\rangle$, $\langle \overline{B}\rangle$, and $\langle \overline{E}\rangle$ suffice, as shown in [24]. Given two HS fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we say that $\mathcal{F}_2$ is *at least as expressive as* $\mathcal{F}_1$, denoted $\mathcal{F}_1 \preceq \mathcal{F}_2$, if each operator $\langle X\rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$, and that $\mathcal{F}_1$ is *strictly less expressive* than $\mathcal{F}_2$, denoted $\mathcal{F}_1 \prec \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ holds but $\mathcal{F}_2 \preceq \mathcal{F}_1$ does not. The notions of *expressively equivalent* fragments and *expressively incomparable* fragments can be defined likewise.

Definition 2 (Optimal definability). *A definability $\langle X\rangle \lhd \mathcal{F}$ is* optimal *if $\langle X\rangle \not\lhd \mathcal{F}'$ for each fragment $\mathcal{F}'$ such that $\mathcal{F}' \prec \mathcal{F}$.*

3.1 Proof techniques to disprove definability

In order to show non-definability of a given modality in a certain fragment, we use the standard notion of *N-bisimulation* [15,17,19], suitably adapted to our setting.

Definition 3. *Let $\mathcal{F}$ be an* HS*-fragment. An $\mathcal{F}_N$-bisimulation between two models $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V'\rangle$ over a set of proposition letters $\mathcal{AP}$ is a sequence of N relations $Z_N, \ldots, Z_1 \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ such that: (i) for every $([x,y],[x',y']) \in Z_h$, with $N \geq h \geq 1$, $M,[x,y] \Vdash p$ if and only if $M',[x',y'] \Vdash p$, for all $p \in \mathcal{AP}$* (local condition)*; (ii) for every $([x,y],[x',y']) \in Z_h$, with $N \geq h > 1$, if $[x,y]R_X[v,w]$ for some $[v,w] \in \mathbb{I}(\mathbb{D})$ and some $\langle X\rangle \in \mathcal{F}$, then there exists $([v,w],[v',w']) \in Z_{h-1}$ such that $[x',y']R_X[v',w']$* (forward condition)*; (iii) for every $([x,y],[x',y']) \in Z_h$, with $N \geq h > 1$, if $[x',y']R_X[v',w']$ for some $[v',w'] \in \mathbb{I}(\mathbb{D}')$ and some $\langle X\rangle \in \mathcal{F}$, then there exists $([v,w],[v',w']) \in Z_{h-1}$ such that $[x,y]R_X[v,w]$* (backward condition).

Given an $\mathcal{F}_N$-bisimulation, the truth of $\mathcal{F}$-formulae of modal depth at most $h-1$ is invariant for pairs of intervals belonging to Z_h, with $N \geq h \geq 1$ (see, e.g., [15]). Thus, to prove that a modality $\langle X\rangle$ is not definable in $\mathcal{F}$, it suffices to provide, for every natural number N, a pair of models M and M', and an $\mathcal{F}_N$-bisimulation between them for which there exists a pair $([x,y],[x',y']) \in Z_N$ such that $M,[x,y] \Vdash \langle X\rangle p$ and $M',[x',y'] \Vdash \neg\langle X\rangle p$, for some $p \in \mathcal{AP}$ (in this case, we say that the $\mathcal{F}_N$-bisimulation *violates* $\langle X\rangle$). To convince oneself that this is enough to ensure that $\langle X\rangle$ is not definable by any $\mathcal{F}$-formula of any modal depth, assume, towards a contradiction, that ϕ is an $\mathcal{F}$-formula of modal depth n such that $\langle X\rangle p \equiv \phi$. Since, for each N, there is an $\mathcal{F}_N$-bisimulation that violates $\langle X\rangle$, there exists, in particular, one such bisimulation for $N = n+1$. Let $([x,y],[x',y']) \in Z_N$ be the pair of intervals that *violates* $\langle X\rangle$, that is, $M,[x,y] \Vdash \langle X\rangle p$ and $M',[x',y'] \Vdash \neg\langle X\rangle p$. Then, the truth value of ϕ over $[x,y]$ (in M) and $[x',y']$ (in M') is the same, and this is in contradiction with the fact that

$M, [x, y] \Vdash \langle X \rangle p$ and $M', [x', y'] \Vdash \neg \langle X \rangle p$. A result obtained following this argument applies to all classes of linear orders that contain (as their elements) both structures on which M and M' are based. Notice that, in some cases, it is convenient to define $\mathcal{F}_N$-bisimulations between a model M and itself.

It is worth pointing out that the standard notion of $\mathcal{F}$*-bisimulation* can be recovered as a special case of $\mathcal{F}_N$-bisimulation. Formally, an $\mathcal{F}$-bisimulation can be thought of as an $\mathcal{F}_N$-bisimulation with $N = 2$ and $Z_1 = Z_2$. In the following, as is customary, we will treat $\mathcal{F}$-bisimulations as relations instead of sequences of two equal relations: if the sequence Z_2, Z_1 is an $\mathcal{F}$-bisimulation, with $Z_1 = Z_2 = Z$, then we will simply refer to it as to the relation Z. It is important to notice that showing that two intervals are related by an $\mathcal{F}$-bisimulation (i.e., they are $\mathcal{F}$*-bisimilar*) is stronger than showing that they are related by a relation Z_N, which belongs to a sequence $Z_N, \ldots, Z_1$ corresponding to an $\mathcal{F}_N$-bisimulation (i.e., the intervals are $\mathcal{F}_N$*-bisimilar*). Indeed, while in the latter case we are only guaranteed invariance of $\mathcal{F}$-formulae of modal depth at most $N - 1$, in the former case the truth of $\mathcal{F}$-formulae of any (possibly unbounded) modal depth is preserved. This means that undefinability results obtained using $\mathcal{F}$-bisimulations are not restricted to the finitary logics we consider in this paper, but also apply to extensions with infinite disjunctions and with fixed-point operators.

Since $\mathcal{F}$-bisimulations are notationally easier to deal with than $\mathcal{F}_N$-bisimulations, it is in principle more convenient to use the former, rather than the latter, when proving an undefinability result. However, while in few cases (see Section 4) a proof based on $\mathcal{F}$-bisimulations is possible, this is not generally the case, because some modalities that cannot be defined in fragments of HS can be expressed in their infinitary versions. In those cases (see Section 5), we resort to a proof via $\mathcal{F}_N$-bisimulations.

For a given modality $\langle X \rangle$ and a given class $\mathcal{C}$ of linear orders, we shall identify a set of definabilities for $\langle X \rangle$, and we shall prove its *soundness*, by shoving that each definability equation is valid in $\mathcal{C}$, and its *completeness*, by arguing that each definability is optimal and that there are no other optimal definabilities for $\langle X \rangle$ in $\mathcal{C}$. Completeness is proved by computing all maximal fragments $\mathcal{F}$ that cannot define $\langle X \rangle$ (in the attempt of defining $\langle X \rangle$ in $\mathcal{F}$, we can obviously make use of the set of known definabilities). For each modality, such fragments are listed in the last column of Fig. 2. Depending on the number of known definabilities, such a task can be time-consuming and error-prone, so an automated procedure has been devised and implemented in [1] to serve the purpose. Then, for each such $\mathcal{F}$ and each $N \in \mathbb{N}$, we provide an $\mathcal{F}_N$-bisimulation that violates $\langle X \rangle$. Notice that all the classes of linear orders we consider in this paper are (left/right) *symmetric*, namely, if a class $\mathcal{C}$ contains a linear order $\mathbb{D} = \langle D, \prec \rangle$, then it also contains (a linear order isomorphic to) its dual linear order $\mathbb{D}^d = \langle D, \succ \rangle$, where $\succ$ is the inverse of $\prec$. This implies that the definabilities for $\langle \overline{L} \rangle$, $\langle \overline{A} \rangle$, $\langle B \rangle$, and $\langle \overline{B} \rangle$ can be immediately deduced (and shown to be sound and optimal) from those for $\langle L \rangle$, $\langle A \rangle$, $\langle E \rangle$, and $\langle \overline{E} \rangle$, respectively.

Fig. 2 depicts the complete sets of optimal definabilities holding in Dis and Fin for the modalities $\langle L \rangle$, $\langle A \rangle$, $\langle D \rangle$, $\langle \overline{D} \rangle$, $\langle E \rangle$, and $\langle \overline{E} \rangle$ (recall that those for $\langle \overline{L} \rangle$, $\langle \overline{A} \rangle$, $\langle B \rangle$, and $\langle \overline{B} \rangle$ follow by symmetry). Section 4 and Section 5 are devoted to proving completeness of such sets. For all the modalities, but $\langle A \rangle$ and $\langle \overline{A} \rangle$, soundness is an immediate consequence of the corresponding soundness in Lin, shown in [13]. For lack of space,

Modalities	Equations	Definabilities	Maximal fragments not defining it
$\langle L\rangle$	$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$	$\langle L\rangle \lhd \mathsf{A}$	$\mathsf{BDO}\overline{\mathsf{ALBEDO}}$ $\mathsf{BEDO}\overline{\mathsf{ALEDO}}$
$\langle A\rangle$	$\langle A\rangle p \equiv \varphi(p) \vee \langle E\rangle\varphi(p)^*$	$\langle A\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$	$\mathsf{LBDO}\overline{\mathsf{ALBEDO}}$ $\mathsf{LBEDO}\overline{\mathsf{ALEDO}}$
	$^*\varphi(p) := [E]\bot \wedge \langle\overline{B}\rangle([E][E]\bot \wedge\langle E\rangle(p \vee \langle\overline{B}\rangle p))$		
$\langle D\rangle$	$\langle D\rangle p \equiv \langle B\rangle\langle E\rangle p$	$\langle D\rangle \lhd \mathsf{BE}$	$\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$ $\mathsf{ALEO}\overline{\mathsf{ALBEDO}}$
$\langle\overline{D}\rangle$	$\langle\overline{D}\rangle p \equiv \langle\overline{B}\rangle\langle\overline{E}\rangle p$	$\langle D\rangle \lhd \overline{\mathsf{BE}}$	$\mathsf{ALBEDO}\overline{\mathsf{ALBO}}$ $\mathsf{ALBEDO}\overline{\mathsf{ALEO}}$
$\langle E\rangle$	no definabilities		$\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}$
$\langle\overline{E}\rangle$	no definabilities		$\mathsf{ALBEDO}\overline{\mathsf{ALBDO}}$

Fig. 2. Optimal definabilities in Dis and Fin. The last column contains the maximal fragments not defining the modality under consideration.

we omit the proofs of the soundness of the definabilities for $\langle A\rangle$ and $\langle\overline{A}\rangle$, which anyway are quite straightforward. Finally, while it is known from [16] that $\langle O\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ (resp., $\langle\overline{O}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$), it is still an open problem whether this is the only optimal definability for $\langle O\rangle$ (resp., $\langle\overline{O}\rangle$) in Dis and in Fin.

3.2 Expressing properties of a model in HS fragments

We give here a short account of meaningful temporal properties, such as counting and (un)boundedness ones, which can be expressed in HS fragments, when they are interpreted over discrete linear orders. The outcomes of such an analysis are summarized in Fig. 3 (other properties can obviously be expressed as Boolean combinations of those displayed). They demonstrate the expressiveness capabilities of HS modalities, which are of interest by themselves. As an example, the ability of constraining the length of intervals is a desirable feature of any formalism for representing and reasoning about temporal knowledge over a discrete domain. As a matter of fact, most HS fragments have many chances to succeed in practical applications, and thus it is definitely worth carrying out a taxonomic study of their expressiveness. As we alreay pointed out, such a study presents various intricacies. For instance, in some fragments, assuming the discreteness of the linear order suffices to constrain the lenght of intervals (this is the case with the fragment E); other fragments rely on additional assumptions (this is the case with the fragment DO, which requires the linear order to be right-unbounded). This gives evidence of how expressiveness results can be affected by the specific class of linear orders under consideration.

Counting properties. When the linear order is assumed to be discrete, some HS fragments are powerful enough to constrain (to some extent) the *length* of an interval, that is, the number of its points minus one. Let $\sim \in \{<, \leq, =, \geq, >\}$. For every $k \in \mathbb{N}$, we define $\ell_{\sim k}$ as a (pre-interpreted) atomic proposition which is true over all and only those intervals whose length is $\sim$-related to k. Moreover, for a modality $\langle X\rangle$, we denote by

Counting properties		Right Unboundedness ($\exists_r$)	
$\ell_{>k}$	$\equiv \langle E\rangle^k \top$		$\langle \overline{B}\rangle\top$, $\langle A\rangle\top$
$\ell_{=k}$	$\equiv \langle E\rangle^{k-1}\top \wedge [E]^k\bot$	(‡)	$\langle O\rangle\top$, $[B]\langle L\rangle\top$
$\ell_{>2\cdot k}$	$\equiv \langle D\rangle^k\top$	(§)	$\langle \overline{D}\rangle\top$, $\langle \overline{E}\rangle\langle O\rangle\top$
$\ell_{\leq 2\cdot k}$	$\equiv [D]^k\bot$	(‡,§)	$[\overline{O}]\langle L\rangle\top$
$\ell_{>1}$	$\equiv^\dagger \langle O\rangle\top$	(♭)	$[D]\langle L\rangle\top$
$\ell_{>2\cdot k+1}$	$\equiv^\dagger \langle D\rangle^k\langle O\rangle\top$		
$\ell_{=2\cdot(k+1)}$	$\equiv^\dagger \langle D\rangle^k\langle O\rangle\top \wedge [D]^{k+1}\bot$		

†: only on right-unbounded domains; ‡: only on intervals longer than 1;
§: only on left-unbounded domains; ♭: only on intervals longer than 2.

Fig. 3. Expressiveness of HS modalities over discrete linear orders.

$\langle X\rangle^k\varphi$ the formula $\langle X\rangle \dots \langle X\rangle\varphi$, with k occurrences of $\langle X\rangle$ before φ. Limiting ourselves to a few examples, we highlight here the ability of some of the HS modalities to express $\ell_{\sim k}$, for any k. It is well known that the fragments E and B can express $\ell_{\sim k}$, for every k and $\sim$ (see, e.g., [16]). As an example, the formulae $\langle E\rangle^k\top$ and $[E]^k\bot$ are equivalent to $\ell_{>k}$ and $\ell_{\leq k}$, respectively. The fragment D features limited counting properties, as, for every k, $\langle D\rangle^k\top \wedge [D]^{k+1}\bot$ is true over intervals whose length is either $2\cdot k+1$ or $2\cdot(k+1)$ (notice that, as a particular instance, $[D]\bot$ is true over intervals whose length is either 1 or 2). In a sense, it is not able to discriminate the parity of an interval. The counting capabilities of the fragment O are limited as well: it allows one to discriminate between *unit intervals* (intervals whose length is 1) and *non-unit intervals* (which are longer than 1), provided that the underlying linear order is right-unbounded, like $\mathbb{Z}$ or $\mathbb{N}$ ($\langle\overline{O}\rangle$ possesses the same capability, provided that the underlying linear order is left-unbounded, like $\mathbb{Z}$ or $\mathbb{Z}^-$). However, quite interestingly, by pairing $\langle D\rangle$ and $\langle O\rangle$, or, symmetrically, $\langle D\rangle$ and $\langle\overline{O}\rangle$, it is possible to express $\ell_{\sim k}$ for every k and $\sim$ over right-unbounded linear order (left-unbounded linear orders if $\langle O\rangle$ is replaced by $\langle\overline{O}\rangle$): it suffices to first use $\langle D\rangle$ to narrow the length down to k or $k+1$, and then $\langle O\rangle$ (or $\langle\overline{O}\rangle$) to discriminate the parity.

(Un)boundedness properties. Let us denote by $\exists_r$ (resp., $\exists_l$) a (pre-interpreted) atomic proposition that is true over all and only the intervals that have a point to their right (resp., left). Various combinations of HS operators can express $\exists_r$. Once again, while in some cases we need to assume only the discreteness of the underlying linear order, there are cases where the validity of the definability relies on additional assumptions. For example, to impose that the current interval has a point to the right within the fragment O, we can use $\langle O\rangle\top$ only on non-unit intervals (otherwise, $\langle O\rangle$ has no effect). Analogously, it is possible to express $\exists_l$, possibly under analogous assumptions.

4 The Easy Cases

In this section, we prove the completeness of the set of definabilities for the modalities $\langle L\rangle$, $\langle\overline{L}\rangle$, $\langle A\rangle$, and $\langle\overline{A}\rangle$, thus strengthening a similar result presented in [12, Theorem 1].

Theorem 1. *The sets of optimal definabilities for $\langle L\rangle$ and $\langle A\rangle$ (listed in Fig. 2), as well as (by symmetry) those for $\langle\overline{L}\rangle$ and $\langle\overline{A}\rangle$, are complete for the classes* Dis *and* Fin.

Proof. The results for $\langle L\rangle$ (and, symmetrically, for $\langle\overline{L}\rangle$) immediately follows from [13], as the completeness proof for $\langle L\rangle$ presented there used a bisimulation between models based on finite linear orders. Notice that $\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ holds in Dis and Fin, as it does in Lin. However, such a definability, which is optimal in Lin, is not optimal in Dis and Fin (and thus it is not listed in Fig. 2), due to the fact that $\langle A\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ (which is not a sound definability in Lin) holds over Dis. As a pleasing consequence, we can extend Venema's result from [24] concerning the expressive completeness of the fragment $\mathsf{BE}\overline{\mathsf{BE}}$ in the non-strict semantics to the strict one under the discreteness assumption.

According to Fig. 2, $\langle A\rangle$ is definable in terms of $\overline{\mathsf{B}}\mathsf{E}$, implying that the maximal fragments not defining $\langle A\rangle$ are, as shown in the last column of Fig. 2, $\mathsf{LBDO}\overline{\mathsf{ALBEDO}}$ and $\mathsf{LBEDO}\overline{\mathsf{ALEDO}}$. Thus, proving that $\langle A\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ is the only optimal definability amounts to providing two bisimulations, namely an $\mathsf{LBDO}\overline{\mathsf{ALBEDO}}$- and an $\mathsf{LBEDO}\overline{\mathsf{ALEDO}}$-bisimulation that violate $\langle A\rangle$. As for the first one, we consider two models M and M', both based on the finite linear order $\{0,1,2\}$. We set $V(p)=\{[1,2]\}$, $V'(p)=\emptyset$, and $Z=\{([0,1],[0,1]),([0,2],[0,2])\}$. It is easy to verify that Z is an $\mathsf{LBDO}\overline{\mathsf{ALBEDO}}$-bisimulation that violates $\langle A\rangle$, as $M,[0,1] \Vdash \langle A\rangle p$ and $M',[0,1] \Vdash \neg\langle A\rangle p$. As for the second one, models and valuations are defined as before, but we take now $Z=\{([0,1],[0,1])\}$. Once again, it is easy to see that Z is an $\mathsf{LBEDO}\overline{\mathsf{ALEDO}}$-bisimulation that violates $\langle A\rangle$, as $M,[0,1] \Vdash \langle A\rangle p$ and $M',[0,1] \Vdash \neg\langle A\rangle p$. Since the result is based on a finite linear order, it holds for both Dis and Fin. □

5 The hard cases

In this section, we provide the completeness result for the modalities $\langle D\rangle$ and $\langle\overline{D}\rangle$ (Theorem 2), as well as for $\langle E\rangle$, $\langle\overline{E}\rangle$, $\langle B\rangle$, and $\langle\overline{B}\rangle$ (Theorem 3). Because of the technical complexity of the proofs, we only provide proof sketches that explain the main ideas behind them at a very intuitive level, and refer the interested reader to [3] for the details.

Theorem 2. *The sets of optimal definabilities for $\langle D\rangle$ and $\langle\overline{D}\rangle$ (listed in Fig. 2) are complete for the classes* Dis *and* Fin.

Proof (sketch). According to Fig. 2, $\langle D\rangle$ is definable in terms of BE; thus there are two maximal fragments not defining it, namely, $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$ and $\mathsf{ALEO}\overline{\mathsf{ALBEDO}}$. First, we observe that it is possible to define $\langle D\rangle$ in infinitary extensions of AB or $\overline{\mathsf{A}}\mathsf{E}$, using, respectively, the following formulae of unbounded modal depths:

$$\langle D\rangle p \equiv \begin{cases} \bigvee_{k\in\mathbb{N}}(\ell_{=k} \wedge \bigvee_{i<k-1}(\langle B\rangle(\ell_{=i} \wedge \langle A\rangle(\ell_{<k-i} \wedge p)))), \\ \bigvee_{k\in\mathbb{N}}(\ell_{=k} \wedge \bigvee_{i<k-1}(\langle E\rangle(\ell_{=i} \wedge \langle\overline{A}\rangle(\ell_{<k-i} \wedge p)))), \end{cases}$$

where length constraints of the form $\ell_{=k}$ and $\ell_{<k}$ can be expressed using either $\langle B\rangle$ or $\langle E\rangle$ (see Section 3.2). It immediately follows that there exists no $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$-bisimulation (resp., $\mathsf{ALEO}\overline{\mathsf{ALBEDO}}$-bisimulation) that violates $\langle D\rangle$, and thus we have to resort to $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulations (resp., $\mathsf{ALEO}\overline{\mathsf{ALBEDO}}_N$-bisimulations). Besides, since the two fragments $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$ and $\mathsf{ALEO}\overline{\mathsf{ALBEDO}}$ are symmetric,

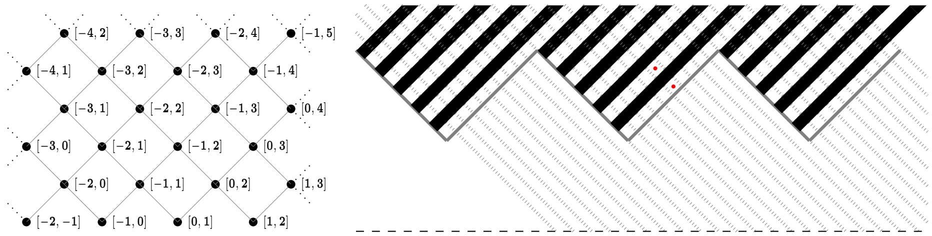

Fig. 4. Grid-based interpretation of intervals (left) and a graphical account of the $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle D\rangle$ (right).

that is, they are indistinguishable over symmetric classes of linear orders, providing an $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle D\rangle$ suffices to prove the result.

For the purposes of the proof, it is convenient to introduce a new interpretation for intervals over grid-like structures (the so-called compass structures [24]), by exploiting the existence of a natural bijection between the intervals $[x, y]$ of an interval model and the points $p = (x, y)$, with $x < y$, of an $N \times N$ grid. A graphical account is given in Fig. 4 (left), where the $N \times N$ grid has been rotated by a 45-degree angle clockwise, so that the bisector of the I and III quadrant is the base of the picture.

First, we define the model M, as depicted in Fig. 4 (right), where intervals satisfying p are all and only the points belonging to the black areas. Thus, intervals satisfying p are grouped into stripes. The dotted lines in the picture are perpendicular to the stripes, more precisely, to (the ideal continuations of) their edges. Each dotted line intersects exactly one such continuation at the base of the picture (dashed line, representing the bisector of the I and III quadrant). Intersections of dotted lines with stripes give rise to small squares. Black (resp., white) squares only contains intervals satisfying p (resp., $\neg p$). Now, let us focus on the gray, zigzag solid line. If we ideally draw the straight lines continuing the segments making up such a zigzag line, their intersections shape bigger squares, each of them containing a (square) number of the above-mentioned small squares.

In order to define an $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation, we focus on the generic hth element of the sequence, namely, the relation Z_h. The idea is to relate points that are either "far enough" from the elements of discontinuity of the model (stripes' edges, dotted lines, dashed line, and gray line) or at the same distance from them. The key element is the notion of "far enough", which can be formalized by means of monotonically increasing *distance functions* on h, representing the number of nested modalities that can still be used to build a formula that discriminates between the related intervals, before reaching the greatest allowed modal depth N. In other words, the notion of distance is induced by h through suitable distance functions, and the distance decreases as h does: in this way, if an interval i_1 is far from a significant element e of the model, according to the notion of distance induced by some h (i.e., i_1 is h-far from e), it is always possible to find another interval i_2, that is closer to e, but still far from e according to the "new" notion of distance induced by $h - 1$ (i.e., i_2 is $(h - 1)$-far from e).

Now, still at a very high level, by exploiting such a notion of “far enough”, we can conclude that the two red circles in the two white stripes in the middle of the picture are Z_h-related, because, according to suitable distance functions, both of them are far from all the elements of discontinuity of the model, that is, the edges of their own small squares (both points are in the middle of a small square, with enough points in between them and the edges), as well as the ones of the big square. Moreover, the relative position of the two small squares in the big one is the same (up to a certain distance from the edges of the big square), with the exception of the position relative to the bottom-right edge of the big square: one of the circles is in the first small square, the other in the third one. This is not a problem, because distances in the bottom-right directions can be ignored as moving in that direction corresponds to using the modality $\langle E \rangle$, which does not belong to $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$. Finally, from Fig. 4, it is clear that the lower circle does not “see” any interval satisfying p (black stripes) in the triangle underneath, and thus $\langle D \rangle p$ is false on it. On the contrary, the higher circle “sees” intervals satisfying p in the triangle underneath, which means that $\langle D \rangle p$ is true over it. Thus, we have an $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle D \rangle$. A similar construction can be done to deal with the modality $\langle \overline{D} \rangle$, which somehow turns the picture upside-down, thus showing that the result holds also for $\langle \overline{D} \rangle$. □

Theorem 3. *There are no definabilities for* $\langle E \rangle$ *and* $\langle \overline{E} \rangle$ *(as shown in Fig. 2), as well as for their transposes* $\langle B \rangle$ *and* $\langle \overline{B} \rangle$*, in the classes* Dis *and* Fin*.*

Proof (sketch). We only give the sketch of the proof for the operators $\langle E \rangle$ and $\langle \overline{E} \rangle$. The result for $\langle B \rangle$ and $\langle \overline{B} \rangle$ follows from a symmetric argument. According to Fig. 2, there are no definabilities for $\langle E \rangle$ when the underlying structure is discrete, and therefore $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}$ is the only maximal fragment not defining it. This is also true on Lin and Den, but on Dis and Fin it is simply harder to prove. An indication of such a difficulty comes from the analysis of the proofs presented in [13], where the density of the models involved plays a major role. Similarly to the case of Theorem 2, $\langle E \rangle$ is definable in an infinitary extension of the language AB:

$$\langle E \rangle p \equiv \bigvee_{k \in \mathbb{N}} (\ell_{=k} \wedge \bigvee_{i<k} (\langle B \rangle (\ell_{=i} \wedge \langle A \rangle (\ell_{=k-i} \wedge p))),$$

since, as stated in Section 3.2, $\langle B \rangle$ can express $\ell_{=k}$, for every $k \in \mathbb{N}$. Thus, there exists no $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}$-bisimulation that violates $\langle E \rangle$, and we need to find an $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation. Unlike the case of Theorem 2, the best way to sketch the construction is by explicitly giving models and relative valuation functions.

Let $\mathbb{D}$ be a finite domain, e.g., an arbitrarily large prefix of $\mathbb{N}$. We define a model M based on it and an $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation between M and itself that violates $\langle E \rangle$. Given $N \in \mathbb{N}$, we make use of $h \leq N$ to refer to the hth component of the N-bisimulation, also called in the following the hth *step* of the N-bisimulation. Building the $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation relies on a very technical construction that allows us to “simulate density” over discrete models up to a certain threshold. To this end, in analogy to what we did in the proof of Theorem 2, we will use monotonically increasing *threshold functions*, which are parametric in h and which characterize a notion of “long interval”, relative to a generic *step* h of the N-bisimulation. Since such functions are monotonic, intervals that are “long” at the step h of the N-bisimulation always contain intervals that are still “long” at the step $h-1$, despite being obviously

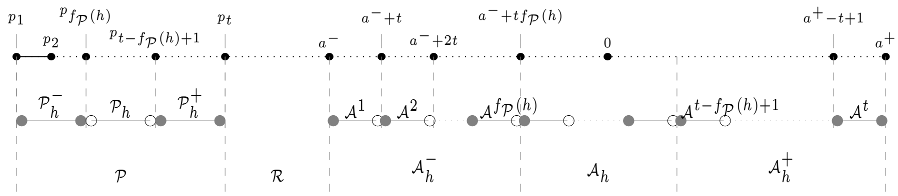

Fig. 5. A graphical account of the $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle E\rangle$.

shorter of the the containing interval. We will also use suitably defined equivalences up to a threshold (given by the aforementioned threshold functions) to recognize when two intervals are "long enough" to be indistinguishable by modal formulae in the fragment $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}$ whose modal depth is less than $h \leq N$.

Now, we define the function $f(h) = h+1$, which will be used as threshold function, and the function $f_{\mathcal{P}}(h) = \sum_{i=1}^{h} f(i)$. Notice that both functions are monotonically increasing. Moreover, we let $t = 2(f_{\mathcal{P}}(1) + N + 4)$, $a^+ = \frac{t^2}{2} - 1$, and $a^- = -\frac{t^2}{2}$. Finally, we consider a partition of $\mathbb{D}$ as in Fig. 5.

Three subsets, from left to right, are clearly identified in Fig. 5:

$\mathcal{P} = \{p_1, \ldots, p_t\}$, $\mathcal{R} = \{x \in \mathbb{D} \mid p_t < x < a^-\}$, $\mathcal{A} = \{x \in \mathbb{D} \mid a^- \leq x \leq a^+\}$, where we let $p_t = a^- - t$ and, for each $i < t$, $p_i = p_{i+1} - 1$.

For each h, we define a further partition of the subsets $\mathcal{P}$ and $\mathcal{A}$, as follows:

$$\mathcal{P} = \bigcup \begin{cases} \mathcal{P}_h^- = \{x \mid p_1 \leq x \leq p_{f_{\mathcal{P}}(h)}\} \\ \mathcal{P}_h^+ = \{x \mid p_{t-f_{\mathcal{P}}(h)+1} \leq x \leq p_t\} \\ \mathcal{P}_h = \{x \mid p_{f_{\mathcal{P}}(h)} < x < p_{t-f_{\mathcal{P}}(h)+1}\}, \end{cases}$$

$$\mathcal{A}^i = \{x \in \mathbb{D} \mid a^- + (i-1)\cdot t \leq x < a^- + i \cdot t\},$$

$$\mathcal{A} = \bigcup \begin{cases} \mathcal{A}_h^- = \bigcup_{i=1}^{f_{\mathcal{P}}(h)} \mathcal{A}^i \\ \mathcal{A}_h^+ = \bigcup_{i=t-f_{\mathcal{P}}(h)+1}^{t} \mathcal{A}^i \\ \mathcal{A}_h = \mathcal{A} \setminus (A_h^- \cup A_h^+) = \bigcup_{i=f_{\mathcal{P}}(h)+1}^{t-f_{\mathcal{P}}(h)} \mathcal{A}^i. \end{cases}$$

Roughly speaking, we can say that stepping from $h+1$ to h, the sets $\mathcal{P}_{h+1}^-, \mathcal{P}_{h+1}^+, \mathcal{A}_{h+1}^-$, and $\mathcal{A}_{h+1}^+$ shrink, while the sets $\mathcal{P}_{h+1}$ and $\mathcal{A}_{h+1}$ expand. Now, let M be a model based on $\mathbb{D}$ described as above. We first define a function $\mathcal{V} : \mathcal{A} \to \mathcal{P}$, and then the valuation function V of M, which uses $\mathcal{V}$:

$$\mathcal{V}(y) = \begin{cases} p_1 + i & \text{if } y = a^- + i, \text{ for each } 0 \leq i < t \\ \mathcal{V}(y-t) & \text{if } a^- + t \leq y \leq a^+, \end{cases}$$

$$V(p) = \{[x,y] \mid y \in \mathcal{A} \text{ implies } x \leq \mathcal{V}(y)\}.$$

In order to define an $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation, we first define a sequence Z_N, $\ldots, Z_1$, which is common to both cases $\langle E\rangle$ and $\langle\overline{E}\rangle$, and then we show how to adjust it to obtain our results. To characterize the generic hth component Z_h of the sequence $Z_N, \ldots, Z_1$ we make use of an equivalence relation $\equiv_h$, parameterized by h, which is defined as follows. Let us denote by x (resp., w) the nth element of $\mathcal{A}_i$ (resp., the mth element of $\mathcal{A}_j$), that is, $x = a_n^i$ and $w = a_m^j$. Then, we have:

$$x \equiv_h w \text{ iff} \begin{cases} x = w \text{ or} \\ x, w \in \mathcal{P}_h \text{ or} \\ x, w \in \mathcal{A} \text{ and} \begin{cases} i = j \vee x, w \in \mathcal{A}_h, \text{ and} \\ m = n \vee f_{\mathcal{P}}(h) < m, n < t - f_{\mathcal{P}}(h) + 1. \end{cases} \end{cases}$$

As already pointed out, to define the desired N-bisimulation, we also need an equivalence up to a threshold. Such a relation, denoted $\simeq_h^f$, relates integers, which represent interval lengths, as follows: $a \simeq_h^f b$ if and only if either $a = b$ or both a and b are greater than the threshold $f(h)$. We can now define Z_h as follows: for each $1 \leq h \leq N$, $([x,y],[w,z]) \in Z_h$ if and only if: *(a)* $x \equiv_h w$ and $y \equiv_h z$, *(b)* $y - x \simeq_h^f z - w$, *(c)* if $x, w \in \mathcal{P}$ and $y, z \in \mathcal{A}$, then $\mathcal{V}(y) - x \simeq_h^f \mathcal{V}(z) - w$, and *(d)* if $x \in \mathcal{A}^i$ and $y \in \mathcal{A}^j$ for some $i, j \in \{1, \ldots, t\}$, then $w \in \mathcal{A}^k$ and $z \in \mathcal{A}^\ell$ for some $k, \ell \in \{1, \ldots, t\}$ such that $j - i \simeq_h^f \ell - k$. As a last step, we define a new sequence of relations $Z_N^E, \ldots, Z_1^E$ such that $Z_N^E \cup Z_N, \ldots, Z_1^E \cup Z_1$ is an $\mathsf{ALBDO\overline{ALBEDO}}_N$-bisimulation (the proof is technically involved, so details are omitted). Consider a point $a = a_m^i$ such that $i = m = \frac{t}{2}$, that is, a is the $\frac{t}{2}$th point of the $\frac{t}{2}$th sub-group of $\mathcal{A}$. It holds that $\mathcal{V}(a) = p_m = p_{\frac{t}{2}}$. Now, for each $1 \leq h \leq N$, let $Z_h^E = \{([\mathcal{V}(a) - (N - h + 1), a], [\mathcal{V}(a) - (N - h), a])\}$. It is easy to see that $M, [\mathcal{V}(a) - 1, a] \Vdash \langle E \rangle p$, $M, [\mathcal{V}(a), a] \Vdash \neg\langle E \rangle p$, and $([\mathcal{V}(a)-1, a], [\mathcal{V}(a), a]) \in Z_N^E$. Thus, $Z_N^E \cup Z_N, \ldots, Z_1^E \cup Z_1$ is an $\mathsf{ALBDO\overline{ALBEDO}}_N$-bisimulation that violates $\langle E \rangle$.

To deal with the modality $\langle \overline{E} \rangle$, a new sequence $Z_N^{\overline{E}}, \ldots, Z_1^{\overline{E}}$ can be defined, following a technique similar to the above-described one, so that $Z_N^{\overline{E}} \cup Z_N, \ldots, Z_1^{\overline{E}} \cup Z_1$ is an $\mathsf{ALBEDO\overline{ALBDO}}_N$-bisimulation that violates $\langle \overline{E} \rangle$. Once again, since the proof only uses a finite linear order, the result holds for both Dis and Fin. □

6 Conclusions

In this paper we studied the expressiveness of fragments of the interval temporal logic HS interpreted over both discrete and finite linear orders. A complete classification of all such fragments with respect to their relative expressive power has been recently given for the classes of all linear orders and all dense linear orders. The cases of discrete and finite linear orders turn out to be much more involved. We illustrated here various non-trivial temporal properties that can be expressed when HS is interpreted over them, and we provided a complete set of definabilities for the modalities corresponding to the Allen's relations *meets*, *later*, *begins*, *finishes*, and *during*, plus their transposes. We leave open the problem of identifying the complete set of definabilities for the modalities corresponding to the Allen relation *overlaps* and to its inverse *overlapped by*.

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