Axiomatizing Weak Simulation Semantics over BCCSP☆

Luca Aceto[a,c,d], David de Frutos Escrig[b,c], Carlos Gregorio-Rodríguez[b,c], Anna Ingolfsdottir[a,c]

[a] *ICE-TCS, School of Computer Science, Reykjavik University, Iceland*
[b] *Departamento de Sistemas Informáticos y Computación, Universidad Complutense de Madrid, Spain*
[c] *Abel Extraordinary Chair (Universidad Complutense-Reykjavik University)*
[d] *SysMA, IMT Lucca Institute for Advanced Studies, Lucca 55100, Italy*

Abstract

This paper is devoted to the study of the (in)equational theory of the largest (pre)congruences over the language BCCSP induced by variations on the classic simulation preorder and equivalence that abstract from internal steps in process behaviours. In particular, the article focuses on the (pre)congruences associated with the weak simulation, the weak complete simulation and the weak ready simulation preorders. We present results on the (non)existence of finite (ground-)complete (in)equational axiomatizations for each of these behavioural semantics. The axiomatization of those semantics using conditional equations is also discussed in some detail.

Keywords: Process algebra, simulation semantics, complete simulation

☆Luca Aceto and Anna Ingolfsdottir have been partially supported by the projects 'New Developments in Operational Semantics' (nr. 080039021) and 'Meta-theory of Algebraic Process Theories' (nr. 100014021) of the Icelandic Research Fund. David de Frutos Escrig and Carlos Gregorio-Rodríguez have been partially supported by the Spanish projects TESIS (TIN2009-14312-C02-01), DESAFIOS10 (TIN2009-14599-C03-01) and PROMETIDOS S2009/TIC-1465. The paper was begun when David de Frutos Escrig and Carlos Gregorio-Rodríguez held Abel Extraordinary Chair positions at Reykjavik University, and finalized while Luca Aceto and Anna Ingolfsdottir held Abel Extraordinary Chairs at Universidad Complutense de Madrid, Spain, supported by the NILS Mobility Project. Luca Aceto was also supported by a visiting professorship at IMT Lucca Institute for Advanced Studies, Italy.

semantics, ready simulation semantics, equational logic, equational axiomatizations, non-finitely based algebras

---

Contents

1. Introduction

Process algebras, such as ACP [13, 15], CCS [43] and CSP [35], are prototype specification languages for reactive systems. Such languages offer a small, but expressive, collection of operators that can be combined to form terms that describe the behaviour of reactive systems. Since terms in a process algebra can be used to describe both specifications of reactive behaviours and their implementations, an important ingredient in the theory of such languages is a notion of behavioural equivalence or preorder over terms. The chosen notion of behavioural semantics can be employed, for instance, to prove whether a term describing an implementation of a system is correct with respect to a given specification. The lack of consensus on what constitutes an appropriate notion of observable behaviour for reactive systems has led to a large number of proposals for behavioural equivalences and preorders for concurrent processes. In his by now classic paper [28], van Glabbeek presented the linear time-branching time spectrum of behavioural preorders and equivalences for finitely branching, concrete, sequential processes. The semantics in this spectrum are based on notions of simulation and on decorated traces.

Since the seminal work by Bergstra and Klop [13, 15], and Hennessy and Milner [34], the search for (in)equational axiomatizations of notions of behavioural semantics for fragments of process algebras has been one of the classic topics of investigation within concurrency theory. A complete axiomatization of a behavioural semantics yields a purely syntactic and model-independent characterization of the semantics of a process algebra, and paves the way to the application of theorem-proving techniques in establishing whether two process descriptions exhibit related behaviours.

There are three types of 'complete' axiomatizations that one meets in the literature on process algebras. An (in)equational axiomatization is called *ground-complete* if it can prove all the valid (in)equivalences relating terms with no occurrences of variables in the process algebra of interest. It is *complete* when it can be used to derive all the valid (in)equivalences. (A complete axiom system is also referred to as a *basis* for the algebra it axiomatizes.) These two notions of completeness relate the semantic notion of process, namely an equivalence class of terms, with the proof-theoretic notion of provability from an (in)equational axiom system. In particular, a basis for an algebra of processes offers a full, purely syntactic view of the semantic notion of 'process' that underlies it. An axiomatization E is *ω-complete* when an inequation can be derived from E if, and only if, all of its closed instantiations can be derived from E. The notion of ω-completeness is therefore a proof-theoretic one. Its connections with completeness are well known, and are discussed in, e.g., [8].

In [28], van Glabbeek studied the semantics in his spectrum in the setting of the process algebra BCCSP, which contains only the basic process algebraic operators from CCS [43] and CSP [35], but is sufficiently powerful to express all finite synchronization trees [41]. In the aforementioned reference, van Glabbeek gave, amongst a wealth of other results, (in)equational ground-complete axiomatizations for the equivalences and preorders in the spectrum. In [22], two of the authors of this paper presented a unification of the axiomatizations of all the semantics in the linear time-branching time spectrum. This unification is achieved by means of conditional axioms that provide a simple and clear picture of the similarities and differences between all the semantics. In [31], Groote obtained ω-completeness results for most of the axiomatizations presented in [28], in case the alphabet of actions is infinite.

The article [8] surveys results on the existence of finite, complete equational axiomatizations of behavioural equivalences over fragments of process algebras up to 2005. Some of the results on the (non)existence of finite, complete (in)equational axiomatizations of behavioural semantics over process algebras that have been obtained since the publication of that survey may be found in [2, 3, 9, 10, 12, 18].

In the setting of BCCSP, in a seminal journal paper that collects and unifies the results in a series of conference articles, Chen, Fokkink, Luttik and Nain have offered in [21] a definitive classification of the status of the finite basis problem—that is, the problem of determining whether a behavioural equivalence has a finite, complete, equational axiomatization over the chosen process algebra—for all the semantics in van Glabbeek's spectrum. Notable later results by Chen and Fokkink, presented in [18], give the first example of a semantics—the so-called *impossible future semantics* from [49]—where the preorder defining the semantics can be finitely axiomatized over BCCSP, but its induced equivalence cannot. The authors of this paper have recently shown in [5] that complete simulation and ready simulation semantics do not afford a finite (in)equational axiomatization even when the set of actions is a singleton.

The collection of results mentioned in the previous paragraph gives a complete picture of the axiomatizability of behavioural semantics in van Glabbeek's spectrum over BCCSP. However, such notions of behavioural semantics are *concrete*, in the sense that they consider each action processes perform as being observable by their environment. Despite the fundamental role they play in the development of a theory of reactive systems, concrete semantics are not very useful from the point of view of applications. For this reason, notions of behavioural semantics that, in some well-defined way, abstract from externally unobservable steps of computation that processes perform have been proposed in the literature—see, e.g., the classic references [26, 30, 34], which offer, amongst many other results, ground-complete axiomatizations of the studied notion of behavioural semantics. (Following Milner, such notions of behavioural semantics are usually called 'weak semantics'.) However, to the best of our knowledge, no systematic study of the axiomatizability properties of variations on the classic notion of *simulation semantics* [40, 46] that abstract away from internal steps of computation in the behaviour of processes has been presented in the literature. This is all the more surprising since simulation semantics is very natural and plays an important role in applications.

The aim of this paper is to offer a detailed study of the axiomatizability properties of the largest (pre)congruences over the language BCCSP induced by variations on the classic simulation preorder and equivalence that abstract from internal steps in process behaviours. In particular, we focus on the (pre)congruences associated with the weak simulation, the weak complete simulation [28] and the weak ready simulation [16, 36] preorders. For each of these behavioural semantics, we present results on the (non)existence of finite (ground-)complete (in)equational axiomatizations. Following [22], we also discuss the axiomatization of those semantics using conditional equations in some detail.

We begin our study of the weak simulation semantics over BCCSP in Section 3 by focusing on the natural extension of the classic simulation preorder to a setting with the internal action τ. Unlike most other notions of semantics for reactive systems that abstract from internal steps of computation, the *weak simulation preorder* and its induced equivalence are preserved by all the operators of BCCSP. Indeed, the equation

$$\tau x = x$$

is sound modulo weak simulation equivalence and, using it, one can remove all occurrences of the symbol τ from terms. This allows us to lift all the known results on the (non)existence of finite (ground-)complete axiomatizations from the setting of the classic simulation semantics to its weak counterpart using, for instance, the approach developed in [10].

In Section 4, we study the notion of *weak complete simulation*, which is the 'weak counterpart' of complete simulation. In the setting without internal actions, a complete simulation is a simulation that can only relate a state without outgoing transitions to states having the same property. In particular, unlike in the setting of the simulation preorder, the inequation

$$\mathbf{0} \leq x$$

does *not* hold in complete simulation semantics. Our definition of the notion of weak complete simulation is based on considering a process 'complete', or 'mute', when it cannot perform any observable action. For instance, τ is mute, but neither τa nor $\tau + a$ is. The resulting preorder is not preserved by non-deterministic choice. However, unlike in the setting of weak bisimilarity and branching bisimilarity [30, 43], in order to characterize the largest precongruence over BCCSP included in the weak complete simulation preorder, one has only to take special care in handling initial τ-labelled transitions when they lead to a mute process. This semantics satisfies the inequation

$$x \leq \tau x,$$

but not $\tau x \leq x$. For example, $\tau\mathbf{0} \leq \mathbf{0}$ does not hold because $\tau\mathbf{0} + a$ may become mute by performing an internal computational step, whereas $\mathbf{0} + a$ cannot do so. On the other hand, the inequation $\tau a \leq a$ does hold because the initial internal step from τa does not lead to a mute process.

We offer finite (un)conditional ground-complete axiomatizations for the weak complete simulation precongruence. In sharp contrast to this positive result, we prove that, in the presence of at least one observable action, the (in)equational theory of the weak complete simulation precongruence over BCCSP does *not* have a finite (in)equational basis. In fact, the collection of (in)equations in at most one variable that hold in weak complete simulation semantics over BCCSP does not have an (in)equational basis of 'bounded depth', let alone a finite one.

Section 5 paints a similar picture for weak ready simulation semantics. However, in this case we must be careful with some technical details, because even the characterization of the largest precongruence included in the weak ready simulation preorder depends on whether the set of observable actions is finite or not. Moreover, the existence of a finite complete axiomatization of this semantics depends crucially on having an infinite alphabet of observable actions.

A *weak ready simulation* is a weak simulation that can only relate states that afford the same sets of observable actions. It turns out that, if the set of observable actions A is finite, the following inequation is sound for each term p:

$$\left(\tau \sum_{a\in A} a\right) + p \leq \left(\sum_{a\in A} a\right) + p.$$

This indicates that one has only to take special care in handling initial τ-labelled transitions when they lead to a process that does not initially afford each action in A.

We offer finite (un)conditional ground-complete axiomatizations for the weak ready simulation precongruence. In sharp contrast to this positive result, we prove that, when the set of observable actions A is finite and non-empty, the (in)equational theory of the weak ready simulation precongruence over BCCSP does *not* have a finite (in)equational basis. In fact, as was the case for weak complete simulation semantics, the collection of (in)equations in at most one variable that hold in weak ready simulation semantics over BCCSP does not have an (in)equational basis of 'bounded depth', let alone a finite one.

The paper is organized as follows. Section 2 presents the syntax and the operational semantics for the language BCCSP, and reviews the necessary background on (in)equational logic as well as classic axiom systems for strong bisimulation equivalence and observational congruence (the largest congruence included in weak bisimulation equivalence). In Section 3, we define the weak simulation preorder and present our results on its (in)equational axiomatization. Sections 4 and 5 are devoted to results on the weak complete and weak ready simulation preorders, respectively. We conclude the paper by discussing further related work and directions for future research in Section 6.

This study is an expanded version of the conference paper [4]. Apart from offering full proofs of the results that were announced without proof in [4], this paper includes a new 18-page section (Section 4) devoted to the axiomatizability of weak complete simulation semantics. (The main technical contributions we give in Section 4 were announced, without proof, in [6].) In addition, Section 3.3, which is devoted to the axiomatization of a natural variation on the weak simulation preorder, Section 5.3, which discusses alternative notions of weak ready simulation semantics, and Appendix A are also new in this paper.

2. Preliminaries

To set the stage for the developments offered in the rest of the paper, we present the syntax and the operational semantics for the language BCCSP, some background on (in)equational logic, and classic axiom systems for strong bisimulation equivalence and observational congruence [43].

Syntax of BCCSP. BCCSP(A_τ) is a basic process algebra for expressing finite process behaviour. Its syntax consists of closed (process) terms p, q, r that are constructed from a constant $\mathbf{0}$, a binary operator $\_ + \_$ called *alternative composition*, or *choice*, and unary *prefix* operators $\alpha\_$, where α ranges over some set A_τ of *actions* of the form $A \cup \{\tau\}$, where τ is a distinguished action symbol that is not contained in A. Following Milner [43], we use τ to denote an internal, unobservable action of a reactive system, and we let a, b, c denote typical elements of A and α range over A_τ. The set of closed terms is named $\mathrm{T}(\text{BCCSP}(A_\tau))$, in short $\mathrm{T}(A_\tau)$.

We write $|A|$ for the cardinality of the set of actions A.

Open terms t, u, v can moreover contain occurrences of variables from a countably infinite set V (with typical elements x, y, z). $\mathbb{T}(\text{BCCSP}(A_\tau))$, in short $\mathbb{T}(A_\tau)$, denotes the set of open terms. The *depth* of a term t, written $|t|$, is the maximum nesting of prefix operators in t. The depth of a term may be easily defined by induction thus: $|\mathbf{0}| = |x| = 0$, $|\alpha t| = 1 + |t|$ and $|t + u| = \max(|t|, |u|)$.

In what follows, for each non-negative integer n and term t, we use $a^n t$ to stand for t when $n = 0$, and for $a(a^{n-1}t)$ otherwise. As usual, trailing occurrences of $\mathbf{0}$ are omitted; for example, we shall usually write α in lieu of $\alpha\mathbf{0}$.

A (closed) substitution maps variables in V to (closed) terms. For every term t and substitution σ, the term $\sigma(t)$ is obtained by replacing every occurrence of a variable x in t by $\sigma(x)$. Note that $\sigma(t)$ is closed if σ is a closed substitution. We say that σ is a $\mathbb{T}(A)$-substitution if its range is included in $\mathbb{T}(A)$.

We sometimes write $[t_1/x_1, \ldots, t_n/x_n]$, where $t_1, \ldots, t_n$ is a sequence of terms and $x_1, \ldots, x_n$ is a sequence of distinct variables, for the substitution that maps each x_i to t_i, $1 \leq i \leq n$, and acts like the identity function on all the other variables.

Transitions and their defining rules. Intuitively, closed BCCSP(A_τ) terms denote finite process behaviours, where $\mathbf{0}$ does not exhibit any behaviour,

$p + q$ is the nondeterministic choice between the behaviours of p and q, and αp executes action α to transform into p. This intuition is captured, in the style of Plotkin [47], by the simple transition rules below, which give rise to A_τ-labelled transitions between closed terms.

$$\frac{}{\alpha x \xrightarrow{\alpha} x} \qquad \frac{x \xrightarrow{\alpha} x'}{x + y \xrightarrow{\alpha} x'} \qquad \frac{y \xrightarrow{\alpha} y'}{x + y \xrightarrow{\alpha} y'}$$

The operational semantics is extended to open terms by assuming that variables do not exhibit any behaviour.

The so-called *weak transition relations* $\stackrel{\alpha}{\Longrightarrow}$ ($\alpha \in A_\tau$) are defined over $\mathbb{T}(A_\tau)$ in the standard fashion as follows.

- We use $\stackrel{\tau}{\Longrightarrow}$ for the reflexive and transitive closure of $\xrightarrow{\tau}$.
- For each $a \in A$ and for all terms $t, u \in \mathbb{T}(A_\tau)$, we have that $t \stackrel{a}{\Longrightarrow} u$ if, and only if, there are $t_1, t_2 \in \mathbb{T}(A_\tau)$ such that $t \stackrel{\tau}{\Longrightarrow} t_1 \xrightarrow{a} t_2 \stackrel{\tau}{\Longrightarrow} u$.

As usual, see, for instance, [43], we extend the weak transition relations to sequences of actions in A thus:

- $t \stackrel{\varepsilon}{\Longrightarrow} u$, where ε denotes the empty string, if, and only if, $t \stackrel{\tau}{\Longrightarrow} u$;
- $t \stackrel{as}{\Longrightarrow} u$, where $a \in A$ and $s \in A^*$, if, and only if, there is some $t' \in \mathbb{T}(A_\tau)$ such that $t \stackrel{a}{\Longrightarrow} t' \stackrel{s}{\Longrightarrow} u$.

For each term t, we define

$$I^*(t) = \{a \mid a \in A \text{ and } t \stackrel{a}{\Longrightarrow} t' \text{ for some } t'\}.$$

Preorders and their kernels. We recall that a *preorder* $\precsim$ is a reflexive and transitive relation. In what follows, any preorder $\precsim$ we consider will first be defined over the set of closed terms $\mathrm{T}(A_\tau)$. For terms $t, u \in \mathbb{T}(A_\tau)$, we define $t \precsim u$ if, and only if, $\sigma(t) \precsim \sigma(u)$ for each closed substitution σ.

The *kernel* $\approx$ of a preorder $\precsim$ is the equivalence relation it induces, and is defined thus:

$$t \approx u \text{ if, and only if, } (t \precsim u \text{ and } u \precsim t).$$

It is easy to see that the kernel of a preorder $\precsim$ is the largest symmetric relation included in $\precsim$.

Inequational logic. An *inequation* (respectively, an *equation*) over the language BCCSP(A_τ) is a formula of the form $t \leq u$ (respectively, $t = u$), where t and u are terms in $\mathbb{T}(A_\tau)$. An *(in)equational axiom system* is a set of (in)equations over the language BCCSP(A_τ). An equation $t = u$ is derivable from an equational axiom system E, written $E \vdash t = u$, if it can be proven from the axioms in E using the rules of equational logic (viz. reflexivity, symmetry, transitivity, substitution and closure under BCCSP(A_τ) contexts).

$$t = t \quad \frac{t = u}{u = t} \quad \frac{t = u \;\; u = v}{t = v} \quad \frac{t = u}{\sigma(t) = \sigma(u)} \quad \frac{t = u}{\alpha t = \alpha u} \quad \frac{t = u \;\; t' = u'}{t + t' = u + u'}$$

For the derivation of an inequation $t \leq u$ from an inequational axiom system E, the rule for symmetry—that is, the second rule above—is omitted. We write $E \vdash t \leq u$ if the inequation $t \leq u$ can be derived from E.

It is well known that, without loss of generality, one may assume that substitutions happen first in (in)equational proofs, i.e., that the fourth rule may only be used when its premise is one of the (in)equations in E. Moreover, by postulating that for each equation in E also its symmetric counterpart is present in E, one may assume that applications of symmetry happen first in equational proofs, i.e., that the second rule is never used in equational proofs. (See, e.g., [21, page 497] for a thorough discussion of this notion of 'normalized equational proof'.) In the remainder of this paper, we shall always tacitly assume that equational axiom systems are closed with respect to symmetry. Note that, with this assumption, there is no difference between the rules of inference of equational and inequational logic. In what follows, we shall consider an equation $t = u$ as a shorthand for the pair of inequations $t \leq u$ and $u \leq t$.

The depth of $t \leq u$ and $t = u$ is the maximum of the depths of t and u. The depth of a collection of (in)equations is the supremum of the depths of its elements. So, the depth of a finite axiom system E is zero, if E is empty, and it is the largest depth of its (in)equations otherwise.

An inequation $t \leq u$ is *sound* with respect to a given preorder relation $\precsim$ if $t \precsim u$ holds. (Sometimes we say that $t \leq u$ *holds modulo* $\precsim$, in lieu of '$t \leq u$ is sound with respect to a given preorder relation $\precsim$.') An (in)equational axiom system E is sound with respect to $\precsim$ if so is each (in)equation in E.

Classic axiomatizations for notions of bisimilarity. The well-known axioms B_1–B_4 for BCCSP(A_τ) given below stem from [34]. These axioms are ω-

complete [45], and sound and ground-complete [34, 43], over BCCSP(A_τ) (over any non-empty set of actions) modulo bisimulation equivalence [43, 46], which is the finest semantics in van Glabbeek's spectrum [28].

$$\begin{array}{lrcl} B_1 & x+y & = & y+x \\ B_2 & (x+y)+z & = & x+(y+z) \\ B_3 & x+x & = & x \\ B_4 & x+\mathbf{0} & = & x \end{array}$$

In what follows, for notational convenience, we consider terms up to the least congruence generated by axioms B_1–B_4, that is, up to bisimulation equivalence. We use *summation* $\sum_{i=1}^{n} t_i$ (with $n \geq 0$) to denote $t_1 + \cdots + t_n$, where the empty sum denotes $\mathbf{0}$. Modulo the equations B_1–B_4 each term $t \in \mathbb{T}(A_\tau)$ can be written in the form $\sum_{i=1}^{n} t_i$, where each t_i is either a variable or is of the form $\alpha t'$, for some action α and term t'.

The following lemma is standard and will be implicitly used in the technical developments to follow.

Lemma 1. *Let t, t', u be terms, let s be a sequence of actions in A and let σ be a substitution.*

1. *If $t \stackrel{s}{\Longrightarrow} t'$ then $\sigma(t) \stackrel{s}{\Longrightarrow} \sigma(t')$.*
2. *If $t \stackrel{s}{\Longrightarrow} x + t'$, for some variable x, and $\sigma(x) \stackrel{s'}{\Longrightarrow} u$ for some $s' \in A^*$ and $u \neq \sigma(x)$, then $\sigma(t) \stackrel{ss'}{\Longrightarrow} u$.*
3. *If $\sigma(t) \stackrel{s}{\Longrightarrow} u$ then*
   (a) *either $t \stackrel{s}{\Longrightarrow} t'$ for some t' such that $\sigma(t') = u$*
   (b) *or there are sequences s' and s'' of actions in A with $s = s's''$, some variable x and some t' such that $t \stackrel{s'}{\Longrightarrow} x + t'$, $\sigma(x) \stackrel{s''}{\Longrightarrow} u$ and $u \neq \sigma(x)$.*

In a setting with internal transitions, the classic work of Hennessy and Milner on *weak bisimulation equivalence* and on the largest congruence included in it, *observational congruence*, shows that the axioms B_1–B_4 together with the axioms W_1–W_3 below are sound and complete over BCCSP(A_τ) modulo observational equivalence. (See [34, 43, 44].)

$$\begin{array}{lrcl} W_1 & \alpha x & = & \alpha\tau x \\ W_2 & \tau x & = & \tau x + x \\ W_3 & \alpha(\tau x + y) & = & \alpha(\tau x + y) + \alpha x \end{array}$$

Natural definition of a weak relation, order and equivalence $\precsim_{RS} \quad \succsim_{RS} \quad \approx_{RS}$
Abstract largest (pre)congruence contained in a weak relation $\sqsubseteq_{RS} \quad \sqsupseteq_{RS} \quad \equiv_{RS}$
Operational characterization of the largest (pre)congruence $\lesssim_{RS} \quad \gtrsim_{RS} \quad \eqsim_{RS}$
Relations defined by axioms $\leq \quad \geq \quad =$

Table 1: General symbol notation used for relations, using ready simulation as a concrete example

The above axioms are often referred to as the τ-*laws*. For ease of reference, we write

$$BW = \{B_1, B_2, B_3, B_4, W_1, W_2, W_3\}.$$

As it is well known, when dealing with process algebras with internal, unobservable actions, usually a 'natural' definition of a behavioural semantics does not yield a (pre)congruence. In this case, it is customary to consider the largest (pre)congruence included in the behavioural relation of interest. Throughout the paper, we use quite a number of relations defined for the language BCCSP(A_τ) and, for the sake of clarity, in Table 1 we summarize the main symbol conventions we use to give them names. The subscripting is used to differentiate between semantics. For instance, $\precsim_{RS}$ is the symbol we will use for the weak ready simulation preorder, while $\precsim_S$ is the one we will use for the weak simulation preorder. In the subsequent sections, we will define these, as well as other, relations and study their (in)equational theories.

3. Weak simulation

We begin our study of the equational theory of weak simulation semantics by considering the natural, τ-abstracting version of the classic simulation preorder [40, 46]. We start by defining the notion of weak simulation preorder and the equivalence relation it induces. We then argue that all the known positive and negative results on the existence of (ground-)complete (in)equational axiomatizations for the concrete simulation semantics over the language BCCSP(A_τ) can be lifted to the corresponding weak semantics.

Definition 1. The *weak simulation preorder*, denoted by $\precsim_S$, is the largest relation over terms in $\mathrm{T}(A_\tau)$ satisfying the following condition whenever $p \precsim_S q$ and $\alpha \in A_\tau$:

- if $p \xrightarrow{\alpha} p'$ then there exists some q' such that $q \xRightarrow{\alpha} q'$ and $p' \precsim_S q'$.

We say that $p, q \in \mathrm{T}(A_\tau)$ are *weak simulation equivalent*, written $p \approx_S q$, iff p and q are related by the kernel of $\precsim_S$, that is when both $p \precsim_S q$ and $q \precsim_S p$ hold.

Unlike many other notions of behavioural relations that abstract away from internal steps in the behaviour of processes, see [30, 41, 50] for classic examples, the weak simulation preorder is a precongruence over the language we consider in this study.

Proposition 1. *The preorder $\precsim_S$ is a precongruence over $\mathrm{T}(A_\tau)$. Hence $\approx_S$ is a congruence over $\mathrm{T}(A_\tau)$. Moreover, the axiom*

$$(\tau e) \quad \tau x = x$$

holds over $\mathrm{T}(A_\tau)$ modulo $\approx_S$.

PROOF. The relation

$$\begin{aligned} \mathcal{R} \;=\; & \{(\alpha p, \alpha q) \mid p \precsim_S q,\ \alpha \in A_\tau\} \cup \\ & \{(p + r, q + r) \mid p \precsim_S q,\ r \in \mathrm{T}(A_\tau)\} \cup \\ & \{(p, q + r) \mid p \precsim_S q,\ r \in \mathrm{T}(A_\tau)\} \cup \precsim_S \end{aligned}$$

satisfies the conditions in Definition 1. Therefore $\precsim_S$ is a precongruence over $\mathrm{T}(A_\tau)$. It is well known that the kernel of a precongruence is a congruence.

To see that the axiom $\tau x = x$ holds modulo $\approx_S$, it suffices to observe that the relation

$$\{(p, \tau p), (\tau p, p), (p, p) \mid p \in \mathrm{T}(A_\tau)\}$$

satisfies the conditions in Definition 1. □

The soundness of equation τe is the key to all the results on the equational theory of the weak simulation semantics we present in the remainder of this section. In establishing the negative results, we shall make use of the reduction technique from the paper [10].

We start by defining the reduction function $\hat{\cdot} : \mathbb{T}(A_\tau) \to \mathbb{T}(A)$ as the unique homomorphism satisfying

$$\begin{aligned} \hat{x} &= x \text{ for each } x \in V, \text{ and} \\ \widehat{\tau t} &= \hat{t} \text{ for each } t \in \mathbb{T}(A_\tau). \end{aligned}$$

The following properties of $\hat{\cdot}$ will be useful in the technical developments to follow.

Lemma 2.

1. *$\hat{\cdot}$ is the identity function over terms in $\mathbb{T}(A)$.*
2. *$\hat{\cdot}$ is structural, in the sense of [10, Definition 3]. (For the sake of completeness, we recall that $\hat{\cdot}$ is structural if it is the identity function over variables, it does not introduce new variables, and it is defined compositionally.)*
3. *For each term $t \in \mathbb{T}(A_\tau)$ and $\mathbb{T}(A_\tau)$-substitution σ, it holds that $\widehat{\sigma(t)} = \widehat{\sigma}(\hat{t})$, where $\widehat{\sigma}$ is the $\mathbb{T}(A)$-substitution mapping each variable x to the term $\widehat{\sigma(x)}$.*

PROOF. The first statement is immediate from the definition of $\hat{\cdot}$. The third follows from the second and Lemma 1 in [10]. To establish the second statement, observe that

- $\hat{\cdot}$ is the identity function over variables,
- for each term $t \in \mathbb{T}(A_\tau)$, the variables occurring in $\hat{t}$ are exactly the variables occurring in t and
- for all terms $t, u \in \mathbb{T}(A_\tau)$, actions $\alpha \in A_\tau$ and distinct variables x, y,

$$\begin{aligned} \widehat{\alpha t} &= (\widehat{\alpha x})[\hat{t}/x] \quad \text{and} \\ \widehat{t+u} &= (\widehat{x+y})[\hat{t}/x, \hat{u}/y]. \end{aligned}$$

Therefore $\hat{\cdot}$ meets the requirements for a structural mapping laid out in [10, Definition 3]. □

Lemma 3. *The following statements hold.*

1. *For each $t \in \mathbb{T}(A_\tau)$, the equation $t = \hat{t}$ is provable using axiom τe, and therefore $t \approx_S \hat{t}$.*
2. *For all $t, u \in \mathbb{T}(A_\tau)$, the inequation $t \leq u$ holds modulo $\precsim_S$ over $\mathbb{T}(A_\tau)$ iff $\hat{t} \leq \hat{u}$ holds over $\mathbb{T}(A)$ modulo the simulation preorder.*

PROOF. Statement 1 can be shown easily by structural induction on t. To prove statement 2, first observe that the claim can be shown to hold for inequations relating closed terms. (See [10] for similar proofs.) Using the validity of the claim over closed terms, statement 1 and Lemma 2(3), it is routine to prove that statement 2 holds for open terms too. □

3.1. Ground-completeness

Besides the equation τe previously stated in Proposition 1, there will be another important equation to consider in order to achieve an axiomatic characterization of the weak simulation preorder, namely

$$(S) \quad x \leq x + y.$$

This equation also plays an essential role in the axiomatization of the simulation preorder in the concrete case [23, 28].

Proposition 2. *The set of equations*

$$E_{S\leq} = \{B_1, B_2, B_3, B_4, S, \tau e\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo $\precsim_S$.*

PROOF. We limit ourselves to showing that the axiom system mentioned in the statement of the proposition is ground-complete. To this end, assume that $p, q \in \mathrm{T}(A_\tau)$ and $p \precsim_S q$. By Lemma 3, using axiom τe, we can prove the equations $p = \hat{p}$ and $q = \hat{q}$. Moreover, the inequation $\hat{p} \leq \hat{q}$ holds modulo the simulation preorder. It is well known that the axiom system $\{B_1, B_2, B_3, B_4, S\}$ is ground-complete for the simulation preorder over the language $\mathrm{T}(A)$. Therefore, the inequation $\hat{p} \leq \hat{q}$ is provable from it. By combining a proof of $\hat{p} \leq \hat{q}$, with proofs for the equations $p = \hat{p}$ and $q = \hat{q}$, one obtains a proof of $p \leq q$. □

The completeness result in Proposition 2 was announced in [29] by van Glabbeek. Since no proof is given in that paper, and for the sake of methodology, we have included here our proof.

Note that the equations W_1–W_3, even if sound for $\precsim_S$, are not needed in order to obtain a ground-complete axiomatization of $\precsim_S$ over BCCSP(A_τ). Those equations can easily be derived from the axiom system in Proposition 2.

To obtain an axiomatization for the weak simulation equivalence, we need the equation

$$(SE) \quad a(x+y) = a(x+y) + ay \qquad (a \in A).$$

This equation is well known from the setting of standard simulation equivalence, where it is known to be the key to a ground-complete axiomatization [28].

Proposition 3. *The set of equations*

$$E_{S^=} = \{B_1, B_2, B_3, B_4, SE, \tau e\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\approx_S$.

PROOF. The algorithm *weak ready to preorder* from [19, pages 107–108] can be applied. A direct proof using Lemma 3 is also immediate. □

3.2. ω-completeness

Propositions 2 and 3 offer ground-complete axiomatizations for the weak simulation preorder and its kernel over BCCSP(A_τ). The inequational axiomatization of the weak simulation preorder is finite, and so is the one for its kernel if the set of actions A is finite. In the presence of an infinite collection of actions, the axiom system in Proposition 3 is finite if we consider a to be an action variable. It is natural to wonder whether the weak simulation semantics afford finite (in)equational axiomatizations that are complete over $\mathbb{T}(A_\tau)$. The following results answer this question.

Proposition 4. *If the set of actions is infinite, then the axiom system*

$$E_{S\leq} = \{B_1, \ldots, B_4, S, \tau e\}$$

is ω-complete over BCCSP(A_τ) *modulo* $\precsim_S$.

PROOF. To prove the result, we use Groote's inverted substitution technique [31]. Actually, we use a variation on that technique appearing in Chen's PhD. thesis [17] and in [20, Section 5], which is valid also for preorders and not only for equivalences.

In the rest of the proof, for the sake of readability, we abbreviate $E_{S\leq}$ by E. Given an inequation $t_0 \leq u_0 \in E$, let σ be the closed substitution $\sigma : V \longrightarrow \mathrm{T}(A_\tau)$ defined as follows: $\sigma(x) = a_x\mathbf{0}$, where a_x is a distinguished action for each variable $x \in V$, and every a_x does not appear in t_0, u_0. (This is possible because A is infinite.)

Let $\rho : \mathrm{T}(A_\tau) \longrightarrow \mathbb{T}(A_\tau)$ be structurally defined as follows.

$$\begin{aligned}
&\rho(\mathbf{0}) = \mathbf{0} \\
&\rho(\alpha u) = \alpha\rho(u), \text{ when } \alpha \neq a_x \text{ for each } x \in V \\
&\rho(a_x u) = x \\
&\rho(u+v) = \rho(u) + \rho(v)
\end{aligned}$$

Next we check the three properties needed to apply Chen's result.

(1) $E \vdash t_0 \leq \rho(\sigma(t_0))$ and $E \vdash \rho(\sigma(u_0)) \leq u_0$.
It can be easily proved by structural induction that if an open term u does not contain action prefix operators of the form a_x, then $u = \rho(\sigma(u))$. From this (1) follows trivially.

(2) $E \vdash \rho(\sigma'(t)) \leq \rho(\sigma'(u))$, for each $t \leq u \in E$ and closed substitution σ'.
Here, as an example, we show the proof for the inequation S.

$$\begin{aligned}
\rho(\sigma'(x)) &\leq \rho(\sigma'(x)) + \rho(\sigma'(y)) \\
&= \rho(\sigma'(x) + \sigma'(y)) \\
&= \rho(\sigma'(x+y))
\end{aligned}$$

The first inequality is an instance of inequation S and the subsequent equalities follow from the fact that ρ and σ' are homomorphisms with respect to the choice operator $+$.

(3) $E \cup \{u_i \leq u_i', \rho(u_i) \leq \rho(u_i') \mid i = 1,2\} \vdash \rho(u_1 + u_2) \leq \rho(v_1 + v_2)$ and $E \cup \{u \leq v, \rho(u) \leq \rho(v)\} \vdash \rho(\alpha u) \leq \rho(\alpha v)$.
These are straightforward. By way of example, we just present the proof of the second claim. If $\alpha = a_x$, for some variable x, then $x \leq x$ can trivially be proved by reflexivity. Otherwise, $\rho(\alpha u) \leq \rho(\alpha v)$ becomes $\alpha\rho(u) \leq \alpha\rho(v)$, which can be proved from $E \cup \{u \leq v, \rho(u) \leq \rho(v)\}$. □

Corollary 1. *If the set of actions is infinite, then the axiom system*

$$E_{S\leq} = \{B_1, \ldots, B_4, S, \tau e\}$$

is complete over BCCSP(A_τ) *modulo* $\precsim_S$.

PROOF. The axiom system $E_{S\leq}$ is both ground-complete (Proposition 2) and ω-complete (Proposition 4). It is well known that an axiom system with these properties is complete—see, for example, [8, Remark 2]. $\square$

So the weak simulation preorder has a finite axiomatization over $\mathbb{T}(A_\tau)$ when A is infinite. This state of affairs changes dramatically when A is a finite collection of actions of cardinality at least two. Chen et al. proved in [21] that this was already true in the concrete case, using a rather involved family of equations (see page 511 in that paper). Since the precise form of that family of equations is immaterial in what follows, we will not describe it in detail here and we will simply refer to it as FESE.

Proposition 5. *If* $1 < |A| < \infty$, *then the weak simulation equivalence does not afford a finite equational axiomatization over* $\mathbb{T}(A_\tau)$. *In particular, no finite axiom system over* $\mathbb{T}(A_\tau)$ *that is sound modulo weak simulation equivalence can prove all the (valid) equations in the family FESE.*

PROOF. By Theorem 28 in [21], there is no finite axiom system over $\mathbb{T}(A)$ that is sound modulo simulation equivalence and can prove all the equations in the family FESE. We will now use the results that we have obtained so far, in combination with the reduction technique presented in [10], to lift this negative result to the setting of weak simulation equivalence over $\mathbb{T}(A_\tau)$.

By Lemma 2(2) and [10, Theorem 2], we have that the mapping $\hat{\cdot}$ preserves provability of equations. This means that, for any axiom system E over $\mathbb{T}(A_\tau)$, if E proves an equation $t = u$ then $\widehat{E}$ proves $\hat{t} = \hat{u}$, where

$$\widehat{E} = \{\widehat{t'} = \widehat{u'} \mid (t' = u') \in E\}.$$

By Lemma 2(1), $\hat{\cdot}$ reflects the family FESE, since those equations relate terms that do not contain occurrences of τ. Lemma 3(2) tells us that $\hat{\cdot}$ preserves the soundness of inequations. We may therefore apply [10, Theorem 1] to infer that no finite axiom system E over $\mathbb{T}(A_\tau)$ that is sound modulo weak simulation equivalence can prove all of the equations in the family FESE. Therefore weak simulation equivalence affords no finite equational axiomatization over $\mathbb{T}(A_\tau)$. $\square$

Weak Simulation	Ground-complete		Complete	
Finite Equations	Order	Equiv.	Order	Equiv.
$\lvert A\rvert = \infty$	$E_{S^{\leq}}$	$E_{S^{=}}$	$E_{S^{\leq}}$	$E_{S^{=}}$
$1 < \lvert A\rvert < \infty$	$E_{S^{\leq}}$	$E_{S^{=}}$	Do not exist	
$\lvert A\rvert = 1$	$E_{S_1^{\leq}}$	$E_{S_1^{=}}$	$E_{S_1^{\leq}}$	$E_{S_1^{=}}$

Table 2: Axiomatizations for the weak simulation semantics

Corollary 2. *If* $1 < |A| < \infty$*, then the weak simulation preorder does not afford a finite inequational axiomatization over* $\mathbb{T}(A_\tau)$*.*

PROOF. If the weak simulation preorder afforded a finite inequational axiomatization over $\mathbb{T}(A_\tau)$ then one could obtain a finite equational axiomatization for weak simulation equivalence by applying the algorithm presented in [19]. The existence of such an axiomatization would contradict Proposition 5. Alternatively, one could replay the proof of Theorem 28 in [21], which also applies essentially unchanged to the (weak) simulation preorder. □

Remark 1. If A is a singleton then the simulation preorder coincides with trace inclusion. In that case, the simulation preorder is finitely based over $\mathbb{T}(A)$, as is simulation equivalence —see, e.g., [8]. Those axiomatizations can be lifted to the setting of weak simulation semantics simply by adding the equation τe to any complete axiomatization of the simulation preorder or equivalence.

Tables 2–3 summarize the positive and negative results on the existence of finite axiomatizations for weak simulation semantics. In Table 2, and in subsequent ones, we write 'Do not exist' to indicate that there is no *finite* (in)equational axiomatization for the corresponding semantic relation.

3.3. Observational equivalence and simulation

For the sake of completeness, in this section we mention a rather natural, simulation-like behavioural relation that has an easy equational characterization. Let us consider the relation $\lesssim'_S$ defined as follows:

$E_{S^{\leq}} = \{B_1\text{–}B_4, \tau e, S\}$	(τe)	$\tau x = x$
$E_{S^{=}} = \{B_1\text{–}B_4, \tau e, SE\}$	(S)	$x \leq x + y$
$E_{S_1^{\leq}} = \{B_1\text{–}B_4, \tau e, S, TE, Sg_{\leq}\}$	$(Sg_{\leq})$	$x \leq ax$
$E_{S_1^{=}} = \{B_1\text{–}B_4, \tau e, TE, Sg\}$	(SE)	$a(x+y) = a(x+y) + ay$
	(TE)	$a(x+y) = ax + ay$
	(Sg)	$ax = ax + x$

Table 3: Axioms for the weak simulation semantics

Definition 2. $\lesssim'_S$ is the largest relation over closed terms in $\mathrm{T}(A_\tau)$ satisfying the following condition whenever $p \lesssim'_S q$ and $\alpha \in A_\tau$:

- if $p \xrightarrow{\alpha} p'$ there exists some q' such that $q \overset{\tau}{\Longrightarrow}\xrightarrow{\alpha}\overset{\tau}{\Longrightarrow} q'$ and $p' \precsim_S q'$, that is, p' is weak simulated by q'.

So, unlike the weak simulation preorder, the relation $\lesssim'_S$ requires that initial internal steps of one process be matched by at least one internal step from the other. This is similar to the extra requirement imposed by observational congruence with respect to weak bisimilarity [43].

Proposition 6. *The relation $\lesssim'_S$ is a precongruence over $\mathrm{T}(A_\tau)$, which is finer than the weak simulation preorder. Besides, for every term t, we have that $t \lesssim'_S \tau t$, but, in general, $\tau t \not\lesssim'_S t$.*

PROOF. It is clear that $p \lesssim'_S q$ implies $p \precsim_S q$ since the condition imposed by the definition of the weak simulation preorder (Definition 1) is weaker than the one in Definition 2 above. That $\lesssim'_S$ is a precongruence can be proved exactly as it was done for $\precsim_S$ in Proposition 1.

The validity of $t \lesssim'_S \tau t$, for every term t, is clear.

Moreover, for any term t that cannot initially perform a τ action, it holds that $\tau t \precsim_S t$ but $\tau t \not\lesssim'_S t$. In particular, $\tau \mathbf{0} \not\lesssim'_S \mathbf{0}$. □

Now we present a technical lemma that will be useful in the proof of Proposition 7 to follow. This lemma establishes a simple relationship between the weak simulation preorder, $\precsim_S$, and $\lesssim'_S$.

Lemma 4. *For all $p, q \in \mathrm{T}(A_\tau)$, we have that $p \precsim_S q$ implies $p \lesssim'_S \tau q$.*

Proposition 7. *The set of equations*

$$E = BW \cup \{S\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim'_S$.

PROOF. Ground-completeness can be shown by induction on the depth of terms, using the fact that, by Lemma 4, $p \precsim_S q$ implies $p \lesssim'_S \tau q$.

Let $p = \sum_{i=1}^{n} \alpha_i p_i$, where $n \geq 0$, and suppose that $p \lesssim'_S q$. Then, by definition of $\lesssim'_S$, we know that, for any transition $p \xrightarrow{\alpha_i} p_i$, there is some q' such that $q \stackrel{\tau}{\Longrightarrow}\xrightarrow{\alpha_i}\stackrel{\tau}{\Longrightarrow} q'$ and $p_i \precsim_S q'$. By Lemma 4, we have that $p_i \lesssim'_S \tau q'$ and, applying the induction hypothesis, also that $E \vdash p_i \leq \tau q'$. Using closure under prefixes, equation W_1 and inequation S, we infer that $E \vdash \alpha p_i \leq \alpha q' + q$. The weak derivatives of q can be absorbed into q by the classic Absorption Lemma (see [34, page 157] and [43, Lemma 16, page 163]), and therefore $E \vdash \alpha p_i \leq q$. As this is true for each i, using B_3 we may conclude that $E \vdash p \leq q$, which was to be shown. □

So $\lesssim'_S$ is axiomatized precisely with the equations for observational equivalence and the simulation inequation S.

Remark 2. In fact, in the presence of at least two actions in A, the axiom system $BW \cup \{S\}$ is sound and complete for BCCSP(A_τ) modulo $\lesssim'_S$.

Therefore, by comparing the ground-complete axiomatizations of both $\lesssim'_S$ and $\lesssim_S$, we see that the only difference is the axiom $\tau x \leq x$, which is not sound for the first. This is due to the added condition in the definition of $\lesssim'_S$. We will see in the following sections that, as soon as we add some standard constraint to the studied simulations, the corresponding preorder will not be a precongruence anymore. In order to obtain the corresponding largest precongruence included in it, we will need to add additional constraints to its operational characterization. These constraints will be reflected in its axiomatization, where the axiom $\tau x \leq x$ will totally disappear, as in the case above, or will only remain valid under some conditions.

4. Weak complete simulation

We now study the notion of complete simulation preorder in a setting with τ actions. Recall that, in the setting without τ, a complete simulation is a simulation relation that is only allowed to relate a state with no outgoing transitions to states with the same property.

Definition 3. For $\mathrm{T}(A_\tau)$ terms, we say that process p *must terminate* (or *is mute*), written $p\Downarrow$, iff there does not exist any $a \in A$ such that $p \stackrel{a}{\Longrightarrow}$, that is, if the set of visible initial actions of process p is empty, written $I^*(p) = \emptyset$.

Note that p is not mute, written $p \not\Downarrow$, if, and only if, there exist $n \geq 0$ and $a \in A$ such that $p(\stackrel{\tau}{\longrightarrow})^n \stackrel{a}{\longrightarrow}$, where $(\stackrel{\tau}{\longrightarrow})^n$ denotes the n-fold composition of the relation $\stackrel{\tau}{\longrightarrow}$.

Definition 4. The *weak complete simulation preorder*, denoted by $\precsim_{CS}$, is the largest relation over terms in $\mathrm{T}(A_\tau)$ satisfying the following conditions whenever $p \precsim_{CS} q$ and $\alpha \in A_\tau$:

- if $p \stackrel{\alpha}{\longrightarrow} p'$ then there exists some term q' such that $q \stackrel{\alpha}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$,
- if $p\Downarrow$ then $q\Downarrow$.

We say that $p, q \in \mathrm{T}(A_\tau)$ are *weak complete simulation equivalent*, written $p \approx_{CS} q$, iff p and q are related by the kernel of $\precsim_{CS}$, that is when both $p \precsim_{CS} q$ and $q \precsim_{CS} p$ hold.

The following result is standard.

Lemma 5. *If $p \precsim_{CS} q$, then*

1. *$I^*(p) \subseteq I^*(q)$, and*
2. *$p\Downarrow$ if, and only if, $q\Downarrow$.*

Example 1. $\precsim_{CS}$ is not a precongruence with respect to the choice operator of BCCSP(A_τ). It is immediate to show that $\tau\mathbf{0} \precsim_{CS} \mathbf{0}$, however $\tau\mathbf{0} + a \not\precsim_{CS} \mathbf{0} + a$. If $\tau\mathbf{0} + a$ performs the τ-transition, the process evolves to $\mathbf{0}$, which satisfies $\mathbf{0}\Downarrow$; however, $\mathbf{0} + a$ can only transform into itself by a $\stackrel{\tau}{\Longrightarrow}$ transition and it does not satisfy the mute predicate, $(\mathbf{0} + a)\not\Downarrow$.

Definition 5. We denote by $\sqsubseteq_{CS}$ the largest precongruence included in $\precsim_{CS}$. That is, $\sqsubseteq_{CS}$ is the largest relation such that

- $p \sqsubseteq_{CS} q$ implies $p \precsim_{CS} q$, and
- $p \sqsubseteq_{CS} q \Rightarrow \forall \alpha \in A_\tau \quad \alpha p \sqsubseteq_{CS} \alpha q$, and
- $p \sqsubseteq_{CS} q \Rightarrow \forall r \in \mathrm{T}(A_\tau) \quad p + r \sqsubseteq_{CS} q + r$.

The definition of the largest precongruence included in $\precsim_{CS}$ is purely algebraic and difficult to use when studying this relation. We next present a behavioural characterization of $\sqsubseteq_{CS}$. In what follows, we use $(\xrightarrow{\tau})^+$ to denote the transitive closure of the relation $\xrightarrow{\tau}$.

Definition 6. The preorder relation $\lesssim_{CS}$ between processes in $\mathrm{T}(A_\tau)$ is defined taking $p \lesssim_{CS} q$ iff

- $p \precsim_{CS} q$, and
- whenever $p \xrightarrow{\tau} p'$ for some p' such that $p' \Downarrow$, there exists some q' such that $q(\xrightarrow{\tau})^+ q'$ and $q' \Downarrow$.

We denote the kernel of $\lesssim_{CS}$ by $\eqsim_{CS}$.

Example 2. It is immediate to see that $\tau\mathbf{0} \not\lesssim_{CS} \mathbf{0}$. On the other hand, $\tau a \lesssim_{CS} a$ does hold because the second requirement in Definition 6 is vacuous. In general, $\tau p \lesssim_{CS} p + q$ holds for all p and q provided that p is *not* $\mathbf{0}$. (Recall that we consider terms up to $B1$–$B4$.)

Lemma 6. *Assume that $p \precsim_{CS} q$ and p is not mute. Then $p \precsim_{CS} q + r$ for each closed term r.*

PROOF. Define the relation $\mathcal{R}$ as follows: $(p, q + r) \in \mathcal{R}$ iff

- p is not mute and
- $p \precsim_{CS} q$.

It is not hard to see that the relation $\mathcal{R} \cup \precsim_{CS}$ is a weak complete simulation. □

Proposition 8 (Behavioural characterization of $\sqsubseteq_{CS}$). *$p \lesssim_{CS} q$ if, and only if, $p \sqsubseteq_{CS} q$, for all $p, q \in \mathrm{T}(A_\tau)$.*

PROOF. We prove the two implications separately.

For the implication from right to left, assume that $p \sqsubseteq_{CS} q$. We shall prove that $p \lesssim_{CS} q$ also holds. To this end, note first that $p \precsim_{CS} q$ because $\sqsubseteq_{CS}$ in included in $\precsim_{CS}$. Moreover, $p + a \precsim_{CS} q + a$, and we shall now prove that this yields that, whenever $p \xrightarrow{\tau} p'$ and $p'\Downarrow$, there exists some q' such that $q(\xrightarrow{\tau})^+ q'$ and $q'\Downarrow$. This will complete the proof that $p \lesssim_{CS} q$. So, assume that $p \xrightarrow{\tau} p'$ and $p'\Downarrow$. Then $p + a \xrightarrow{\tau} p'$. Since $p + a \precsim_{CS} q + a$, there is some q' such that $q + a \stackrel{\tau}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$. Since $p'\Downarrow$ and $q + a$ is not mute, it must be the case that $q + a(\xrightarrow{\tau})^+ q'$. Hence $q(\xrightarrow{\tau})^+ q'$. As $p' \precsim_{CS} q'$ and $p'\Downarrow$, we have that q' is mute, and we are done.

We now prove the implication from left to right. By the definition of $\lesssim_{CS}$, we have that $\lesssim_{CS}$ is included in $\precsim_{CS}$. It therefore suffices to show that $\lesssim_{CS}$ is a congruence. To this end, assume that $p \lesssim_{CS} q$. It is easy to see that $\alpha p \lesssim_{CS} \alpha q$ for each action α. We claim that $p + r \lesssim_{CS} q + r$ also holds for each closed term r. To establish this claim we consider each of the conditions in Definition 6 in turn.

- We first prove that $p + r \precsim_{CS} q + r$. Take $p + r \xrightarrow{\alpha} p'$ for some p'. The only interesting case is that when this transition stems from p, so we have $p \xrightarrow{\alpha} p'$. We will prove that $q + r \stackrel{\alpha}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$, for some q'. Since $p \precsim_{CS} q$ because $p \lesssim_{CS} q$, this is clear in all cases apart from when

  - $\alpha = \tau$ and
  - q is the only τ-derivative of itself for which $p' \precsim_{CS} q$.

  This means that p' is *not* mute. Indeed, if p' were mute then, as $p \lesssim_{CS} q$, there would be some q' such that $q + r(\xrightarrow{\tau})^+ q'$ and $q'\Downarrow$. For such a q', we would have $p' \precsim_{CS} q'$. So, $p \xrightarrow{\tau} p'$ and p' is not mute as claimed. It then follows that $p' \precsim_{CS} q + r$ by Lemma 6, and we are done, since $q + r \stackrel{\tau}{\Longrightarrow} q + r$.

- We now show that if $p + r \xrightarrow{\tau} p'$ and $p'\Downarrow$, then there exists some q' such that $q + r(\xrightarrow{\tau})^+ q'$ and $q'\Downarrow$. To this end, assume that $p + r \xrightarrow{\tau}$

p' and $p' \Downarrow$. Then either $p \xrightarrow{\tau} p'$ or $r \xrightarrow{\tau} p'$. The latter case is immediate; in the former, since $p \lesssim_{CS} q$, we have $q(\xrightarrow{\tau})^+ q'$ and $q' \Downarrow$, for some q'. Therefore, it holds that $q + r(\xrightarrow{\tau})^+ q'$ and $q' \Downarrow$, which was to be shown.

This completes the proof. □

The following result is an immediate consequence of Lemma 5.

Lemma 7. *If $p \lesssim_{CS} q$, then $p\Downarrow$ if, and only if, $q\Downarrow$.*

We shall now provide an alternative characterization of the preorder $\lesssim_{CS}$, and therefore of $\sqsubseteq_{CS}$, over $\mathrm{T}(A_\tau)$. This definition of $\lesssim_{CS}$ bears a strong resemblance to the characterization of the largest precongruence included in the weak ready simulation preorder, when A is finite and non-empty, that we shall present in Section 5.2.

Definition 7. The preorder $\lesssim_{CS}^N$ over $\mathrm{T}(A_\tau)$ is defined taking $p \lesssim_{CS}^N q$ iff

- whenever $p \xrightarrow{a} p'$, there exists some q' such that $q \stackrel{a}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$;
- whenever $p \xrightarrow{\tau} p'$,
  - either there exists some q' such that $q(\xrightarrow{\tau})^+ q'$ and $p' \precsim_{CS} q'$
  - or $p' \not\Downarrow$ and $p' \precsim_{CS} q$;
- if $p\Downarrow$ then $q\Downarrow$.

Proposition 9. *$p \lesssim_{CS}^N q$ iff $p \lesssim_{CS} q$, for all $p, q \in \mathrm{T}(A_\tau)$.*

PROOF. We prove the two implications separately. First of all, note that the implication from left to right follows immediately from the definition of the relations $\lesssim_{CS}^N$ and $\lesssim_{CS}$.

Assume now that $p \lesssim_{CS} q$ and $p \xrightarrow{\tau} p'$ for some p'. By the definition of $\lesssim_{CS}$, we have that $p \precsim_{CS} q$. Therefore, there exists some q' such that $q \stackrel{\tau}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$. Assume that $q' = q$ is the only state that q can reach via $\stackrel{\tau}{\Longrightarrow}$ that weakly complete simulates p'. We claim that $p' \not\Downarrow$. Indeed, if $p' \Downarrow$ then, by the definition of $\lesssim_{CS}$, there would be some q'' such that

$q(\xrightarrow{\tau})^+ q''$ and $q'' \Downarrow$. For that q'', it would hold that $p' \precsim_{CS} q''$, and this would contradict our assumption that q is the only state that it can reach via $\xRightarrow{\tau}$ that weakly complete simulates p'. The other two clauses in the definition of $\lesssim_{CS}^{N}$ follow immediately from the definition of $\lesssim_{CS}$. □

4.1. Ground-completeness

In order to find a set of equations that gives a ground-complete axiomatization for the largest precongruence included in the weak complete simulation preorder, it is natural to consider the following (conditional) equations.

$$(CS_\tau) \qquad (x\Downarrow \Leftrightarrow y\Downarrow) \Rightarrow x \leq x + y$$
$$(CS_{\tau e}) \qquad \tau(ax + y) = ax + y$$

The first equation, CS_τ, is similar to the key axiom in the axiomatization for the complete simulation preorder in the concrete case, see e.g. [23]. However, in our setting, the mute predicate takes into account the silent steps of processes. This conditional equation restricts the applicability of inequation S, which is only sound in (weak) complete simulation semantics when the terms substituted for the variables x and y have the same 'termination status'.

The second equation, $CS_{\tau e}$, is a restricted version of equation τe, which we used for the weak simulation preorder, but is unsound for the weak complete simulation semantics. Intuitively, equation $CS_{\tau e}$ expresses the fact that a process of the form τp, for some term p that is not mute, is weak complete simulation equivalent to p. In fact, equation $CS_{\tau e}$ could 'equivalently' be formulated by the following conditional equation:

$$x\not\Downarrow \Rightarrow \tau x = x.$$

Lemma 8. *For every term p such that $p\not\Downarrow$, we can prove using $CS_{\tau e}$ that $\tau p = p$.*

Proof. Assume that $p(\xrightarrow{\tau})^n \xrightarrow{a}$ for some $n \geq 0$ and $a \in A$. We prove the lemma by induction on n.

- Base case, $n = 0$. Then $p = ap_1 + p_2$, for some p_1 and p_2, and we may apply directly the axiom $CS_{\tau e}$.
- Induction step, $n > 0$. Then $p = \tau p_1 + p_2$, for some p_1 and p_2, with $p_1(\xrightarrow{\tau})^{n-1} \xrightarrow{a}$. By the induction hypothesis, we derive $p_1 = \tau p_1$ and $p = p_1 + p_2$. We again apply the induction hypothesis to derive $\tau(p_1 + p_2) = (p_1 + p_2)$, thus getting $\tau p = p$. □

Proposition 10. *The set of equations*

$$E^c_{CS\leq} = BW \cup \{CS_{\tau e}, CS_\tau\},$$

where CS_τ is conditional, is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim_{CS}$.

PROOF. The soundness of the axioms is obvious. To prove ground completeness we establish that $p \lesssim_{CS} q$ implies $E^c_{CS\leq} \vdash p \leq q$, by structural induction on p. In the rest of the proof, for the sake of readability, we abbreviate $E^c_{CS\leq}$ by E.

$p = \mathbf{0}$. Then $q\Downarrow$ and $E \vdash \mathbf{0} \leq q$, by application of CS_τ.

$p = \alpha p'$. Considering $p \xrightarrow{\alpha} p'$, using Proposition 9 and Definition 7, we have that

1. either $q \stackrel{\tau}{\Longrightarrow}\stackrel{\alpha}{\longrightarrow}\stackrel{\tau}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$, for some q',
2. or $\alpha = \tau$, $p'\not\Downarrow$ and $p' \precsim_{CS} q$.

In the first case, using W_1–W_3 we can infer $q = q + \alpha q'$, and from $p' \precsim_{CS} q'$ we obtain $p' \lesssim_{CS} \tau q'$. By the induction hypothesis, $E \vdash p' \leq \tau q'$. Therefore, we have that $E \vdash \alpha p' \leq \alpha\tau q'$ and thus $E \vdash \alpha p' \leq \alpha q'$. So

$$E \vdash p + q = \alpha p' + q \leq \alpha q' + q = q.$$

Finally, given that $p\Downarrow \Leftrightarrow q\Downarrow$, using CS_τ and transitivity, we obtain $E \vdash p \leq q$.

In the second case, from $p'\not\Downarrow$ and $p' \precsim_{CS} q$ we conclude that $q\not\Downarrow$. Using Lemma 8 we have $E \vdash \tau q = q$ and $E \vdash p = \tau p' = p'$. Since $p' \precsim_{CS} q$, we have $p' \lesssim_{CS} \tau q$. By the induction hypothesis, $E \vdash p' \leq \tau q$, and we are done.

$p = p_1 + p_2$. In this case we have that, for $i = 1, 2$, either $p_i = \mathbf{0}$ or $p_i \lesssim_{CS} q$. The result follows immediately by applying the induction hypothesis. □

Axiom CS_τ highlights the similarities with the concrete version of complete simulation and with the general theory of constrained simulations developed in [23]. However, it is natural to wonder whether it is possible to find a finite, non-conditional and ground-complete axiomatization for $\lesssim_{CS}$ over BCCSP(A_τ). Indeed, this is possible; it is enough to substitute the conditional equation CS_τ by the following two inequations:

$$\begin{array}{ll} (CS) & ax \leq ax + y \\ (\tau N) & \mathbf{0} \leq \tau\mathbf{0} \end{array}$$

Lemma 9. *The following claims hold for each closed term p.*

1. *If $p\not\Downarrow$ then there exist some $a \in A$ and p_1 such that $p = ap_1 + p$.*
2. *If $p\Downarrow$ then either $p = \mathbf{0}$ or the equation $p = \tau\mathbf{0}$ can be proved using W_1 and W_2.*

Proposition 11. *The set of unconditional inequations*

$$E_{CS^{\leq}} = BW \cup \{CS_{\tau e}, CS, \tau N\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim_{CS}$.

PROOF. Observe, first of all, that both $ax \leq ax + y$ and $\mathbf{0} \leq \tau\mathbf{0}$ can be derived using CS_τ. Therefore, we only need to prove that any use of the conditional axiom CS_τ in a proof of an inequation $p \leq q$ can be replaced by a combination of the two unconditional axioms above and the rest of axioms in $E_{CS^{\leq}}$.

For the case $p\not\Downarrow$ and $q\not\Downarrow$, from $p\not\Downarrow$ we infer, by Lemma 9, that there exist some $a \in A$ and p_1 such that $p = ap_1 + p$. Next, using CS, we obtain $ap_1 \leq ap_1 + q$, and finally $p \leq p + q$.

For the case $p\Downarrow$ and $q\Downarrow$, we reason as follows. Since $p\Downarrow$, by Lemma 9 we have that either $p = \mathbf{0}$ or the equation $p = \tau\mathbf{0}$ can be proved using W_1 and W_2. Similarly, either $q = \mathbf{0}$ or the equation $q = \tau\mathbf{0}$ can be proved using W_1 and W_2. If $p = \mathbf{0}$ the inequation $p \leq q$ can be proven by possibly applying τN. If $p = \tau\mathbf{0}$ then, by the soundness of the axiom system $E^c_{CS^{\leq}}$, we have that $q = \tau\mathbf{0}$, and we are done. □

Remark 3. It is clear that, in the axiomatization above, we could substitute equation τN by

$$(\tau g) \qquad x \leq \tau x,$$

since the inequation τg is sound for BCCSP(A_τ) modulo $\lesssim_{CS}$ and is more general than τN.

Let us now move on to the ground-complete axiomatization of the largest congruence included in complete simulation equivalence. In order to axiomatize that congruence, it is natural to consider the following equation.

$$(CSE_\tau) \quad (x\Downarrow \Leftrightarrow y\Downarrow) \ \Rightarrow a(x+y) = a(x+y) + ax$$

This equation is essentially the same one that was used in earlier conditional axiomatizations for the complete simulation equivalence in the concrete case [23]. However, we remark that now the mute predicate deals with silent transitions, although we only allow visible actions as prefixes of the terms in the constrained equation CSE_τ. As a matter of fact, the conditional equation

$$(x\Downarrow \Leftrightarrow y\Downarrow) \ \Rightarrow \tau(x+y) = \tau(x+y) + \tau x$$

is also sound modulo $\eqsim_{CS}$. However, as the following lemma states, each of its closed instantiations can be derived using the axiom system $BW \cup \{CS_{\tau e}\}$. This observation will be useful in the proof of Proposition 12 to follow.

Lemma 10. *Let p and q two processes such that $p\Downarrow$ if, and only if, $q\Downarrow$. Then we have*

$$BW \cup \{CS_{\tau e}\} \vdash \tau(p+q) = \tau(p+q) + \tau p.$$

PROOF. For the case $p\not\Downarrow$ and $q\not\Downarrow$, from $p\not\Downarrow$ we infer, by Lemma 8, that

$$\tau(p+q) = \tau(p+q) + p + q = \tau(p+q) + \tau p.$$

For the case $p\Downarrow$ and $q\Downarrow$, we reason as follows. Since $p\Downarrow$, we have that either $p = \mathbf{0}$ or the equation $p = \tau\mathbf{0}$ can be proved using W_1 and W_2. Similarly, either $q = \mathbf{0}$ or the equation $q = \tau\mathbf{0}$ can be proved using W_1 and W_2. In all cases, $\tau(p+q) = \tau(p+q) + \tau p$ follows, by possibly using W_1. □

Proposition 12. *The set of conditional equations*

$$E^c_{CS^=} = BW \cup \{CS_{\tau e}, CSE_\tau\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\eqsim_{CS}$.

PROOF. We prove, by induction on the depth of p, that

$$p \lesssim_{CS} q \text{ implies } E^c_{CS^=} \vdash q = q + p,$$

from which the claim follows immediately. Let $p = \sum_{i=1}^{n} \alpha_i p_i$, where $n \geq 0$, and $i \in \{1, \ldots, n\}$. Then $p \xrightarrow{\alpha_i} p_i$. By the definition of $\lesssim_{CS}$, we know that there is some q' such that $q \stackrel{\alpha_i}{\Longrightarrow} q'$ and $p_i \precsim_{CS} q'$. There are two possible cases:

1. either $q \stackrel{\tau}{\Longrightarrow}\xrightarrow{\alpha_i}\stackrel{\tau}{\Longrightarrow} q'$, or
2. $\alpha_i = \tau$, $p_i \not\Downarrow$ and $p_i \precsim_{CS} q$.

We proceed with the proof by examining these two cases in turn.

1. Assume that $q \stackrel{\tau}{\Longrightarrow}\xrightarrow{\alpha_i}\stackrel{\tau}{\Longrightarrow} q'$. By possibly applying axiom W_3, we derive $q = q + \alpha_i q'$. Since $p_i \precsim_{CS} q'$, we have that $p_i \lesssim_{CS} \tau q'$. Therefore, by the induction hypothesis,

   $$E^c_{CS^=} \vdash \tau q' = \tau q' + p_i.$$

   From the above equation and axiom W_1 we may now derive

   $$\alpha_i q' = \alpha_i \tau q' = \alpha_i(\tau q' + p_i). \qquad (1)$$

   Using $p_i \lesssim_{CS} \tau q'$, we infer $p_i \Downarrow$ iff $\tau q' \Downarrow$. Therefore, if $\alpha_i \in A$, we may apply the conditional equation CSE_τ to infer

   $$E^c_{CS^=} \vdash \alpha_i(\tau q' + p_i) = \alpha_i(\tau q' + p_i) + \alpha_i p_i.$$

   By transitivity, we may now conclude that, when $\alpha_i \in A$,

   $$\alpha_i q' = \alpha_i q' + \alpha_i p_i.$$

   Therefore,

   $$E^c_{CS^=} \vdash q = q + \alpha_i q' = q + \alpha_i p_i.$$

   If $\alpha_i = \tau$, then the above equation follows from (1) by Lemma 10.
2. Assume that $\alpha_i = \tau$, $p_i \not\Downarrow$ and $p_i \precsim_{CS} q$. Since $p_i \precsim_{CS} q$, we have that $q \not\Downarrow$ and $p_i \lesssim_{CS} \tau q$. Therefore, by the induction hypothesis,

   $$E^c_{CS^=} \vdash \tau q = \tau q + p_i.$$

   As $p_i \not\Downarrow$ and $q \not\Downarrow$, by Lemma 8, we have that $p_i = \tau p_i$ and $q = \tau q$. Therefore,

   $$E^c_{CS^=} \vdash q = q + \tau p_i.$$

Concluding, for each $1 \leq i \leq n$,

$$E^c_{CS^=} \vdash q = q + \alpha_i p_i.$$

Therefore, $E^c_{CS^=} \vdash q = q + p$, as required. □

To turn the previous axiomatization into one without conditional equations we consider the equation

$$(CSE) \quad a(bx + y + z) = a(bx + y + z) + a(bx + z) \qquad (a, b \in A).$$

This is the same equation that is used when axiomatizing complete simulation equivalence in a setting without silent moves.

Proposition 13. *The set of unconditional equations*

$$E_{CS^=} = BW \cup \{CS_{\tau e}, CSE\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\eqsim_{CS}$.

PROOF. In light of the above result, we only need to show that any use of the conditional axiom CSE_τ in a proof of an equation $p = q$ can be replaced by the application of some axioms in $E_{CS^=}$.

By the classic Absorption Lemma (see [34, page 157] and Lemma 16 on page 163 of [43]), if $p \stackrel{a}{\Longrightarrow} p'$ then $p = p + ap'$ can be proved using the τ-laws. Therefore, the pattern $ax + z$ characterizes the set of processes p such that $p \not\Downarrow$. For such processes, any application of the conditional equation CSE_τ can therefore be simulated by using CSE.

The other possible case is when $p\Downarrow$ and $q\Downarrow$. In this case, both p and q are either $\mathbf{0}$ or are provably equal to $\tau\mathbf{0}$, using W_1 and W_2 in the latter case. (See Lemma 9.) But then

$$a(p + q) = a(p + q) + ap$$

can be proved in all cases, by possibly using W_1. □

4.2. Nonexistence of finite complete axiomatizations

We shall now prove that if A contains at least one action, then the (in)equational theory of $\lesssim_{CS}$ over BCCSP(A_τ) does not have a finite basis. (The assumption that A be non-empty is necessary for such a result. In the trivial case that A is empty, the relation $\lesssim_{CS}$ has a finite inequational axiomatization, which we describe in Appendix A. If A is empty, then the inequation $x \leq y$ suffices to obtain a complete axiomatization for $\sqsubseteq_{CS}=\precsim_{CS}$. Indeed, if τ is the only action, $\precsim_{CS}$ is easily seen to be the universal relation over terms.)

For the sake of clarity, we recall one more time that we consider terms up to (strong) bisimulation.

Our proof of the claimed nonfinite axiomatizability result will consider the following infinite family of inequations, which are all sound modulo $\lesssim_{CS}$:

$$a^n x \leq a^n\mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

To see that each of the inequations in the above family is indeed sound, it suffices to observe that if $p \approx_{CS} \mathbf{0}$ then $a^n p \lesssim_{CS} a^n\mathbf{0}$, for each $n \geq 0$; while for $p \not\approx_{CS} \mathbf{0}$ we have $a^n p \lesssim_{CS} a^n(p+a)$, for all $n \geq 1$. (Note that the assumption that $n \geq 1$ is necessary for the soundness of the above inequation. Indeed, the inequation $x \leq \mathbf{0} + (x+a)$ is *not* sound modulo $\lesssim_{CS}$ because $\mathbf{0} \not\lesssim_{CS} \mathbf{0} + (\mathbf{0}+a)$.)

Proposition 14. *If $|A| \geq 1$ then the (in)equational theory of the precongruence $\lesssim_{CS}$ over* BCCSP(A_τ) *does not have a finite (in)equational basis. In particular, the following statements hold true:*

1. *No finite set of sound inequations over* BCCSP(A_τ) *modulo $\lesssim_{CS}$ can prove all of the sound inequations in the family*

$$a^n x \leq a^n\mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

2. *No finite set of sound (in)equations over* BCCSP(A_τ) *modulo $\lesssim_{CS}$ can prove all of the sound equations in the family*

$$a^n x + a^n\mathbf{0} + a^n(x+a) = a^n\mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

Before we embark on the proof of the above result, let us point out that the families of (in)equations that lie at the heart of the negative results in Proposition 14, as well as the structure of the proof of that result, stem from [5, 21], where it is shown that, in the setting without τ, the complete simulation preorder and the induced equivalence afford no finite inequational axiomatization. The details of our argument are based on the developments in [5], but we need to take carefully into account the role played by the internal action τ, since it could be the case that, once we have introduced it, we could obtain the desired finite basis, even if such a basis does not exist for the concrete case with no silent moves.

Proposition 14 is a corollary of the following result.

Proposition 15. *Assume that $|A| \geq 1$. Let E be a collection of inequations whose elements are sound modulo $\lesssim_{CS}$ and have depth smaller than n. Suppose furthermore that the inequation $t \leq u$ is derivable from E and that $u \precsim_{CS} a^n\mathbf{0} + a^n(x+a)$. Then $t \stackrel{a^n}{\Longrightarrow} x$ implies $u \stackrel{a^n}{\Longrightarrow} x$.*

Having shown the above result, we can prove statement 1 in Proposition 14 as follows. Let E be a finite inequational axiom system that is sound modulo $\lesssim_{CS}$. Pick n larger than the depth of any axiom in E. Then, by Proposition 15, E cannot prove the valid inequation

$$a^n x \leq a^n\mathbf{0} + a^n(x+a),$$

and is therefore incomplete. Indeed,

$$a^n x \stackrel{a^n}{\Longrightarrow} x.$$

On the other hand, the only terms t for which

$$a^n\mathbf{0} + a^n(x+a) \stackrel{a^n}{\Longrightarrow} t$$

holds are $\mathbf{0}$ and $x+a$. And therefore, $a^n\mathbf{0} + a^n(x+a) \stackrel{a^n}{\Longrightarrow} x$ does not hold.

Statement 2 in Proposition 14 is a corollary of Proposition 14(1). To see this, assume Proposition 14(1) and suppose, towards a contradiction, that there is a finite set of sound (in)equations over BCCSP(A_τ) modulo $\lesssim_{CS}$ that can prove all of the equations in the family

$$a^n x + a^n\mathbf{0} + a^n(x+a) = a^n\mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

Recall that we may assume that E is closed with respect to symmetry and that, under this assumption, there is no difference between the rules of inference of equational and inequational logic. Thus E can prove all the inequations

$$a^n x + a^n \mathbf{0} + a^n(x+a) \leq a^n \mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

Now the sound inequation CS can be used to derive

$$a^n x \leq a^n x + a^n \mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

Therefore, by transitivity, the finite set of sound inequations $E \cup \{CS\}$ can prove all of the inequations in the family

$$a^n x \leq a^n \mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

This, however, contradicts Proposition 14(1).

In order to show Proposition 15, we shall first prove that the property mentioned in that statement holds true for the instantiations of any sound inequation whose depth is smaller than n. Next we use this fact to argue that the stated property is preserved by arbitrary inequational derivations from a collection of inequations whose elements are sound modulo $\lesssim_{CS}$ and have depth smaller than n.

Definition 8. We say that a term t contains an occurrence of variable x reachable via a sequence s of visible actions if there is some term t' such that $t \stackrel{s}{\Longrightarrow} x + t'$.

For example, $ax + a\mathbf{0}$ contains an occurrence of x reachable via $s = a$, because $ax + a\mathbf{0} \stackrel{a}{\longrightarrow} x$ and $x = x + \mathbf{0}$ can be shown using B4.

Lemma 11. *Assume that $t \lesssim_{CS} u$ and that t contains an occurrence of variable x reachable via a sequence s of visible actions. Then u also contains an occurrence of variable x reachable via s.*

PROOF. Assume that $t \lesssim_{CS} u$ and that t has an occurrence of variable x reachable via a sequence s of visible actions. Let m be larger than the depth of u. Consider the closed substitution σ mapping x to a^m and every other variable to $\mathbf{0}$. Since t has an occurrence of variable x reachable via s, it is easy to see that $\sigma(t) \stackrel{sa^m}{\Longrightarrow} \mathbf{0}$. As $\sigma(t) \lesssim_{CS} \sigma(u)$, because $t \lesssim_{CS} u$ by assumption, it must be the case that $\sigma(u) \stackrel{sa^m}{\Longrightarrow} p$ for some p such that $\mathbf{0} \precsim_{CS} p$. Such a p is mute. As the depth of u is smaller than m, σ maps all variables different from x to $\mathbf{0}$, $\sigma(u) \stackrel{sa^m}{\Longrightarrow} p$ and p is mute, it follows that $u \stackrel{s}{\Longrightarrow} x + u'$ for some u', which was to be shown. □

Lemma 12. *Suppose that $t \lesssim_{CS} u$ and that n is larger than the depth of t. Then $\sigma(t) \stackrel{a^n}{\Longrightarrow} x$ implies $\sigma(u) \stackrel{a^n}{\Longrightarrow} x$.*

PROOF. Assume that $\sigma(t) \stackrel{a^n}{\Longrightarrow} x$. Since n is larger than the depth of t, there are some $0 \leq i < n$ and some variable z such that t has an occurrence of variable z reachable via a^i and $\sigma(z) \stackrel{a^{n-i}}{\Longrightarrow} x$. As $t \lesssim_{CS} u$, Lemma 11 yields that u has an occurrence of variable z reachable via a^i. Therefore $\sigma(u) \stackrel{a^n}{\Longrightarrow} x$, which was to be shown. □

Lemma 13. *Let p be a closed term such that $p\Downarrow$. Then $p \stackrel{\tau}{\Longrightarrow} \mathbf{0}$.*

PROOF. By structural induction on p. □

The following lemma will allow us to handle closure under action prefixing when proving that the property stated in Proposition 15 is preserved by arbitrary inequational derivations from a collection of inequations whose elements are sound modulo $\lesssim_{CS}$ and have depth smaller than n.

Lemma 14. *Assume that $at \precsim_{CS} au \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$, and that $at \stackrel{a^n}{\Longrightarrow} x$. Then, $au \stackrel{a^n}{\Longrightarrow} x$.*

PROOF. Let t and u be such that

1. $at \precsim_{CS} au \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$, and
2. $at \stackrel{a^n}{\Longrightarrow} x$.

Since $at \stackrel{a^n}{\Longrightarrow} x$, at contains some occurrence of x reachable via a^n. Therefore, by Lemma 11, so does au. This means that $au \stackrel{a^n}{\Longrightarrow} x + u'$ for some u'. Observe now that $au \stackrel{a^n}{\Longrightarrow} \mathbf{0}$ cannot hold, because this would contradict the fact that $au \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$. Indeed, assume, towards a contradiction, that $au \stackrel{a^n}{\Longrightarrow} \mathbf{0}$. Consider a closed substitution σ that maps x to a. Then $\sigma(au) \stackrel{a}{\longrightarrow} \sigma(u)$. The only terms that can be reached from $\sigma(a^n\mathbf{0} + a^n(x + a))$ via $\stackrel{a}{\Longrightarrow}$ are $a^{n-1}\mathbf{0}$ and $a^{n-1}(a + a)$. However, neither $\sigma(u) \precsim_{CS} a^{n-1}\mathbf{0}$ nor $\sigma(u) \precsim_{CS} a^{n-1}(a + a)$ holds. Indeed, the former fails because

$$\sigma(u) \stackrel{a^{n-1}}{\Longrightarrow} a + \sigma(u') \not\precsim_{CS} \mathbf{0},$$

and the latter because $\sigma(u) \stackrel{a^{n-1}}{\Longrightarrow} \mathbf{0} \not\precsim_{CS} a + a$.

Consider now the closed substitution $\sigma_{\mathbf{0}}$ that maps all variables to $\mathbf{0}$. Then $\sigma_{\mathbf{0}}(at) \stackrel{a^n}{\Longrightarrow} \mathbf{0}$ because $at \stackrel{a^n}{\Longrightarrow} x$ by the proviso of the lemma. As $at \precsim_{CS} au$, we have that $\sigma_{\mathbf{0}}(at) \precsim_{CS} \sigma_{\mathbf{0}}(au)$. Therefore, $\sigma_{\mathbf{0}}(au) \stackrel{a^n}{\Longrightarrow} p$ for some closed term p such that $\mathbf{0} \precsim_{CS} p$. Using Lemma 13, $\sigma_{\mathbf{0}}(au) \stackrel{a^n}{\Longrightarrow} p \stackrel{\tau}{\Longrightarrow} \mathbf{0}$. Since, by our earlier observation, $au \stackrel{a^n}{\Longrightarrow} \mathbf{0}$ cannot hold, we have that $au \stackrel{a^n}{\Longrightarrow} u''$ for some u'' such that $u'' \neq \mathbf{0}$ and $\sigma_{\mathbf{0}}(u'') = \mathbf{0}$. Such a u'' can only contain occurrences of the variable x (by Lemma 11 and the assumption that $au \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$). Therefore $u'' = x$ and we are done. □

We now have all the necessary ingredients to complete the proof of Proposition 15, and therefore also that of statement 1 in Proposition 14.

PROOF. *(of Proposition 15)* Assume that E is a collection of inequations whose elements have depth smaller than n and are sound modulo $\lesssim_{CS}$. Suppose furthermore that

- the inequation $t \leq u$ is derivable from E,
- $u \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$, and
- $t \stackrel{a^n}{\Longrightarrow} x$.

(Observe that n is positive because it is larger than the depth of E.) We shall prove that $u \stackrel{a^n}{\Longrightarrow} x$ by induction on the derivation of $t \leq u$ from E. We proceed by examining the last rule used in that derivation. The case of reflexivity is trivial, and that of transitivity follows by applying the inductive hypothesis twice. If $t \leq u$ is proved by instantiating an inequation in E, then the claim follows by Lemma 12. We are therefore left with the congruence rules, which we consider one by one below.

- Suppose that E proves $t \leq u$ because we have $t = \tau t'$, $u = \tau u'$ and E proves $t' \leq u'$ by a shorter inference. Observe that $t' \stackrel{a^n}{\Longrightarrow} x$, since $t = \tau t' \stackrel{a^n}{\Longrightarrow} x$. Moreover, $u' \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$. Then the induction hypothesis yields $u' \stackrel{a^n}{\Longrightarrow} x$. Therefore, we obtain that $u = \tau u' \stackrel{a^n}{\Longrightarrow} x$, as required.

- Suppose that E proves $t \leq u$ because we have $t = at'$, $u = au'$ and E proves $t' \leq u'$ by a shorter inference. By the soundness of E, the fact that $\lesssim_{CS}$ is included in $\precsim_{CS}$ and the proviso of the proposition, we have
$$t = at' \precsim_{CS} u = au' \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$$
and $t \stackrel{a^n}{\Longrightarrow} x$. Lemma 14 now yields $u \stackrel{a^n}{\Longrightarrow} x$, as required.

- Suppose that E proves $t \leq u$ because we have $t = t_1 + t_2$, $u = u_1 + u_2$ and E proves both $t_1 \leq u_1$ and $t_2 \leq u_2$, by shorter inferences. Since $t \stackrel{a^n}{\Longrightarrow} x$ and n is positive, we may assume, without loss of generality, that $t_1 \stackrel{a^n}{\Longrightarrow} x$. Using soundness of E and the fact that $t_1 \not\approx_{CS} \mathbf{0}$, it is not hard to see that
$$u_1 \precsim_{CS} a^n\mathbf{0} + a^n(x + a). \qquad (2)$$
Indeed, let σ be a closed substitution. We claim that
$$\sigma(u_1) \precsim_{CS} \sigma(a^n\mathbf{0} + a^n(x + a)).$$
To see this, note, first of all, that $\sigma(u_1)\not\Downarrow$ because $t_1 \lesssim_{CS} u_1$, by the soundness of E, and $t_1 \stackrel{a^n}{\Longrightarrow} x$ for some $n > 0$. Moreover, if $\sigma(u_1) \stackrel{\alpha}{\longrightarrow} p$, for some α and p, then $\sigma(u) \stackrel{\alpha}{\longrightarrow} p$ also holds. Since
$$u \precsim_{CS} a^n\mathbf{0} + a^n(x + a),$$
we have that $\sigma(u) \precsim_{CS} \sigma(a^n\mathbf{0} + a^n(x + a))$. Thus, there is some q such that $\sigma(a^n\mathbf{0} + a^n(x + a)) \stackrel{\alpha}{\Longrightarrow} q$ and $p \precsim_{CS} q$.
Therefore, since (2) holds, we may apply the induction hypothesis to $t_1 \leq u_1$ to infer $u_1 \stackrel{a^n}{\Longrightarrow} x$. Hence, as n is positive, $u \stackrel{a^n}{\Longrightarrow} x$, as required.

This completes the proof. □

Corollary 3. *If* $|A| \geq 1$ *then the collection of (in)equations in at most one variable that hold over* BCCSP(A_τ) *modulo* $\lesssim_{CS}$ *does not have a finite (in)equational basis. Moreover, for each n, the collection of all sound (in)equations of depth at most n cannot prove all the valid (in)equations in at most one variable that hold in weak complete simulation semantics over* BCCSP(A_τ).

Tables 4–5 summarize the positive and negative results on the existence of finite axiomatizations for weak complete simulation semantics.

Weak Complete Simulation	Ground-complete		Complete	
Finite Equations	Order	Equiv.	Order	Equiv.
$1 \leq \lvert A \rvert = \infty$	$E_{CS^{\leq}}$	$E_{CS^{=}}$	Do not exist	

Table 4: Axiomatizations for the largest (pre)congruence included in the weak complete simulation semantics

Unconditional	
$E_{CS^{\leq}} = BW \cup \{CS_{\tau e}, CS, \tau N\}$ $E_{CS^{=}} = BW \cup \{CS_{\tau e}, CSE\}$	$(CS_{\tau e})$ $\tau(ax+y) = ax+y$ (CS) $ax \leq ax+y$ (τN) $\mathbf{0} \leq \tau\mathbf{0}$ (CSE) $a(bx+y+z) =$ $a(bx+y+z)+a(bx+z)$
Conditional	
$E^{c}_{CS^{\leq}} = BW \cup \{CS_{\tau e}, CS_{\tau}\}$ $E^{c}_{CS^{=}} = BW \cup \{CS_{\tau e}, CSE_{\tau}\}$	(CS_{τ}) $(x{\Downarrow} \Leftrightarrow y{\Downarrow}) \Rightarrow x \leq x+y$ (CSE_{τ}) $(x{\Downarrow} \Leftrightarrow y{\Downarrow}) \Rightarrow$ $a(x+y) = a(x+y)+ax$

Table 5: Axioms for the largest (pre)congruence included in the weak completed simulation semantics

4.3. Observational equivalence and complete simulation

As we did in Section 3.3 for weak simulation semantics, we now study the 'forced to be' precongruence, associated with the weak complete simulation preorder, based on the requirements for Milner's observational congruence. Let us consider the following definition of a new preorder between processes.

Definition 9. $\lesssim'_{CS}$ is the largest relation over closed terms in $\mathrm{T}(A_\tau)$ satisfying the following condition whenever $p \lesssim'_{CS} q$ and $\alpha \in A_\tau$:

- if $p \xrightarrow{\alpha} p'$ then there exists some q' such that $q \stackrel{\tau}{\Longrightarrow}\xrightarrow{\alpha}\stackrel{\tau}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$;
- if $p\Downarrow$ then $q\Downarrow$.

Proposition 16. *The relation $\lesssim'_{CS}$ is a precongruence over $\mathrm{T}(A_\tau)$, which is finer than the largest precongruence included in the weak complete simulation preorder, that is, $\lesssim'_{CS} \subset \sqsubseteq_{CS}$.*

PROOF. The proof is similar to that of Proposition 6. □

The following result is similar to that in Lemma 4 in Section 3.3, and will be useful to find an axiomatization for $\lesssim'_{CS}$.

Lemma 15. *We have that $p \precsim_{CS} q$ implies $p \lesssim'_{CS} \tau q$, for all $p, q \in \mathrm{T}(A_\tau)$.*

Proposition 17. *The set of equations*

$$E = BW \cup \{CS_\tau\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo $\lesssim'_{CS}$.*

PROOF. To prove ground-completeness we can proceed as in the proof of Proposition 7, observing that Lemma 15 and axiom CS_τ are applicable. □

The set of axioms in Proposition 17 is similar to that characterizing $\lesssim_{CS}$ in Proposition 10. However, to axiomatize $\lesssim'_{CS}$ we do not need the equation $CS_{\tau e}$, which is unsound. Note that the inequation

$$ax + y \leq \tau(ax + y)$$

is valid modulo $\lesssim'_{CS}$. However, the closed instances of that inequation can all be derived using CS_τ and W_2.

In fact, we can also provide a non-conditional axiomatization for $\lesssim'_{CS}$ in the same way we did for the relation $\lesssim_{CS}$.

Proposition 18. *The set of equations*

$$E = BW \cup \{CS, \tau N\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim'_{CS}$.

PROOF. The same proof strategy we adopted in Proposition 11 to obtain a ground-complete unconditional axiomatization of $\lesssim_{CS}$ from the conditional one can be used here. Let us note that, in that proof, we did not make any use of axiom $CS_{\tau e}$, which is the axiom needed for $\lesssim_{CS}$, but missing in the axiomatization of $\lesssim'_{CS}$. □

We conclude our study of this variation on the weak complete simulation preorder by showing that, like $\lesssim_{CS}$, it does not afford a finite (in)equational basis.

Proposition 19. *If* $|A| \geq 1$ *then the (in)equational theory of the precongruence* $\lesssim'_{CS}$ *over* BCCSP(A_τ) *does not have a finite (in)equational basis.*

PROOF. Observe that the family of inequations

$$a^n x \leq a^n \mathbf{0} + a^n (x + a) \qquad (n \geq 1)$$

is sound modulo $\lesssim'_{CS}$. Since $\lesssim'_{CS}$ is included in $\lesssim_{CS}$, by statement 1 in Proposition 14 no finite axiom system that is sound modulo $\lesssim'_{CS}$ can prove all the inequations in the above family. Therefore, no finite axiom system that is sound modulo $\lesssim'_{CS}$ can be complete. □

5. Weak ready simulation

In this section, we shall study the equational theory of the largest precongruence included in the weak ready simulation preorder. We start by defining the notion of weak ready simulation that we will consider. We then proceed to study its induced precongruence, first in the case in which the set of actions A is infinite and then when A is finite.

In order to define the weak ready simulation semantics, we recall the definition of the function I^*, presented in Section 2, that returns the set of initial visible actions of a term.

$$I^*(t) = \{a \mid a \in A \text{ and } t \stackrel{a}{\Longrightarrow} t' \text{ for some } t'\}.$$

Definition 10. The *weak ready simulation preorder* $\precsim_{RS}$ is the largest relation over terms in $\mathrm{T}(A_\tau)$ satisfying the following conditions whenever $p \precsim_{RS} q$ and $\alpha \in A_\tau$:

- if $p \stackrel{\alpha}{\longrightarrow} p'$ then there exists some term q' such that $q \stackrel{\alpha}{\Longrightarrow} q'$ and $p' \precsim_{RS} q'$,
- $I^*(p) = I^*(q)$.

We say that $p, q \in \mathrm{T}(A_\tau)$ are *weak ready simulation equivalent*, written $p \approx_{RS} q$, iff p and q are related by the kernel of $\precsim_{RS}$, that is when both $p \precsim_{RS} q$ and $q \precsim_{RS} p$ hold.

Example 3. $\precsim_{RS}$ is not a precongruence with respect to the choice operator of BCCSP(A_τ). It is immediate to show that $\tau a \precsim_{RS} a$. However, $\tau a + b \not\precsim_{RS} a + b$. Indeed, by performing a τ-transition, $\tau a + b$ evolves into a, while it is not possible for $a + b$ to transform itself into a process able to weak ready simulate the process a.

Definition 11. We denote by $\sqsubseteq_{RS}$ the largest precongruence included in $\precsim_{RS}$. That is, $\sqsubseteq_{RS}$ is the largest relation such that

- $p \sqsubseteq_{RS} q \Rightarrow p \precsim_{RS} q$, and
- $p \sqsubseteq_{RS} q \Rightarrow \forall \alpha \in A_\tau \quad \alpha p \sqsubseteq_{RS} \alpha q$, and
- $p \sqsubseteq_{RS} q \Rightarrow \forall r \in \mathrm{T}(A_\tau) \quad p + r \sqsubseteq_{RS} q + r$.

Once more, the definition of the relation $\sqsubseteq_{RS}$ is purely algebraic and difficult to use when studying this relation. In what follows we look for an operational characterization of $\sqsubseteq_{RS}$. Unlike in the setting of complete simulation semantics, the behavioural characterization of the relation $\sqsubseteq_{RS}$ and its axiomatic properties will depend crucially on whether the set of visible actions A is finite or infinite.

5.1. Infinite alphabet of actions

We start by studying the equational theory of the precongruence relation $\sqsubseteq_{RS}$ when A is infinite. To this end, we first provide an explicit characterization of $\sqsubseteq_{RS}$ in this case.

Definition 12. We define the relation $\lesssim_{RS}$ taking $p \lesssim_{RS} q$ iff

- for any $\alpha \in A_\tau$ and p' such that $p \xrightarrow{\alpha} p'$, there is some q' such that $q \overset{\tau}{\Longrightarrow}\xrightarrow{\alpha}\overset{\tau}{\Longrightarrow} q'$ and $p' \precsim_{RS} q'$, and
- $I^*(p) = I^*(q)$.

We denote the kernel of $\lesssim_{RS}$ by $\eqsim_{RS}$.

Proposition 20 (Behavioural characterization of $\sqsubseteq_{RS}$). *If A is infinite then we have $p \lesssim_{RS} q$ if, and only if, $p \sqsubseteq_{RS} q$, for all $p, q \in \mathrm{T}(A_\tau)$. As a consequence, $\eqsim_{RS}$ is also the kernel of $\sqsubseteq_{RS}$.*

PROOF. It is routine to show that $\lesssim_{RS}$ is a precongruence included in $\precsim_{RS}$. Therefore the implication from left to right holds because $\sqsubseteq_{RS}$ is the largest precongruence included in $\precsim_{RS}$. To show that $p \sqsubseteq_{RS} q$ implies $p \lesssim_{RS} q$, we use the fact that, since A is infinite, there is some action $a \in A \setminus I^*(p+q)$. From $p \sqsubseteq_{RS} q$, we obtain $p + a \precsim_{RS} q + a$. Then if $p \xrightarrow{\tau} p'$, we have that $a \notin I^*(p')$. Since $q + a \overset{\tau}{\Longrightarrow} q'$ for some q' such that $p' \precsim_{RS} q'$, and $a \notin I^*(p')$, process q will need to execute at least one τ-labelled transition when reaching q', as required by the definition of $\lesssim_{RS}$. $\square$

5.1.1. Ground-completeness

We shall now provide ground-complete (conditional) axiomatizations of the relations $\lesssim_{RS}$ and $\eqsim_{RS}$.

To axiomatize $\lesssim_{RS}$ using conditional inequations, the key axiom is

$$(RS_\tau) \quad I^*(x) = I^*(y) \Rightarrow x \leq x + y.$$

This axiom mirrors the one used in the concrete setting in [23, 28].

The following technical lemma shows the relation between the weak ready simulation preorder and its induced precongruence, by means of the operational characterization provided in Definition 12. This lemma will be useful in the proof of Proposition 21.

Lemma 16. *If $p \precsim_{RS} q$ then $p \lesssim_{RS} \tau q$.*

Proposition 21. *The set of equations*

$$E^c_{RS\leq} = BW \cup \{RS_\tau\},$$

in which RS_τ is conditional, is sound and ground-complete for BCCSP(A_τ) *modulo $\lesssim_{RS}$.*

PROOF. Checking the soundness of the axioms is straightforward. Let us therefore concentrate on ground-completeness:

$$p \lesssim_{RS} q \Rightarrow E^c_{RS\leq} \vdash p \leq q.$$

To simplify the notation, in the rest of the proof we use E instead of $E^c_{RS\leq}$. We proceed by induction on the depth of p.

If $|p| = 0$ then p is $\mathbf{0}$. Since $I^*(p) = I^*(q)$, using RS_τ, B1 and B4, we have $E \vdash \mathbf{0} \leq \mathbf{0} + q = q$.

Let us assume that $|p| = n+1$. As $p \lesssim_{RS} q$, we know that for each $p \xrightarrow{\alpha} p'$ there exists some q' such that $q \stackrel{\tau}{\Longrightarrow}\xrightarrow{\alpha}\stackrel{\tau}{\Longrightarrow} q'$ and $p' \precsim_{RS} q'$. By Lemma 16 we know that $p' \lesssim_{RS} \tau q'$ and by the induction hypothesis we obtain $E \vdash p' \leq \tau q'$. Therefore, $E \vdash \alpha p' \leq \alpha\tau q'$ and using W_1 we also obtain $E \vdash \alpha p' \leq \alpha q'$. That is, for every α-derivative of p we can prove using E that there exists a larger α-weak derivative of q. Therefore, if $p = \sum \alpha_i p_i$ by applying the axioms in BW we can rewrite q into $q + \sum \alpha_i q_i$ where for each index i we have $E \vdash p_i \leq q_i$ and then $E \vdash \sum \alpha_i p_i \leq \sum \alpha_i q_i$, so that we also obtain $E \vdash p + q \leq \sum \alpha_i q_i + q$, and the weak derivatives of q in the term on the right-hand side can be absorbed using the τ-laws thus obtaining $E \vdash p + q \leq q$.

Finally, given that $p \lesssim_{RS} q$, we have that $I^*(p) = I^*(q)$ and we can use RS_τ to derive $E \vdash p \leq p + q$, and then by transitivity we conclude $E \vdash p \leq q$. □

Next we turn the axiomatization above into a ground-complete and unconditional axiomatization for the weak ready simulation preorder in this case. The axiom system will include the following inequations

$$\begin{array}{ll} (RS) & ax \leq ax + ay \\ (\tau g) & x \leq \tau x \end{array}$$

Equation RS is well known in the algebraic theory of process semantics. In particular, together with B_1–B_4, it characterizes the ready simulation preorder in the concrete case. RS also appears as a necessary condition that process semantics have to fulfil in many general results in process theory—see, e.g. [7, 24, 25].

As for the axiom τg, this is a quite simple and natural one that is satisfied by any 'natural' precongruence on processes with silent moves.

Proposition 22. *The set of non-conditional equations defined by*

$$E_{RS\leq} = BW \cup \{RS,\ \tau g\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim_{RS}$.

PROOF. The soundness of the equations in $E_{RS\leq}$ is straightforward.

To prove their ground-completeness we use Proposition 21 and show that any application of the axiom RS_τ can be mimicked using $E_{RS\leq}$. More precisely, we show that whenever $I^*(p) = I^*(q)$ we have $E_{RS\leq} \vdash p \leq p + q$.

In fact we will prove an apparently stronger result which only imposes the weaker hypothesis $I^*(p) \supseteq I^*(q)$, although it is easy to see that both results are indeed equivalent since we have $I^*(p) \supseteq I^*(q)$ if, and only if, $I^*(p) = I^*(p+q)$. We prove this result by structural induction on q.

- If $q = \mathbf{0}$ then it is obvious that $E_{RS\leq} \vdash p \leq p + q$.

- If $q = aq'$, then $a \in I^*(p)$ and applying the τ-laws we can derive $p = p + ap'$, for some $p' \in \mathrm{T}(A_\tau)$. Applying RS we get $E_{RS\leq} \vdash ap' \leq ap' + aq'$ and therefore $E_{RS\leq} \vdash p + ap' \leq p + ap' + aq'$, from which $E_{RS\leq} \vdash p \leq p + q$ follows by using the τ-laws.

- If $q = \tau q'$ then $I^*(q') = I^*(q)$, and therefore $I^*(q') \subseteq I^*(p)$. Then by applying the induction hypothesis we have $E_{RS\leq} \vdash p \leq p + q'$, and using the axiom τg we conclude $E_{RS\leq} \vdash p \leq p + q$, as desired.

- If $q = q_1 + q_2$, then $I^*(q_1) \subseteq I^*(p)$ and $I^*(q_2) \subseteq I^*(p + q_1)$. Then by applying the induction hypothesis we obtain $E_{RS\leq} \vdash p \leq p + q_1$ and $E_{RS\leq} \vdash p + q_1 \leq p + q_1 + q_2$, from which $E_{RS\leq} \vdash p \leq p + q$ follows by transitivity. □

To obtain a ground-complete axiomatization of the largest congruence included in weak ready simulation equivalence, it would be pleasing to use a general 'ready-to-preorder result' [7, 25] as the one we have for the concrete case. Chen, Fokkink and van Glabbeek have presented a similar result for weak semantics in [19], but unfortunately it is not general enough to cover the case of the weak ready simulation congruence in Definition 12.

We provide a direct proof of ground-completeness for an axiomatization of this relation in which the equation

$$(RSE_\tau) \quad I^*(x) = I^*(y) \Rightarrow \alpha(x+y) = \alpha(x+y) + \alpha y,$$

which is quite similar to the equation needed for the concrete case, plays a key role.

Proposition 23. *The set of equations*

$$E^c_{RS=} = BW \cup \{RSE_\tau\},$$

in which RSE_τ is conditional, is sound and ground-complete for BCCSP(A_τ) *modulo $\eqsim_{RS}$.*

PROOF. Let us recall that $p \eqsim_{RS} q$ iff $p \lesssim_{RS} q$ and $q \lesssim_{RS} p$. We need to prove that

$$p \eqsim_{RS} q \Leftrightarrow E^c_{RS=} \vdash p = q.$$

To prove soundness of the equations in $E^c_{RS=}$, the only non-trivial case is to show that $\alpha(x+y) + \alpha y \lesssim_{RS} \alpha(x+y)$ whenever $I^*(x) = I^*(y)$. But, if $I^*(p) = I^*(q)$, then the transition $\alpha(p+q) + \alpha q \xrightarrow{\alpha} q$ can be simulated by $\alpha(p+q) \overset{\alpha}{\Longrightarrow} p+q$. Indeed, $q \lesssim_{RS} p+q$, because $I^*(p) = I^*(q)$.

To prove ground-completeness we show by induction on the depth of p that $p \lesssim_{RS} q$ implies $E^c_{RS=} \vdash q = q + p$. Then, by symmetry, $q \lesssim_{RS} p$ implies $E^c_{RS=} \vdash p = q + p$, and this immediately gives us $p \eqsim_{RS} q$ implies $E^c_{RS=} \vdash p = q$.

- The base case is trivial, we have $p = \mathbf{0}$ and then $q = \mathbf{0}$, and $E^c_{RS=} \vdash \mathbf{0} = \mathbf{0} + \mathbf{0}$.

- For the inductive case, let us assume that $p \lesssim_{RS} q$. Then for any transition $p \xrightarrow{\alpha} p'$, we have some $q \xRightarrow{\alpha} q'$ with $p' \precsim_{RS} q'$. By using Lemma 16 we know that $p' \lesssim_{RS} \tau q'$. Applying the induction hypothesis, we can assume that

$$E^c_{RS^=} \vdash \tau q' = \tau q' + p'.$$

As we know that $I^*(p') = I^*(\tau q')$, we can use RSE_τ to get

$$E^c_{RS^=} \vdash \alpha(\tau q' + p') = \alpha(\tau q' + p') + \alpha p'.$$

Since $\tau q' = \tau q' + p'$, we can simplify the term $\alpha(\tau q' + p') + \alpha p'$ in the above equation, yielding

$$E^c_{RS^=} \vdash \alpha(\tau q') = \alpha(\tau q') + \alpha p',$$

and using W_1 we obtain $E^c_{RS^=} \vdash \alpha q' = \alpha q' + \alpha p'$. We can now add q on both sides to get

$$E^c_{RS^=} \vdash q + \alpha q' = q + \alpha q' + \alpha p'.$$

Finally the τ-laws allow the absorption of α-derivatives, and we can conclude that $E^c_{RS^=} \vdash q = q + \alpha p'$, for every transition $p \xrightarrow{\alpha} p'$.

Adding up every possible α-derivative of process p, we get $E^c_{RS^=} \vdash q = q + p$ as desired.

□

In order to give an unconditional axiomatization of $\eqsim_{RS}$, we consider the following equations:

$$\begin{array}{ll} (RSE) & \alpha(bx + z + by) = \alpha(bx + z + by) + \alpha(bx + z) \\ (RSE_{\tau e}) & \alpha(x + \tau y) = \alpha(x + \tau y) + \alpha(x + y) \end{array}$$

Proposition 24. *The set of equations*

$$E_{RS^=} = BW \cup \{RSE, RSE_{\tau e}\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\eqsim_{RS}$.

PROOF. Soundness can be proved by noticing that both RSE and $RSE_{\tau e}$ are particular instances of the conditional equation RSE_τ.

To prove ground-completeness we use the same ideas as in the proof of Proposition 22, showing that whenever $I^*(p) = I^*(q)$ we have $E_{RS^=} \vdash \alpha(p + q) = \alpha(p + q) + \alpha p$. □

5.1.2. A complete axiomatization

We shall now provide an axiomatization for the relation $\lesssim_{RS}$ that is also complete. As a matter of fact, this is just the same axiomatization that was proved to be ground-complete in Proposition 22.

Proposition 25. *If the set of actions A is infinite, then the axiom system*

$$E_{RS\leq} = BW \cup \{RS, \tau g\}$$

is ω-complete for BCCSP(A_τ) *modulo $\lesssim_{RS}$.*

PROOF. The proof is analogous to that of Proposition 4. and therefore we omit it. □

We immediately obtain the following corollary.

Corollary 4. *If the set of actions is infinite, then the axiom system*

$$E_{RS\leq} = BW \cup \{RS, \tau g\}$$

is complete for BCCSP(A_τ) *modulo $\lesssim_{RS}$.*

5.2. Axiomatizing $\sqsubseteq_{RS}$ when A is finite

In this section, we study the (in)equational theory of $\sqsubseteq_{RS}$ when the set of observable actions A is finite and non-empty.

First of all, note that, if A is finite then the relation $\lesssim_{RS}$ defined in Definition 12 is *not* anymore the largest precongruence included in the weak ready simulation preorder. Indeed, if we consider the terms

$$p = \tau \sum_{a \in A} a \text{ and } q = \sum_{a \in A} a, \tag{3}$$

it is easy to check that we have $p + r \precsim_{RS} q + r$, for any $r \in$ BCCSP(A_τ). (This is so because $I^*(q) = I^*(p) = A$ and thus $I^*(r) \subseteq I^*(p)$.) It follows that $p \sqsubseteq_{RS} q$. On the other hand, $p \not\lesssim_{RS} q$ because q cannot initially perform a τ-labelled transition, unlike p.

As a matter of fact, for any $r \in$ BCCSP(A_τ), we also have $p \precsim_{RS} q + r$.

Definition 13. We define the relation $\lesssim^F_{RS}$ taking $p \lesssim^F_{RS} q$ iff

- for each transition $p \xrightarrow{a} p'$ there exists some q' such that $q \xRightarrow{a} q'$ with $p' \precsim_{RS} q'$;
- for each transition $p \xrightarrow{\tau} p'$,
  - either there exists some q' such that $q(\xrightarrow{\tau})^+q'$ with $p' \precsim_{RS} q'$, or
  - $I^*(p') = A$ and $p' \precsim_{RS} q$; and
- $I^*(q) \subseteq I^*(p)$.

Note that $p \lesssim^F_{RS} q$, for the processes p and q defined in (3). Indeed, since $I^*(q) = A$, the definition allows q to match the initial τ-transition of p by remaining idle.

Lemma 17. *For all closed terms p and q, if $p \lesssim^F_{RS} q$ then $p \precsim_{RS} q$.*

PROOF. Assume that $p \lesssim^F_{RS} q$ and $p \xrightarrow{\tau} p'$ for some p'. Then, By the second clause in the definition of $\lesssim^F_{RS}$, there is some q' such that $q \xRightarrow{\tau} q'$ and $p' \precsim_{RS} q'$. Moreover, by the first and the third clause in the definition of $\lesssim^F_{RS}$, if $p \lesssim^F_{RS} q$ then $I^*(p) = I^*(q)$. □

Proposition 26 (Behavioural characterization of $\sqsubseteq_{RS}$). *Assume that A is finite. Then $p \lesssim^F_{RS} q$ if, and only if, $p \sqsubseteq_{RS} q$, for all $p, q \in \mathrm{T}(A_\tau)$.*

PROOF. To establish the 'if' implication, it suffices only to show that, for all $p, q \in$ BCCSP(A_τ),

1. $p \precsim_{RS} q$ and
2. $p + \sum_{a \in A} a \precsim_{RS} q + \sum_{a \in A} a$

imply $p \lesssim^F_{RS} q$. In order to prove this claim, in light of the assumption that $p \precsim_{RS} q$, we only need to prove that if $p \xrightarrow{\tau} p'$ and $I^*(p') \neq A$ then there exists some q' such that $q(\xrightarrow{\tau})^+q'$ with $p' \precsim_{RS} q'$. However, this is an immediate consequence of the assumption that $p + \sum_{a \in A} a \precsim_{RS} q + \sum_{a \in A} a$ because $I^*(q + \sum_{a \in A} a) = A$.

To establish the 'only if' implication, since $\lesssim^F_{RS}$ is included in $\precsim_{RS}$, it suffices to prove that $\lesssim^F_{RS}$ is a precongruence. It is clear that $\lesssim^F_{RS}$ is preserved by action prefixing. We shall therefore focus on showing that $\lesssim^F_{RS}$

is preserved by $+$. To this end, assume that $p \lesssim^F_{RS} q$ and let r be a closed term. We shall now prove that $p + r \lesssim^F_{RS} q + r$, and focus on the only interesting case of the argument.

Suppose that $p + r \xrightarrow{\tau} p'$ because $p \xrightarrow{\tau} p'$. Since $p \lesssim^F_{RS} q$, we have that

- either there exists some q' such that $q(\xrightarrow{\tau})^+q'$ with $p' \precsim_{RS} q'$,
- or $I^*(p') = A$ and $p' \precsim_{RS} q$.

In the former case, $q + r(\xrightarrow{\tau})^+q'$ also holds, and we are done. In the latter case, we claim that $p' \precsim_{RS} q + r$ also holds, and we are done. To see that our claim does hold, observe that the relation

$$\mathcal{R} = \{(p_1, q_1 + r) \mid p_1 \precsim_{RS} q_1 \text{ and } I^*(p_1) = A\} \cup \precsim_{RS}$$

is a weak ready simulation. Indeed, suppose that $p_1 \mathcal{R}\, q_1 + r$ and $p_1 \xrightarrow{\tau} p_1'$. If $I^*(p_1') = A$ then $p_1' \mathcal{R}\, q_1 + r$, and we are done. Otherwise, it must be the case that there exists some q_1' such that $q_1(\xrightarrow{\tau})^+q_1'$ and $p_1' \precsim_{RS} q_1'$. This follows because, since $p_1 \precsim_{RS} q_1$ and $I^*(p_1) = A$ yield that $I^*(q_1) = A$, it cannot be the case that $p_1' \precsim_{RS} q_1$. Checking that every observable transition from p_1 can be matched by $q_1 + r$ in the sense of Definition 10 is immediate. □

We collect below some observations on the relationships between $\precsim_{RS}$ and $\lesssim^F_{RS}$.

Proposition 27. *For all p, q, the following statements hold.*

1. *If $p \lesssim^F_{RS} \tau q$ then $p \precsim_{RS} q$.*
2. *Assume that $p \lesssim^F_{RS} q$, $p \xrightarrow{\tau} p'$, $I^*(p') = A$ and $p' \precsim_{RS} q$. Then $p' \lesssim^F_{RS} q$.*
3. *$p \precsim_{RS} q$ iff $p \lesssim^F_{RS} \tau q$.*

PROOF. The first and the second claims are immediate from the definitions. The implication from right to left in the third claim holds because $\tau q \precsim_{RS} q$. To establish the implication from left to right, we only need to observe that even if q has remained idle to simulate a τ-transition from p, then by executing the τ-prefix in τq we get a non trivial weak τ-transition of this last process, thus fulfilling the conditions in the definition of $\lesssim^F_{RS}$. □

5.2.1. Ground-completeness

In order to give a ground-complete axiomatization of the relation $\lesssim_{RS}^{F}$, we consider the equation

$$(RS_\Sigma) \quad \tau(\sum_{a\in A} ax_a + y) = \sum_{a\in A} ax_a + y,$$

where $\{x_a \mid a \in A\}$ is an indexed set of pairwise distinct variables.

Proposition 28. *The set of equations*

$$E_{RS\leq}^{Fc} = BW \cup \{RS_\tau, RS_\Sigma\},$$

in which RS_τ is conditional, is sound and ground-complete for BCCSP(A_τ) *modulo $\lesssim_{RS}^{F}$.*

PROOF. To prove soundness, we only need to check the validity of axiom RS_Σ, and more exactly that $\tau(\sum_A ap_a + q) \lesssim_{RS}^{F} \sum_A ap_a + q$, for all p_a ($a \in A$) and q, which is immediate since $\tau(\sum_A ap_a + q) \xrightarrow{\tau} \sum_A ap_a + q$ can be mimicked by the process $\sum_A ap_a + q$ by simply staying idle, because we have $I^*(\sum_A ap_a + q) = A$.

To prove the ground-completeness of the proposed axiomatization, we follow exactly the same procedure that we used in the proof of Proposition 21. The only difference appears when we consider a transition $p \xrightarrow{\tau} p'$ with $I^*(p') = A$, so that q can mimic that move by remaining idle, because we have $p' \precsim_{RS} q$. Then by applying statement 2 in Proposition 27, we have $p' \lesssim_{RS}^{F} q$. Now, applying the induction hypothesis we get $E_{RS\leq}^{Fc} \vdash p' \leq q$. But since $I^*(p') = A$, by the classic Absorption Lemma (see [34, page 157] and [43, Lemma 16, page 163]) we can obtain

$$E_{RS\leq}^{Fc} \vdash p' = p' + \sum_A ap'_a$$

taking any a-derivative p'_a for each $a \in A$. And applying RS_Σ we obtain $\tau p' = p'$. Finally, we put everything together to conclude $E_{RS\leq}^{Fc} \vdash \tau p' \leq q$. □

Remark 4. Since the axiomatization $E_{RS\leq}^{Fc}$ is already conditional, it is illustrative to substitute in it the axiom RS_Σ by its conditional form

$$(RS_\Sigma^c) \quad (I^*(x) = A) \Rightarrow \tau p = p,$$

where we immediately recognize the restricted form of the axiom τe, as it was the case for the axiom $CS_{\tau e}$, for the weak complete simulation preorder.

Proposition 29. *The set of equations*

$$E^F_{RS^{\leq}} = BW \cup \{RS, \tau g, RS_\Sigma\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim^F_{RS}$.

PROOF. Since RS_Σ is an (unconditional) equation, we can simply replay here the proof of Proposition 22. □

We now proceed to offer (un)conditional axiomatizations of $\eqsim^F_{RS}$, the kernel of the preorder $\lesssim^F_{RS}$.

Proposition 30. *The set of equations*

$$E^{Fc}_{RS^{=}} = BW \cup \{RSE_\tau, RS_\Sigma\},$$

in which RSE_τ is conditional, is sound and ground-complete for BCCSP(A_τ) *modulo* $\eqsim^F_{RS}$.

PROOF. Similar to the proof of Proposition 28. □

Proposition 31. *The set of equations*

$$E^F_{RS^{=}} = BW \cup \{RSE, RS_{\tau e}, RS_\Sigma\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\eqsim^F_{RS}$.

PROOF. The proof is identical to that of Proposition 24. □

Remark 5. Since, in the case $|A| < \infty$, the preorder $\lesssim^F_{RS}$ is the largest precongruence included in $\precsim_{RS}$, all the axiomatizations above are also sound and ground-complete for $\sqsubseteq_{RS}$ and its kernel, in this case. Note that the situation is similar to those for both the weak simulation and weak complete simulation preorders, where the preorders $\lesssim'_S$ and $\lesssim'_{CS}$ were finer than the corresponding largest precongruences. As in those cases, the corresponding restricted version of the axiom τe shows the difference with the finer preorder, which in this case obviously coincides with the relation $\lesssim_{RS}$, that was the largest precongruence contained in $\precsim_{RS}$ when the alphabet is infinite.

5.2.2. Nonexistence of finite complete axiomatizations

We shall now prove that, if the set of actions A is finite and non-empty, then neither $\lesssim^F_{RS}$ nor its kernel afford a finite (in)equational axiomatization. The following proposition was shown in [21]—see page 516 in that reference.

Proposition 32. *For each $n \geq 0$, the equation*

$$a^n x + a^n \mathbf{0} + \sum_{b \in A} a^n (x + b) = a^n \mathbf{0} + \sum_{b \in A} a^n (x + b) \tag{4}$$

is sound modulo ready simulation equivalence, and therefore modulo the kernel of $\lesssim^F_{RS}$.

The family of equations (4) plays a crucial role in the proof of Theorem 36 in [21], to the effect that the equational theory of ready simulation equivalence is not finitely based over BCCSP(A_τ) when the set of actions is finite and contains at least two distinct actions. (In fact, as we showed in [5], ready simulation semantics is not finitely based, even when the set of actions is a singleton.) In what follows, we will follow the strategy underlying the proof of Proposition 14 to show the following result.

Proposition 33. *If $|A| \geq 1$ then the (in)equational theory of the precongruence $\lesssim^F_{RS}$ over* BCCSP(A_τ) *does not have a finite (in)equational basis. In particular, the following statements hold true:*

1. *No finite set of sound inequations over* BCCSP(A_τ) *modulo $\lesssim^F_{RS}$ can prove all of the sound inequations in the family*

$$a^n x \leq a^n \mathbf{0} + \sum_{b \in A} a^n (x + b) \qquad (n \geq 1).$$

2. *No finite set of sound (in)equations over* BCCSP(A_τ) *modulo $\lesssim^F_{RS}$ can prove all of the sound equations in the family (4).*

Proposition 33 is a corollary of the following result. As usual, we will consider processes up to strong bisimilarity.

Proposition 34. *Assume that $|A| \geq 1$. Let E be a collection of inequations whose elements are sound modulo $\lesssim^F_{RS}$ and have depth smaller than n. Suppose furthermore that the inequation $t \leq u$ is derivable from E and that $u \lesssim^F_{RS} a^n \mathbf{0} + \sum_{b \in A} a^n (x + b)$. Then $t \stackrel{a^n}{\Longrightarrow} x$ implies $u \stackrel{a^n}{\Longrightarrow} x$.*

Using the above result, Proposition 33 (statement 1) can be shown following the same reasoning described on page 33 after Proposition 15.

Statement 2 in Proposition 33 is a corollary of statement 1 in Proposition 33. To see this, assume Proposition 33(1) and suppose, towards a contradiction, that there is a finite set of sound (in)equations over BCCSP(A_τ) modulo $\lesssim_{RS}^{F}$ that can prove all of the equations in the family (4). Recall that we may assume that E is closed with respect to symmetry and that, under this assumption, there is no difference between the rules of inference of equational and inequational logic. Thus E can prove all the inequations

$$a^n x + a^n \mathbf{0} + \sum_{b\in A} a^n(x+b) \leq a^n \mathbf{0} + \sum_{b\in A} a^n(x+b) \qquad (n \geq 1).$$

Observe now that the sound inequation RS, that is

$$ax \leq ax + ay,$$

can be used to show that

$$a^n x \leq a^n x + a^n \mathbf{0} + \sum_{b\in A} a^n(x+b) \qquad (n \geq 1).$$

Therefore, by transitivity, the finite set of sound inequations $E \cup \{RS\}$ can prove all of the inequations in the family

$$a^n x \leq a^n \mathbf{0} + \sum_{b\in A} a^n(x+b) \qquad (n \geq 1).$$

This, however, contradicts Proposition 33(1).

In order to show Proposition 34, we shall follow the strategy we used in the proof of Proposition 15. The crux of the proof is again to argue that the stated property is preserved by arbitrary inequational derivations from a collection of inequations whose elements are sound modulo $\lesssim_{RS}^{F}$ and have depth smaller than n.

The following key lemma can be shown by mimicking the proof of Lemma 14.

Lemma 18. *Assume that* $at \precsim_{RS} au \precsim_{RS} a^n\mathbf{0} + \sum_{b\in A} a^n(x+b)$, *and that* $at \stackrel{a^n}{\Longrightarrow} x$. *Then* $au \stackrel{a^n}{\Longrightarrow} x$.

We now have all the necessary ingredients to complete the proof of Proposition 34, and therefore that of statement 1 in Proposition 33.

PROOF. *(of Proposition 34)* Assume that E is a collection of inequations whose elements have depth smaller than n and are sound modulo $\lesssim^F_{RS}$. Suppose furthermore that

- the inequation $t \leq u$ is derivable from E,
- $u \precsim_{RS} a^n\mathbf{0} + \sum_{b\in A} a^n(x+b)$, and
- $t \stackrel{a^n}{\Longrightarrow} x$.

We shall prove that $u \stackrel{a^n}{\Longrightarrow} x$ by induction on the derivation of $t \leq u$ from E. We proceed by examining the last rule used in the proof of $t \leq u$. The case of reflexivity is trivial and that of transitivity follows by applying the induction hypothesis twice. If $t \leq u$ is proved by instantiating an inequation in E, then the claim follows by Lemma 12, because $\lesssim^F_{RS}$ is included in $\lesssim_{CS}$. We are therefore left with the congruence rules, which we examine separately below:

- Suppose that E proves $t \leq u$ because $t = \tau t'$, $u = \tau u'$ and E proves $t' \leq u'$ by a shorter inference. Observe that $t' \stackrel{a^n}{\Longrightarrow} x$, since $t = \tau t' \stackrel{a^n}{\Longrightarrow} x$. Moreover,
$$u' \precsim_{RS} a^n\mathbf{0} + \sum_{b\in A} a^n(x+b).$$
The induction hypothesis yields that $u' \stackrel{a^n}{\Longrightarrow} x$. Therefore we have that $u = \tau u' \stackrel{a^n}{\Longrightarrow} x$, as required.

- Suppose that E proves $t \leq u$ because $t = at'$, $u = au'$ and E proves $t' \leq u'$ by a shorter inference. By the soundness of E, the fact that $\lesssim^F_{RS}$ is included in $\precsim_{RS}$ and the proviso of the proposition, we have that
$$t = at' \precsim_{RS} u = au' \precsim_{RS} a^n\mathbf{0} + \sum_{b\in A} a^n(x+b),$$
and $t \stackrel{a^n}{\Longrightarrow} x$. Lemma 18 now yields $u \stackrel{a^n}{\Longrightarrow} x$, as required.

Weak Ready Simulation	Ground-complete		Complete	
Finite Equations	Order	Equiv.	Order	Equiv.
$\lvert A\rvert = \infty$	$E_{RS\leq}$	$E_{RS=}$	$E_{RS\leq}$	$E_{RS=}$
$1 \leq \lvert A\rvert < \infty$	$E^F_{RS\leq}$	$E^F_{RS=}$	Do not exist	

Table 6: Axiomatizations for the largest (pre)congruence included in the weak ready simulation semantics

- Suppose that E proves $t \leq u$ because $t = t_1 + t_2$, $u = u_1 + u_2$ and E proves $t_i \leq u_i$, $1 \leq i \leq 2$, by shorter inferences. Since $t \stackrel{a^n}{\Longrightarrow} x$ and n is positive, we may assume, without loss of generality, that $t_1 \stackrel{a^n}{\Longrightarrow} x$. This means that $I^*(t_1) = \{a\}$. (Indeed, $I^*(t) = \{a\}$ because $t \precsim_{RS} a^n\mathbf{0} + \sum_{b\in A} a^n(x+b)$.) Therefore, by the soundness of E, $I^*(u_1) = \{a\}$ also holds. It is now not hard to see that

$$u_1 \precsim_{RS} a^n\mathbf{0} + \sum_{b\in A} a^n(x+b).$$

  Thus we may apply the induction hypothesis to infer that $u_1 \stackrel{a^n}{\Longrightarrow} x$. Hence, as n is positive, $u \stackrel{a^n}{\Longrightarrow} x$, as required.

This completes the proof. □

Corollary 5. *Assume that $1 \leq |A| < \infty$. Then the collection of (in)equations in at most one variable that hold over* BCCSP(A_τ) *modulo $\precsim^F_{RS}$ does not have a finite (in)equational basis. Moreover, for each n, the collection of all sound (in)equations of depth at most n cannot prove all the valid (in)equations in at most one variable that hold in weak ready simulation semantics over* BCCSP(A_τ).

Tables 6–7 summarize the positive and negative results on the existence of finite axiomatizations for weak ready simulation semantics.

5.3. Alternative notions of weak ready simulation

Certainly, there are many possible ways to define a weak ready simulation semantics. The enormous collection of weak semantics discussed

Unconditional	
	(RS) $\quad ax \leq ax + ay$
$E_{RS^{\leq}} = BW \cup \{RS, \tau g\}$	(τg) $\quad x \leq \tau x$
$E_{RS^{=}} = BW \cup \{RSE, RS_{\tau e}\}$	(RSE) $\quad \alpha(bx + z + by) =$
$E^{F}_{RS^{\leq}} = BW \cup \{RS, \tau g, RS_{\Sigma}\}$	$\alpha(bx + z + by) + \alpha(bx + z)$
$E^{F}_{RS^{=}} = BW \cup$	$(RS_{\tau e})$ $\quad \alpha(x + \tau y) =$
$\{RSE, RS_{\tau e}, RS_{\Sigma}\}$	$\alpha(x + \tau y) + \alpha(x + y)$
	(RS_{Σ}) $\quad \tau(\sum_A ax_a + y) = \sum_A ax_a + y$
Conditional	
	(RS_{τ}) $\quad (I^*(x) \Leftrightarrow I^*(y)) \Rightarrow$
$E^{c}_{RS^{\leq}} = BW \cup \{RS_{\tau}\}$	$x \leq x + y$
$E^{c}_{CS^{=}} = BW \cup \{RSE_{\tau}\}$	(RSE_{τ}) $\quad (I^*(x) \Leftrightarrow I^*(y)) \Rightarrow$
	$\alpha(x + y) = \alpha(x + y) + \alpha x$

Table 7: Axioms for the largest (pre)congruence included in the weak ready simulation semantics

in [27] gives us an idea of the wealth of options at one's disposal, but there are even more (reasonable, why not?) possibilities. If we concentrate on the aspects of the definition of these semantics that relate to non-determinism, leaving aside divergence and related features, the main reason that causes this multiplicity of proposals is the double essence of invisible actions, usually represented by τ when we give the operational description of the processes. These invisible actions are either produced by the decision to abstract from the execution of some actions, or just represent non-deterministic choices, that therefore should have better a 'static' meaning, reflecting the 'specification level' at which these non-deterministic choices find their sense.

Our definition of the weak ready simulation preorder looks for the simplest generalization of the original one (without τ's), which uses the function I, which associates with each process its collection of initial actions, as main ingredient. It seems clear that the consideration of I^* instead of I is the easiest way to obtain a constraint [23] that generalizes that for the strong case, capturing the internal invisible character of τ's in an adequate way. We expected, and as we have seen above, this is indeed the case, that in this way all the algebraic (good) properties of the strong semantics would be transferred in a (more or less) easy way to the weak case: indeed, a 'symbolic' substitution of I by I^* in the axiom (RS), together with the addition of the axioms for weak bisimulation WB, produce the desired axiomatization of our weak ready simulation semantics.

There are, however, several objections that could be posed to our proposal. It is true, that our weak semantics is not coarser than the strong ready simulation that can be defined considering τ just as another observable action. Under that strong semantics, we would immediately have $\tau x \leq \tau x + \tau y$, since $I(\tau x) = I(\tau x + \tau y) = \{\tau\}$; instead, under our weak semantics we will have for sure $\tau x \not\lesssim \tau x + \tau y$ as soon as $I^*(x) \not\subset I^*(y)$, because in this case we have $I^*(\tau x) \neq I^*(\tau x + \tau y)$.

It is true that weak bisimulation is (and was expected to be) coarser than strong bisimulation, and guided by this fact one could assume that to preserve this situation is a must when considering any other semantics. Obviously, there are also some practical reasons supporting this procedure. For instance, in order to prove that two processes are weakly bisimilar, it is enough to prove that they are strongly bisimilar, when this is indeed the case. However, we can also give some other reasons that justify the fact that the strong semantics that 'sees' the τ's will not be finer than the corresponding weak semantics that 'hides' their execution. This is related with the double (possible) meaning of these internal transitions, that either come from the abstraction of some operational details, that we want to hide to the external observer, or just represent non-deterministic choices, so that the expected meaning of these τ's is absolutely 'static', and their operational interpretation (certainly totally unnatural) could produce some undesired effects. When our τ's come from the application of hiding then we should look for an alternative definition of weak ready simulations which will be coarser than the strong version; but instead when τ's express non-determinism our proposal is perfectly supportable, even if it is not coarser than the original strong semantics.

One could also argue that if we are interested in a simple algebraic characterization of a weak ready simulation semantics, we should have started by considering the axiom

$$(RS'_\tau) \;\; \alpha x \leq \alpha x + \alpha y,$$

which would yield a notion of weak ready simulation semantics that is preserved by hiding. We tried indeed to follow this path, but, unfortunately, we were unable to obtain an attractive operational characterization of any 'weak ready simulation semantics' that satisfies this axiom. In particular, if we consider the semantics that is algebraically defined by adding either (RS'_τ), or its slightly stronger version

$$(RS^+_\tau) \quad \alpha x \leq \alpha x + y,$$

to the set of axioms WB, then the obtained semantics need cumbersome ad-hoc 'up-to' mechanisms to take into account the syntactic presentation of the processes when defining (operationally) the simulation semantics.

Chen, Fokkink and van Glabbeek presented (RS'_τ) in [19] (which they simply denote by (RS)) as 'the weak ready simulation axiom' that guides their 'ready to preorder' algorithm for the weak case. This algorithm translates to the weak case the one previously developed in [7, 25]. The results in that paper are technically sound, and therefore can be applied to any semantics that satisfies the axiom (RS'_τ). Unfortunately, as mentioned above, it seems that there are not many such semantics that can be operationally defined in an appealing way. Possibly, the weak failures semantics is the only notable exception. Chen, Fokkink and van Glabbeek studied weak failures semantics in [18], obtaining an ω-complete axiomatization that includes (RS'_τ), and is quite close to that for the must-testing semantics in [26, 33], since these two semantics coincide, once the differences between the syntax used in both presentations are adequately taken into account.

However, failures semantics is not a simulation semantics, but instead the coarsest linear semantics at the layer of ready simulation, as one can see from the extended linear-time/branching-time spectrum in [22]. As a consequence, when we consider its weak version, the fact that it satisfies (RS'_τ) does not (necessarily) mean that weak failures semantics 'comes from' a weak ready simulation semantics satisfying this axiom. As a matter of fact, we have obtained the same weak failures semantics when we have looked for the coarsest linear semantics attached to our weak ready simulation semantics! Therefore, the fact that weak failures semantics satisfies (RS'_τ) could be due to the particular character of failures. In fact, it is quite 'suspicious' that no other weak version of a linear semantics in the layer of ready simulation (e.g. readiness, failure traces and ready traces) had been satisfactorily axiomatized, and we would not expect any appealing operational characterization of such a semantics if axiom (RS'_τ) is sound for them.

Another weak ready simulation semantics that also appeared in [27] has been recently used in a collection of papers [37, 39, 38], that investigate the use of disjunction in the specification of constraints to limit the behaviour of the desired implementations. This is *stable ready simulation*, that only takes into account the offers made at stable states, namely those from which no τ-labelled transition is available. This amounts to considering that τ's are *urgent* with respect to the observable actions. As a consequence, when studying this semantics we can restrict ourselves to the use of pure non-deterministic choices, that become associative, so that we can consider a model where external and internal choices alternate, and are never mixed. Moreover, any action prefixing an internal choice can be distributed over the choice producing an external choice between several branches that start with the same action. In this way, all the internal choices, but those at the root of the process, disappear and then this stable ready simulation can be 'encoded' into the strong ready simulation, just representing a minor variant of it, that in particular can be easily axiomatized by means of the axioms that claim priority of τ's, associativity of internal choices, and distributivity of prefix over internal choices.

As a conclusion, we do not claim at all that our weak ready simulation is 'the right one', but after a thorough study of the question we postulate that it is the simplest weak ready simulation notion that has, at the same time, both a simple operational definition and good algebraic properties, as we have shown in this section.

6. Conclusion

In this paper, we have offered a detailed study of the axiomatizability properties of the largest (pre)congruences over the language BCCSP induced by the 'weak' versions of the classic simulation, complete simulation and ready simulation preorders and the induced equivalences. For each of these semantics we have presented results on the (non)existence of finite (ground-)complete (in)equational axiomatizations. As in [21], the finite axiomatizability of the studied notions of semantics depends crucially on the cardinality of the set of observable actions. Following [22], we have also discussed ground-complete axiomatizations of those semantics using conditional (in)equations in some detail. In particular, we have shown how to prove ground-completeness results for (in)equational axiom systems from similar results for conditional axiomatizations in a fairly systematic way.

The results presented in this article paint a rather complete picture of the axiomatic properties of the above-mentioned weak simulation semantics over BCCSP. However, in the cases in which the studied semantics do not afford finite complete axiomatizations, it would be interesting to obtain some infinite, but 'finitely described', complete axiomatizations. This is a topic that we leave for future research.

Our results complement those offered in, e.g., [11, 48, 50], which offer ground-complete inequational axiomatizations for several notions of divergence-sensitive preorders based on variations on prebisimilarity [32, 42] or on the refusal simulation preorder. They are just a first step in the study of the equational characterization of the semantics that abstract from internal steps in computations [27]. However, we have presented a comprehensive study of the basic weak simulation semantics, and now it is time to investigate the equational characterization of weak linear semantics that are based on the notion of decorated trace. We have already started working on this topic and we plan to report on our results in a forthcoming article.

We find it pleasing that all the known results on the existence of finite bases for the weak, complete and plain simulation semantics (see [21]) in the 'concrete' case, that is without silent moves, are 'lifted' to the weak version of the simulation semantics we have presented. However, it is natural to wonder whether our results for the weak semantics can be obtained in a uniform fashion from those for the concrete ones by applying some 'metatheorems' linking the equational theories of concrete and weak semantics over some process algebra. Examples of such result are offered in [10, 20]. The paper [10] presents a general technique for obtaining new results pertaining to the non-finite axiomatizability of behavioural (pre)congruences over process algebras from known ones. The technique is based on establishing translations between languages that preserve sound (in)equations and (in)equational proofs over the source language, and reflect families of (in)equations responsible for the non-finite axiomatizability of the target language. In Section 3, we used the reduction method from [10] to lift known axiomatizability results for simulation semantics to the weak setting. So far, however, we have been unable to apply the reduction technique to obtain axiomatizability results for weak semantics that, unlike weak simulation semantics, do not satisfy the equation τe.

The development of general links between axiomatizations for weak and concrete semantics, along the lines of those presented in [20], is a very interesting line for future research. That reference and the doctoral dissertation [17] present an algorithm to turn an axiomatization of a semantics for concrete processes into one for 'its induced weak semantics'. However, the scope of that algorithm is a bit limited, and an extension of it that could be applicable to larger classes of weak semantics would be a significant advance on the state of the art in the study of axiomatizability results for process semantics over process algebras.

Following the developments in [1, 14, 48], it would also be interesting to study rule formats for operational semantics that provide congruence formats for the semantics considered in this paper, and to give procedures for generating ground-complete axiomatizations for them, for process languages in the given formats.

Finally, since the results presented in [19], to provide 'ready to preorder' procedures for generating the axiomatizations of process equivalences from those for their underlying preorders, cannot be applied, for instance, to our weak ready simulation equivalence, we plan to work on other generalizations of the known results for the concrete semantics that could be applicable to larger families of weak semantics.

Acknowledgements. We thank the anonymous referees of the conference papers [4] and [6], as well as the journal reviewers, for their suggestions that led to improvements in the presentation of this paper. We also thank Wan Fokkink for his insightful comments on [6] and for bringing [20] to our attention.

References

[1] L. Aceto, B. Bloom, and F. W. Vaandrager. Turning SOS rules into equations. *Information and Compututation*, 111(1):1–52, 1994.

[2] L. Aceto, S. Capobianco, A. Ingolfsdottir, and B. Luttik. The equational theory of prebisimilarity over basic CCS with divergence. *Information Processing Letters*, 108(5):284–289, 2008.

[3] L. Aceto, T. Chen, W. Fokkink, and A. Ingolfsdottir. On the axiomatisability of priority. *Mathematical Structures in Computer Science*, 18(1):5–28, 2008.

[4] L. Aceto, D. de Frutos-Escrig, C. Gregorio-Rodríguez, and A. Ingolfsdottir. Axiomatizing weak ready simulation semantics over BCCSP. In A. Cerone and P. Pihlajasaari, editors, *Proceedings of the 8th International Colloquium on Theoretical Aspects of Computing, ICTAC 2011*, volume 6916 of *Lecture Notes in Computer Science*, pages 7–24. Springer-Verlag, 2011.

[5] L. Aceto, D. de Frutos-Escrig, C. Gregorio-Rodríguez, and A. Ingolfsdottir. Complete and ready simulation semantics are not finitely based over BCCSP, even with a singleton alphabet. *Information Processing Letters*, 111(9):408–413, 2011.

[6] L. Aceto, D. de Frutos-Escrig, C. Gregorio-Rodríguez, and A. Ingolfsdottir. The equational theory of weak complete simulation semantics over BCCSP. In M. Bieliková, G. Friedrich, G. Gottlob, S. Katzenbeisser, and G. Turán, editors, *SOFSEM 2012: Theory and Practice of Computer Science, 38th Conference on Current Trends in Theory and Practice of Computer Science*, volume 7147 of *Lecture Notes in Computer Science*, pages 141–152. Springer-Verlag, 2012.

[7] L. Aceto, W. Fokkink, and A. Ingolfsdottir. Ready to preorder: get your BCCSP axiomatization for free! In *Algebra and Coalgebra in Computer Science, Second International Conference, CALCO 2007*, volume 4624 of *Lecture Notes in Computer Science*, pages 65–79. Springer, 2007.

[8] L. Aceto, W. Fokkink, A. Ingolfsdottir, and B. Luttik. Finite equational bases in process algebra: Results and open questions. In *Processes, Terms and Cycles*, volume 3838 of *Lecture Notes in Computer Science*, pages 338–367. Springer, 2005.

[9] L. Aceto, W. Fokkink, A. Ingolfsdottir, and B. Luttik. A finite equational base for CCS with left merge and communication merge. *ACM Transactions on Compututational Logic*, 10(1), 2009.

[10] L. Aceto, W. Fokkink, A. Ingolfsdottir, and M. R. Mousavi. Lifting non-finite axiomatizability results to extensions of process algebras. *Acta Informatica*, 47(3):147–177, 2010.

[11] L. Aceto and M. Hennessy. Termination, deadlock and divergence. *Journal of the ACM*, 39(1):147–187, 1992.

[12] L. Aceto, A. Ingolfsdottir, B. Luttik, and P. van Tilburg. Finite equational bases for fragments of CCS with restriction and relabelling. In *5th IFIP International Conference on Theoretical Computer Science*, volume 273 of *IFIP*, pages 317–332. Springer, 2008.

[13] J. Baeten, T. Basten, and M. Reniers. *Process Algebra: Equational Theories of Communicating Processes*, volume 50 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, November 2009.

[14] J. C. M. Baeten and E. P. de Vink. Axiomatizing GSOS with termination. *Journal of Logic and Algebraic Programming*, 60-61:323–351, 2004.

[15] J. A. Bergstra and J. W. Klop. Process algebra for synchronous communication. *Information and Control*, 60(1-3):109–137, 1984.

[16] B. Bloom, S. Istrail, and A. R. Meyer. Bisimulation can't be traced. *Journal of the ACM*, 42(1):232–268, 1995.

[17] T. Chen. *Clocks, Dice and Processes*. PhD thesis, Centrum voor Wiskunde en Informatica (CWI), Vrije Universiteit Amsterdam, 2009.

[18] T. Chen and W. Fokkink. On the axiomatizability of impossible futures: Preorder versus equivalence. In *LICS*, pages 156–165. IEEE Computer Society, 2008.

[19] T. Chen, W. Fokkink, and R. van Glabbeek. Ready to preorder: The case of weak process semantics. *Information Processing Letters*, 109(2):104–111, 2008.

[20] T. Chen, W. Fokkink, and R. van Glabbeek. On the axiomatizability of impossible futures. Unpublished manuscript, 2011.

[21] T. Chen, W. Fokkink, B. Luttik, and S. Nain. On finite alphabets and infinite bases. *Information and Computation*, 206(5):492–519, 2008.

[22] D. de Frutos-Escrig, C. Gregorio-Rodríguez, and M. Palomino. On the unification of process semantics: Equational semantics. *Electronic Notes in Theoretical Computer Science*, 249:243–267, 2009.

[23] D. de Frutos-Escrig and C. Gregorio-Rodríguez. Universal coinductive characterizations of process semantics. In *5th IFIP International Conference on Theoretical Computer Science*, volume 273 of *IFIP*, pages 397–412. Springer, 2008.

[24] D. de Frutos-Escrig and C. Gregorio-Rodríguez. (Bi)simulations up-to characterise process semantics. *Information and Computation*, 207(2):146–170, 2009.

[25] D. de Frutos-Escrig, C. Gregorio-Rodríguez, and M. Palomino. Ready to preorder: an algebraic and general proof. *Journal of Logic and Algebraic Programming*, 78(7):539–551, 2009.

[26] R. De Nicola and M. Hennessy. Testing equivalences for processes. *Theoretical Computer Science*, 34:83–133, 1984.

[27] R. van Glabbeek. The linear time - branching time spectrum II. In *CONCUR '93, 4th International Conference on Concurrency Theory*, volume 715 of *Lecture Notes in Computer Science*, pages 66–81. Springer, 1993.

[28] R. van Glabbeek. The linear time – branching time spectrum I; the semantics of concrete, sequential processes. In *Handbook of Process Algebra*, chapter 1, pages 3–99. Elsevier, 2001.

[29] R. van Glabbeek. A characterisation of weak bisimulation congruence. In *Processes, Terms and Cycles: Steps on the Road to Infinity*, volume 3838 of *Lecture Notes in Computer Science*, pages 26–39. Springer, 2005.

[30] R. van Glabbeek and P. Weijland. Branching time and abstraction in bisimulation semantics. *Journal of the ACM*, 43(3):555–600, 1996.

[31] J. F. Groote. A new strategy for proving omega-completeness applied to process algebra. In *CONCUR '90, First International Conference on Concurrency Theory*, volume 458 of *Lecture Notes in Computer Science*, pages 314–331. Springer, 1990.

[32] M. Hennessy. A term model for synchronous processes. *Informationa and Control*, 51(1):58–75, 1981.

[33] M. Hennessy. *Algebraic Theory of Processes*. MIT Press, 1988.

[34] M. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32:137–161, 1985.

[35] C. Hoare. *Communicating Sequential Processes*. Prentice Hall, 1985.

[36] K. G. Larsen and A. Skou. Bisimulation through probabilistic testing. *Information and Computation*, 94(1):1–28, 1991.

[37] G. Lüttgen and W. Vogler. Ready simulation for concurrency: It's logical! In *Automata, Languages and Programming, 34th International Colloquium, ICALP 2007*, volume 4596 of *Lecture Notes in Computer Science*, pages 752–763. Springer, 2007.

[38] G. Lüttgen and W. Vogler. Safe reasoning with logic LTS. In *SOFSEM 2009: Theory and Practice of Computer Science, 35th Conference on Current Trends in Theory and Practice of Computer Science*, volume 5404 of *Lecture Notes in Computer Science*, pages 376–387. Springer, 2009.

[39] G. Lüttgen and W. Vogler. Ready simulation for concurrency: It's logical! *Information and Computation*, 208(7):845–867, 2010.

[40] R. Milner. An algebraic definition of simulation between programs. In *Proceedings 2nd Joint Conference on Artificial Intelligence*, pages 481–489. BCS, 1971. Also available as Report No. CS-205, Computer Science Department, Stanford University.

[41] R. Milner. *A Calculus of Communicating Systems*. LNCS 92. Springer, 1980.

[42] R. Milner. A modal characterisation of observable machine behaviour. In E. Astesiano and C. Böhm, editors, *CAAP '81: 6th Colloquium on Trees in Algebra and Programming*, volume 112 of *Lecture Notes in Computer Science*, pages 25–34. Springer, 1981.

[43] R. Milner. *Communication and Concurrency*. Prentice Hall, 1989.

[44] R. Milner. A complete axiomatisation for observational congruence of finite-state behaviors. *Information and Computation*, 81(2):227–247, 1989.

[45] F. Moller. *Axioms for Concurrency*. PhD thesis, Report CST-59-89, Department of Computer Science, University of Edinburgh, 1989.

[46] D. M. Park. Concurrency and automata on infinite sequences. In *Theoretical Computer Science, 5th GI-Conference*, volume 104 of *Lecture Notes in Computer Science*, pages 167–183. Springer, 1981.

[47] G. D. Plotkin. A structural approach to operational semantics. *Journal of Logic and Algebraic Programming*, 60–61:17–139, 2004.

[48] I. Ulidowski. Axiomatisations of weak equivalences for De Simone languages. In *CONCUR '95: Concurrency Theory, 6th International Conference*, volume 962 of *Lecture Notes in Computer Science*, pages 219–233. Springer, 1995.

[49] M. Voorhoeve and S. Mauw. Impossible futures and determinism. *Information Processing Letters*, 80(1):51–58, 2001.

[50] D. Walker. Bisimulation and divergence. *Information and Computation*, 85(2):202–241, 1990.

Appendix A. A finite complete axiomatization for $\lesssim_{CS}$ when $A_\tau = \{\tau\}$

Assume, throughout this appendix, that $A = \emptyset$. Our aim is to show that, under this assumption, $\lesssim_{CS}$ has a finite inequational axiomatization that is given by the axioms $B1$–$B4$ and the inequations

$$\begin{array}{llcl} (L_\tau) & x & \leq & \tau y + z \quad \text{and} \\ (S) & x & \leq & x + y. \end{array}$$

In order to show that the axiom system E^{τ}_{CS} consisting of the above inequations is sound and complete with respect to $\lesssim_{CS}$, we first provide a characterization of $\lesssim_{CS}$ over open terms. In what follows, for a term t, we write $t \stackrel{\tau}{\nrightarrow}$ when there is no t' such that $t \xrightarrow{\tau} t'$.

Proposition 35. *For all terms t, u, we have that $t \lesssim_{CS} u$ if, and only if,*

1. *either $u \xrightarrow{\tau} u'$, for some u',*
2. *or $t \stackrel{\tau}{\nrightarrow}$, $u \stackrel{\tau}{\nrightarrow}$ and* $\mathrm{vars}(t) \subseteq \mathrm{vars}(u)$.

PROOF. Note, first of all, that, when τ is the only action, $\precsim_{CS}$ is the universal relation over closed terms. Therefore, $p \lesssim_{CS} q$ holds for closed terms p and q if, and only if,

$$\text{if } p \xrightarrow{\tau} p', \text{ for some } p', \text{ then } q \xrightarrow{\tau} q', \text{ for some } q'.$$

We now prove the two implications in the statement of the proposition separately.

In order to prove the implication from left to right, assume that $t \lesssim_{CS} u$ and $u \stackrel{\tau}{\nrightarrow}$. We claim that $t \stackrel{\tau}{\nrightarrow}$. Indeed, consider the closed substitution $\sigma_{\mathbf{0}}$ that maps each variable to $\mathbf{0}$. Since $t \lesssim_{CS} u$, it follows that $\sigma_{\mathbf{0}}(t) \lesssim_{CS} \sigma_{\mathbf{0}}(u)$. As $u \stackrel{\tau}{\nrightarrow}$, we have that $\sigma_{\mathbf{0}}(u) \stackrel{\tau}{\nrightarrow}$. Therefore, $\sigma_{\mathbf{0}}(t) \stackrel{\tau}{\nrightarrow}$ and this yields that $t \stackrel{\tau}{\nrightarrow}$, as claimed.

To complete the proof of the implication from left to right, we are left to show that $\mathrm{vars}(t) \subseteq \mathrm{vars}(u)$. To see this, let $x \in \mathrm{vars}(t)$ and consider the closed substitution σ mapping x to $\tau\mathbf{0}$ and mapping all the other variables to $\mathbf{0}$. Since $t \lesssim_{CS} u$, it follows that $\sigma(t) \lesssim_{CS} \sigma(u)$. As $x \in \mathrm{vars}(t)$ and $t \stackrel{\tau}{\nrightarrow}$, it is easy to see that $\sigma(t) \xrightarrow{\tau} \mathbf{0}$. Therefore, $\sigma(u) \xrightarrow{\tau} p$ for some p. Hence, $x \in \mathrm{vars}(u)$, because $u \stackrel{\tau}{\nrightarrow}$ by assumption.

We now prove the implication from right to left. Assume first that $u \xrightarrow{\tau} u'$, for some u'. Let σ be a closed substitution. It is easy to see that $\sigma(u) \xrightarrow{\tau} \sigma(u')$, and this yields that $\sigma(t) \lesssim_{CS} \sigma(u)$.

Suppose now that $t \stackrel{\tau}{\nrightarrow}$, $u \stackrel{\tau}{\nrightarrow}$ and $\mathrm{vars}(t) \subseteq \mathrm{vars}(u)$. Let σ be a closed substitution. If $\sigma(u) \xrightarrow{\tau} p$ for some p then $\sigma(t) \lesssim_{CS} \sigma(u)$. Assume thus that $\sigma(u) \stackrel{\tau}{\nrightarrow}$. This means that $\sigma(x) \stackrel{\tau}{\nrightarrow}$, for each $x \in \mathrm{vars}(u)$, because $u \stackrel{\tau}{\nrightarrow}$ by assumption. As $\mathrm{vars}(t) \subseteq \mathrm{vars}(u)$, we also have that $\sigma(x) \stackrel{\tau}{\nrightarrow}$, for each $x \in \mathrm{vars}(t)$. Since $t \stackrel{\tau}{\nrightarrow}$, this yields that $\sigma(t) \stackrel{\tau}{\nrightarrow}$. We may therefore conclude that $\sigma(t) \lesssim_{CS} \sigma(u)$. □

The following lemma will be useful in the proof of the promised completeness theorem.

Lemma 19. *Let t be an open term.*

1. *If $t \xrightarrow{\tau} t'$ for some t' then there is some t'' such that the equation $t = \tau t' + t''$ can be proved using $B1$, $B2$ and $B4$.*
2. *If $t \stackrel{\tau}{\nrightarrow}$ then $t = \sum_{x \in \mathrm{vars}(t)} x$ can be proved using $B1$–$B3$.*

Using the characterization of $\lesssim_{CS}$ presented in Proposition 35 and the above lemma, it is now a simple matter to show that the axiom system E^{τ}_{CS} is sound and complete with respect to $\lesssim_{CS}$.

Theorem 1. *For all terms t, u, we have that $t \lesssim_{CS} u$ if, and only if, $t \leq u$ can be proved using the axiom system E^{τ}_{CS}.*

PROOF. The equations $B1$–$B4$ are sound modulo bisimilarity and therefore modulo $\lesssim_{CS}$. The soundness of the inequations (L_τ) and (S) is immediate from Proposition 35.

To show completeness, assume that $t \lesssim_{CS} u$ for some terms t, u. By Proposition 35, we have that

1. either $u \xrightarrow{\tau} u'$, for some u',
2. or $t \stackrel{\tau}{\nrightarrow}$, $u \stackrel{\tau}{\nrightarrow}$ and $\mathrm{vars}(t) \subseteq \mathrm{vars}(u)$.

In the former case, by the first statement in Lemma 19, there is some u'' such that the equation $u = \tau u' + u''$ can be proved using $B1$, $B2$ and $B4$. Thus, applying (L_τ),

$$t \leq \tau u' + u'' = u,$$

and we are done.

In the latter case, by Lemma 19(2), $t = \sum_{x \in \mathrm{vars}(t)} x$ and $u = \sum_{y \in \mathrm{vars}(u)} y$ can be proved using $B1$–$B3$. Since $\mathrm{vars}(t) \subseteq \mathrm{vars}(u)$, applying (S) and possibly $B4$,

$$t = \sum_{x \in \mathrm{vars}(t)} x \leq \sum_{y \in \mathrm{vars}(u)} y = u,$$

and we are done. □