Axiomatizing Weak Simulation Semantics over BCCSP*

Luca Aceto[1,3], David de Frutos Escrig[2,3], Carlos Gregorio-Rodríguez[2,3], and Anna Ingolfsdottir[1,3]

[1] ICE-TCS, School of Computer Science, Reykjavik University, Iceland
[2] Departamento de Sistemas Informáticos y Computación, Universidad Complutense de Madrid, Spain
[3] Abel Extraordinary Chair (Universidad Complutense-Reykjavik University)

Abstract. This paper is devoted to the study of the (in)equational theory of the largest (pre)congruences over the language BCCSP induced by variations on the classic simulation preorder and equivalence that abstract from internal steps in process behaviours. In particular, the article focuses on the (pre)congruences associated with the weak simulation, the weak complete simulation and the weak ready simulation preorders. For each of these behavioural semantics, results on the (non)existence of finite (ground-)complete (in)equational axiomatizations are given. The axiomatization of those semantics using conditional equations is also discussed in some detail.

1 Introduction

Process algebras, such as ACP [11, 13], CCS [39] and CSP [31], are prototype specification languages for reactive systems. Such languages offer a small, but expressive, collection of operators that can be combined to form terms that describe the behaviour of reactive systems. Since terms in a process algebra can be used to

* Luca Aceto and Anna Ingolfsdottir have been partially supported by the projects 'New Developments in Operational Semantics' (nr. 080039021) and 'Meta-theory of Algebraic Process Theories' (nr. 100014021) of the Icelandic Fund for Research. David de Frutos Escrig and Carlos Gregorio-Rodríguez have been partially supported by the Spanish projects TESIS (TIN2009-14312-C02-01), DESAFIOS10 (TIN2009-14599-C03-01) and PROMETIDOS S2009/TIC-1465. The paper was begun when David de Frutos Escrig and Carlos Gregorio-Rodríguez held Abel Extraordinary Chair positions at Reykjavik University, and finalized while Luca Aceto held an Abel Extraordinary Chair at Universidad Complutense de Madrid, Spain, supported by the NILS Mobility Project.

describe both specifications of reactive behaviours and their implementations, an important ingredient in the theory of such languages is a notion of behavioural equivalence or preorder over terms. The chosen notion of behavioural semantics can be employed, for instance, to prove whether a term describing an implementation of a system is correct with respect to a given specification. The lack of consensus on what constitutes an appropriate notion of observable behaviour for reactive systems has led to a large number of proposals for behavioural equivalences and preorders for concurrent processes. In his by now classic paper [25], van Glabbeek presented the linear time-branching time spectrum of behavioural preorders and equivalences for finitely branching, concrete, sequential processes. The semantics in this spectrum are based on notions of simulation and on decorated traces.

Since the seminal work by Bergstra and Klop [11, 13], and Hennessy and Milner [30], the search for (in)equational axiomatizations of notions of behavioural semantics for fragments of process algebras has been one of the classic topics of investigation within concurrency theory. A complete axiomatization of a behavioural semantics yields a purely syntactic and model-independent characterization of the semantics of a process algebra, and paves the way to the application of theorem-proving techniques in establishing whether two process descriptions exhibit related behaviours.

There are three types of 'complete' axiomatizations that one meets in the literature on process algebras. An (in)equational axiomatization is called *ground-complete* if it can prove all the valid (in)equivalences relating terms with no occurrences of variables in the process algebra of interest. It is *complete* when it can be used to derive all the valid (in)equivalences. (A complete axiom system is also referred to as a *basis* for the algebra it axiomatizes.) These two notions of completeness relate the semantic notion of process, namely an equivalence class of terms, with the proof-theoretic notion of provability from an (in)equational axiom system. In particular, a basis for an algebra of processes offers a full, purely syntactic view of the semantic notion of 'process' that underlies it. An axiomatization E is *ω-complete* when an inequation can be derived from E if, and only if, all of its closed instantiations can be derived from E. The notion of ω-completeness is therefore a proof-theoretic one. Its connections with completeness are well known, and are discussed in, e.g., [6].

In [25], van Glabbeek studied the semantics in his spectrum in the setting of the process algebra BCCSP, which contains only the basic process algebraic operators from CCS [39] and CSP [31], but is sufficiently powerful to express all finite synchronization trees [37]. In the aforementioned reference, van Glabbeek gave, amongst a wealth of other results, (in)equational ground-complete axiomatizations for the preorders and equivalences in the spectrum. In [19], two of the authors of this paper presented a unification of the axiomatizations of all the semantics in the linear time-branching time spectrum. This unification is achieved by means of conditional axioms that provide a simple and clear picture of the similarities and differences between all the semantics. In [27], Groote obtained ω-completeness results for most of the axiomatizations presented in [25], in case the alphabet of actions is infinite.

The article [6] surveys results on the existence of finite, complete equational axiomatizations of behavioural equivalences over fragments of process algebras up to 2005. Some of the results on the (non)existence of finite, complete (in)equational axiomatizations of behavioural semantics over process algebras that have been obtained since the publication of that survey may be found in [2, 3, 7, 8, 10, 16].

In the setting of BCCSP, in a seminal journal paper that collects and unifies the results in a series of conference articles, Chen, Fokkink, Luttik and Nain have offered in [17] a definitive classification of the status of the finite basis problem—that is, the problem of determining whether a behavioural equivalence has a finite, complete, equational axiomatization over the chosen process algebra—for all the semantics in van Glabbeek's spectrum. Notable later results by Chen and Fokkink presented in [16] give the first example of a semantics—the so-called *impossible future semantics* from [46]—where the preorder defining the semantics can be finitely axiomatized over BCCSP, but its induced equivalence cannot. The authors of this paper have recently shown in [4] that complete simulation and ready simulation semantics do not afford a finite (in)equational axiomatization even when the set of actions is a singleton.

The collection of results mentioned in the previous paragraph gives a complete picture of the axiomatizability of behavioural semantics in van Glabbeek's spectrum over BCCSP. However, such notions of behavioural semantics are *concrete*, in the sense that they consider each action processes perform as being observable by their environment. Despite the fundamental role they play in the development of a theory of reactive systems, concrete semantics are not very useful from the point of view of applications. For this reason, notions of behavioural semantics that, in some well-defined way, abstract from externally unobservable steps of computation that processes perform have been proposed in the literature—see, e.g., the classic references [23, 26, 30], which offer, amongst many other results, ground-complete axiomatizations of the studied notion of behavioural semantics. (Following Milner, such notions of behavioural semantics are usually called 'weak semantics'.) However, to the best of our knowledge, no systematic study of the axiomatizability properties of variations on the classic notion of *simulation semantics* [36, 42] that abstract away from internal steps of computation in the behaviour of processes has been presented in the literature. This is all the more surprising since simulation semantics is very natural and plays an important role in applications.

The aim of this paper is to offer a detailed study of the axiomatizability properties of the largest (pre)congruences over the language BCCSP induced by variations on the classic simulation preorder and equivalence that abstract from internal steps in process behaviours. In particular, we focus on the (pre)congruences associated with the weak simulation, the weak complete simulation [25] and the weak ready simulation [14, 32] preorders. For each of these behavioural semantics, we present results on the (non)existence of finite (ground-)complete (in)equational axiomatizations. Following [19], we also discuss the axiomatization of those semantics using conditional equations in some detail.

We begin our study of the weak simulation semantics over BCCSP in Section 3 by focusing on the natural extension of the classic simulation preorder to a setting with the internal action τ. Unlike most other notions of semantics for reactive systems that abstract from internal steps of computation, the *weak simulation* *preorder* and its induced equivalence are preserved by all the operators of BCCSP. Indeed, the equation

$$\tau x = x$$

is sound modulo weak simulation equivalence and, using it, one can remove all occurrences of the symbol τ from terms. This allows us to lift all the known results on the (non)existence of finite (ground-)complete axiomatizations from the setting of the classic simulation semantics to its weak counterpart using, for instance, the approach developed in [8].

In Section 4, we study the notion of *weak complete simulation*, which is the 'weak counterpart' of complete simulation. In the setting without internal actions, a complete simulation is a simulation that can only relate a state without outgoing transitions to states having the same property. In particular, unlike in the setting of the simulation preorder, the inequation

$$\mathbf{0} \leq x$$

does not hold in complete simulation semantics. Our definition of the notion of weak complete simulation is based on considering a process 'complete', or 'mute', when it cannot perform any observable action. For instance, τ is mute, but neither τa nor $\tau + a$ is. The resulting preorder is not preserved by non-deterministic choice. However, unlike in the setting of weak bisimilarity and branching bisimilarity [26, 39], in order to characterize the largest precongruence over BCCSP included in the weak complete simulation preorder, one has only to take special care in handling initial τ-labelled transitions when they lead to a mute process. This semantics satisfies the inequation

$$x \leq \tau x,$$

but not $\tau x \leq x$. For example, $\tau\mathbf{0} \leq \mathbf{0}$ does not hold because $\tau\mathbf{0} + a$ may become mute by performing an internal computational step, whereas $\mathbf{0} + a$ cannot do so. On the other hand, the inequation $\tau a \leq a$ does hold because the initial internal step from τa does not lead to a mute process.

We offer finite (un)conditional ground-complete axiomatizations for the weak complete simulation precongruence. In sharp contrast to this positive result, we prove that, in the presence of at least one observable action, the (in)equational theory of the weak complete simulation precongruence over BCCSP does *not* have a finite (in)equational basis. In fact, the collection of (in)equations in at most one variable that hold in weak complete simulation semantics over BCCSP does not have an (in)equational basis of 'bounded depth', let alone a finite one.

Section 5 paints a similar picture for weak ready simulation semantics. However, the finite axiomatizability of this semantics depends crucially on the presence of an infinite set of observable actions. Moreover, the characterization of the largest precongruence included in the weak ready simulation preorder depends on whether the set of observable actions is finite. A *weak ready simulation* is a weak simulation that can only relate states that afford the same sets of observable actions. It turns out that, if the set of observable actions A is finite, the following inequivalence is sound for each term p:

$$\left(\tau \sum_{a \in A} a\right) + p \leq \left(\sum_{a \in A} a\right) + p.$$

This indicates that one has only to take special care in handling initial τ-labelled transitions when they lead to a process that does not initially afford each action in A.

We offer finite (un)conditional ground-complete axiomatizations for the weak ready simulation precongruence. In sharp contrast to this positive result, we prove that, when the set of observable actions A is finite and non-empty, the (in)equational theory of the weak ready simulation precongruence over BCCSP does *not* have a finite (in)equational basis. In fact, as was the case for weak complete simulation semantics, the collection of (in)equations in at most one variable that hold in weak ready simulation semantics over BCCSP does not have an (in)equational basis of 'bounded depth', let alone a finite one.

The paper is organized as follows. Section 2 presents the syntax and the operational semantics for the language BCCSP, and reviews the necessary background on (in)equational logic as well as classic axiom systems for strong bisimulation equivalence and observational congruence (the largest congruence included in weak bisimulation equivalence). In Section 3, we define the weak simulation preorder and present our results on its (in)equational axiomatization. Sections 4 and 5 are devoted to results on the weak complete and weak ready simulation preorders, respectively. We conclude the paper by discussing further related work and directions for future research in Section 6.

2 Preliminaries

To set the stage for the developments offered in the rest of the paper, we present the syntax and the operational semantics for the language BCCSP, some background on (in)equational logic, and classic axiom systems for strong bisimulation equivalence and observational congruence [39].

Syntax of BCCSP BCCSP(A_τ) is a basic process algebra for expressing finite process behaviour. Its syntax consists of closed (process) terms p, q, r that are constructed from a constant $\mathbf{0}$, a binary operator $\_ + \_$ called *alternative composition*, or *choice*, and unary *prefix* operators $\alpha\_$, where α ranges over some set A_τ of *actions* of the form $A \cup \{\tau\}$, where τ is a distinguished action symbol that is not contained in A. Following Milner [39], we use τ to denote an internal, unobservable action of a reactive system, and we let a, b, c denote typical elements of A and α range over A_τ. The set of closed terms is named $\mathrm{T}(\mathrm{BCCSP}(A_\tau))$, in short $\mathrm{T}(A_\tau)$.

We write $|A|$ for the cardinality of the set of actions A.

Open terms t, u, v can moreover contain occurrences of variables from a countably infinite set V (with typical elements x, y, z). We use $\mathbb{T}(\mathrm{BCCSP}(A_\tau))$, in short $\mathbb{T}(A_\tau)$, to denote the set of open terms. The *depth* of a term t, written $|t|$, is the maximum nesting of prefix operators in t. The depth of a term may be easily defined by induction thus: $|\mathbf{0}| = |x| = 0$, $|\alpha t| = 1 + |t|$ and $|t + u| = \max(|t|, |u|)$.

In what follows, for each non-negative integer n and term t, we use $a^n t$ to stand for t when $n = 0$, and for $a(a^{n-1}t)$ otherwise. As usual, trailing occurrences of **0** are omitted; for example, we shall usually write α in lieu of $\alpha\mathbf{0}$.

A (closed) substitution maps variables in V to (closed) terms. For every term t and substitution σ, the term $\sigma(t)$ is obtained by replacing every occurrence of a variable x in t by $\sigma(x)$. Note that $\sigma(t)$ is closed if σ is a closed substitution. We say that σ is a $\mathbb{T}(A)$-substitution if its range is included in $\mathbb{T}(A)$.

We sometimes write $[t_1/x_1, \ldots, t_n/x_n]$, where $t_1, \ldots, t_n$ is a sequence of terms and $x_1, \ldots, x_n$ is a sequence of distinct variables, for the substitution that maps each x_i to t_i, $1 \leq i \leq n$, and acts like the identity function on all the other variables.

Transitions and their defining rules Intuitively, closed BCCSP(A_τ) terms represent finite process behaviours, where **0** does not exhibit any behaviour, $p + q$ is the nondeterministic choice between the behaviours of p and q, and αp executes action α to transform into p. This intuition is captured, in the style of Plotkin [43], by the simple transition rules below, which give rise to A_τ-labelled transitions between closed terms.

$$\frac{}{\alpha x \xrightarrow{\alpha} x} \qquad \frac{x \xrightarrow{\alpha} x'}{x + y \xrightarrow{\alpha} x'} \qquad \frac{y \xrightarrow{\alpha} y'}{x + y \xrightarrow{\alpha} y'}$$

The operational semantics is extended to open terms by assuming that variables do not exhibit any behaviour.

The so-called *weak transition relations* $\stackrel{\alpha}{\Longrightarrow}$ ($\alpha \in A_\tau$) are defined over $\mathbb{T}(A_\tau)$ in the standard fashion as follows.

- We use $\stackrel{\tau}{\Longrightarrow}$ for the reflexive and transitive closure of $\xrightarrow{\tau}$.
- For each $a \in A$ and for all terms $t, u \in \mathbb{T}(A_\tau)$, we have that $t \stackrel{a}{\Longrightarrow} u$ if, and only if, there are $t_1, t_2 \in \mathbb{T}(A_\tau)$ such that $t \stackrel{\tau}{\Longrightarrow} t_1 \xrightarrow{a} t_2 \stackrel{\tau}{\Longrightarrow} u$.

As usual, see, for instance, [39], we extend the weak transition relations to sequences of actions in A thus:

- $t \stackrel{\varepsilon}{\Longrightarrow} u$, where ε denotes the empty string, if, and only if, $t \stackrel{\tau}{\Longrightarrow} u$;

– $t \stackrel{as}{\Longrightarrow} u$, where $a \in A$ and $s \in A^*$, if, and only if, there is some $t' \in \mathbb{T}(A_\tau)$ such that $t \stackrel{a}{\Longrightarrow} t' \stackrel{s}{\Longrightarrow} u$.

For each term t, we define

$$I^*(t) = \{a \mid a \in A \text{ and } t \stackrel{a}{\Longrightarrow} t' \text{ for some } t'\}.$$

Preorders and their kernels We recall that a *preorder* $\precsim$ is a reflexive and transitive relation. In what follows, any preorder $\precsim$ we consider will first be defined over the set of closed terms $\mathrm{T}(A_\tau)$. For terms $t, u \in \mathbb{T}(A_\tau)$, we define $t \precsim u$ if, and only if, $\sigma(t) \precsim \sigma(u)$ for each closed substitution σ.

The *kernel* $\approx$ of a preorder $\precsim$ is the equivalence relation it induces, and is defined thus:

$$t \approx u \text{ if, and only if, } (t \precsim u \text{ and } u \precsim t).$$

Inequational logic An *inequation* (respectively, an *equation*) over the language BCCSP(A_τ) is a formula of the form $t \leq u$ (respectively, $t = u$), where t and u are terms in $\mathbb{T}(A_\tau)$. An *(in)equational axiom system* is a set of (in)equations over the language BCCSP(A_τ). An equation $t = u$ is derivable from an equational axiom system E, written $E \vdash t = u$, if it can be proven from the axioms in E using the rules of equational logic (viz. reflexivity, symmetry, transitivity, substitution and closure under BCCSP(A_τ) contexts).

$$t = t \quad \frac{t = u}{u = t} \quad \frac{t = u \quad u = v}{t = v} \quad \frac{t = u}{\sigma(t) = \sigma(u)} \quad \frac{t = u}{\alpha t = \alpha u} \quad \frac{t = u \quad t' = u'}{t + t' = u + u'}$$

For the derivation of an inequation $t \leq u$ from an inequational axiom system E, the rule for symmetry—that is, the second rule above—is omitted. We write $E \vdash t \leq u$ if the inequation $t \leq u$ can be derived from E.

It is well known that, without loss of generality, one may assume that substitutions happen first in (in)equational proofs, i.e., that the fourth rule may only be used when its premise is one of the (in)equations in E. Moreover, by postulating that for each equation in E also its symmetric counterpart is present in E, one may assume that applications of symmetry happen first in equational proofs, i.e., that the second rule is never used in equational proofs. (See, e.g., [17, page 497] for a thorough discussion of this notion of 'normalized equational proof'.) In the remainder of this paper, we shall always tacitly assume that equational axiom systems are closed with respect to symmetry. Note that, with this assumption, there is no difference between the rules of inference of equational and inequational logic. In what follows, we shall consider an equation $t = u$ as a shorthand for the pair of inequations $t \leq u$ and $u \leq t$.

The depth of $t \leq u$ and $t = u$ is the maximum of the depths of t and u. The depth of a collection of (in)equations is the supremum of the depths of its elements. So, the depth of a finite axiom system E is zero, if E is empty, and it is the largest depth of its (in)equations otherwise.

An inequation $t \leq u$ is *sound* with respect to a given preorder relation $\precsim$ if $t \precsim u$ holds. An (in)equational axiom system E is sound with respect to $\precsim$ if so is each (in)equation in E.

Classic axiomatizations for notions of bisimilarity The well-known axioms B_1–B_4 for BCCSP(A_τ) given below stem from [30]. They are ω-complete [41], and sound and ground-complete [30, 39], over BCCSP(A_τ) (over any nonempty set of actions) modulo bisimulation equivalence [39, 42], which is the finest semantics in van Glabbeek's spectrum [25].

$$
\begin{array}{ll}
B_1 & x + y = y + x \\
B_2 & (x + y) + z = x + (y + z) \\
B_3 & x + x = x \\
B_4 & x + \mathbf{0} = x
\end{array}
$$

In what follows, for notational convenience, we consider terms up to the least congruence generated by axioms B_1–B_4, that is, up to bisimulation equivalence. We use *summation* $\sum_{i=1}^{n} t_i$ (with $n \geq 0$) to denote $t_1 + \cdots + t_n$, where the empty sum denotes $\mathbf{0}$. Modulo the equations B_1–B_4 each term $t \in \mathbb{T}(A_\tau)$ can be written in the form $\sum_{i=1}^{n} t_i$, where each t_i is either a variable or is of the form $\alpha t'$, for some action α and term t'.

The following lemma is standard and will be implicitly used in the technical developments to follow.

Lemma 1. *Let t, t', u be terms, let s be a sequence of actions in A and let σ be a substitution.*

1. *If $t \stackrel{s}{\Longrightarrow} t'$ then $\sigma(t) \stackrel{s}{\Longrightarrow} \sigma(t')$.*
2. *If $t \stackrel{s}{\Longrightarrow} x + t'$, for some variable x, and $\sigma(x) \stackrel{s'}{\Longrightarrow} u$ for some $s' \in A^*$ and $u \neq \sigma(x)$, then $\sigma(t) \stackrel{ss'}{\longrightarrow} u$.*
3. *If $\sigma(t) \stackrel{s}{\Longrightarrow} u$ then*
   (a) *either $t \stackrel{s}{\Longrightarrow} t'$ for some t' such that $\sigma(t') = u$*
   (b) *or there are sequences s' and s'' of actions in A with $s = s's''$, some variable x and some t' such that $t \stackrel{s'}{\Longrightarrow} x + t'$, $\sigma(x) \stackrel{s''}{\Longrightarrow} u$ and $u \neq \sigma(x)$.*

In a setting with internal transitions, the classic work of Hennessy and Milner on *weak bisimulation equivalence* and on the largest precongruence included in it, *observational congruence*, shows that the axioms B_1–B_4 together with the axioms W_1–W_3 below are sound and complete over BCCSP(A_τ) modulo observational equivalence. (See [30, 39, 40].)

$$\begin{array}{lr} W_1 & \alpha x = \alpha \tau x \\ W_2 & \tau x = \tau x + x \\ W_3 & \alpha(\tau x + y) = \alpha(\tau x + y) + \alpha y \end{array}$$

The above axioms are often referred to as the *τ-laws*. For ease of reference, we write

$$BW = \{B_1, B_2, B_3, B_4, W_1, W_2, W_3\}.$$

As it is well known, when dealing with process algebras with internal, unobservable actions, usually a 'natural' definition of a behavioural semantics does not yield a (pre)congruence. In this case, it is customary to consider the largest (pre)congruence included in the behavioural relation of interest. Throughout the paper, we use quite a number of relations defined for the language BCCSP(A_τ) and, for the sake of clarity, in Table 1 we summarize the main symbol conventions we use to give them names. The subscripting is used to differentiate between semantics. For instance, $\precsim_{RS}$ is the symbol we will use for the weak ready simulation preorder while $\precsim_S$ is the one we will use for the weak simulation preorder.

Natural definition of a weak relation, order and equivalence
$\precsim_{RS}$ $\succsim_{RS}$ $\approx_{RS}$
Abstract largest (pre)congruence contained in a weak relation
$\sqsubseteq_{RS}$ $\sqsupseteq_{RS}$ $\equiv_{RS}$
Operational characterization of the largest (pre)congruence
$\lesssim_{RS}$ $\gtrsim_{RS}$ $\eqsim_{RS}$
Relations defined by axioms
$\leq$ $\geq$ $=$

Table 1. General symbol notation used for relations, using ready simulation as a concrete example

3 Weak Simulation

We begin our study of the equational theory of weak simulation semantics by considering the natural, τ-abstracting version of the classic simulation preorder [36, 42]. We start by defining the notion of weak simulation preorder and the equivalence relation it induces. We then argue that all the known positive and negative results on the existence of (ground-)complete (in)equational axiomatizations for the concrete simulation semantics over the language BCCSP(A_τ) can be lifted to the corresponding weak semantics.

Definition 1. *The* weak simulation preorder*, denoted by $\precsim_S$, is the largest relation over terms in* $\mathrm{T}(A_\tau)$ *satisfying the following condition whenever $p \precsim_S q$ and $\alpha \in A_\tau$:*

- *if $p \xrightarrow{\alpha} p'$ then there exists some q' such that $q \xRightarrow{\alpha} q'$ and $p' \precsim_S q'$.*

We say that $p, q \in \mathrm{T}(A_\tau)$ are weak simulation equivalent*, written $p \approx_S q$, iff p and q are related by the kernel of $\precsim_S$, that is when both $p \precsim_S q$ and $q \precsim_S p$ hold.*

Unlike many other notions of behavioural relations that abstract away from internal steps in the behaviour of processes, see [26, 37, 47] for classic examples, the weak simulation preorder is a precongruence over the language we consider in this study.

Proposition 1. *The preorder $\precsim_S$ is a precongruence over* $\mathrm{T}(A_\tau)$*. Hence* $\approx_S$ *is a congruence over* $\mathrm{T}(A_\tau)$*. Moreover, the axiom*

$$(\tau e) \quad \tau x = x$$

holds over $\mathrm{T}(A_\tau)$ *modulo* $\approx_S$*.*

Proof. The relation

$$\begin{aligned}\mathcal{R} = &\{(\alpha p, \alpha q) \mid p \precsim_S q,\ \alpha \in A_\tau\} \cup \\ &\{(p + r, q + r) \mid p \precsim_S q,\ r \in \mathrm{T}(A_\tau)\} \cup \\ &\{(p, q + r) \mid p \precsim_S q,\ r \in \mathrm{T}(A_\tau)\} \cup \precsim_S\end{aligned}$$

satisfies the conditions in Definition 1. Therefore $\precsim_S$ is a precongruence over $\mathrm{T}(A_\tau)$. It is well known that the kernel of a precongruence is a congruence.

To see that the axiom $\tau x = x$ holds modulo $\approx_S$, it suffices to observe that the relation

$$\{(p, \tau p), (\tau p, p), (p, p) \mid p \in \mathrm{T}(A_\tau)\}$$

satisfies the conditions in Definition 1. □

The soundness of equation τe is the key to all the results on the equational theory of the weak simulation semantics we present in the remainder of this section. In establishing the negative results, we shall make use of the reduction technique from the paper [8].

We start by defining the reduction function $\hat{\cdot} : \mathbb{T}(A_\tau) \to \mathbb{T}(A)$ as the unique homomorphism satisfying

$$\begin{aligned}&\hat{x} = x \text{ for each } x \in V, \text{ and}\\ &\widehat{\tau t} = \hat{t} \text{ for each } t \in \mathbb{T}(A_\tau).\end{aligned}$$

The following properties of $\hat{\cdot}$ will be useful in the technical developments to follow.

Lemma 2.

1. *$\hat{\cdot}$ is the identity function over terms in $\mathbb{T}(A)$.*
2. *$\hat{\cdot}$ is structural, in the sense of [8, Definition 3].*

3. *For each term* $t \in \mathbb{T}(A_\tau)$ *and* $\mathbb{T}(A_\tau)$*-substitution* σ*, it holds that* $\widehat{\sigma(t)} = \widehat{\sigma}(\hat{t})$*, where* $\widehat{\sigma}$ *is the* $\mathbb{T}(A)$*-substitution mapping each variable* x *to the term* $\widehat{\sigma(x)}$*.*

Proof. The first statement is immediate from the definition of $\hat{\cdot}$. The third follows from the second and Lemma 1 in [8]. To establish the second statement, observe that

- $\hat{\cdot}$ is the identity function over variables,
- for each term $t \in \mathbb{T}(A_\tau)$, the variables occurring in $\hat{t}$ are exactly the variables occurring in t and
- for all terms $t, u \in \mathbb{T}(A_\tau)$, actions $\alpha \in A_\tau$ and distinct variables x, y,

$$\widehat{\alpha t} = (\widehat{\alpha x})[\hat{t}/x] \quad \text{and}$$
$$\widehat{t+u} = (\widehat{x+y})[\hat{t}/x, \hat{u}/y].$$

Therefore $\hat{\cdot}$ meets the requirements for a structural mapping laid out in [8, Definition 3]. □

Lemma 3. *The following statements hold.*

1. *For each* $t \in \mathbb{T}(A_\tau)$*, the equation* $t = \hat{t}$ *is provable using axiom* τe*, and therefore* $t \approx_S \hat{t}$*.*
2. *For all* $t, u \in \mathbb{T}(A_\tau)$*, the inequation* $t \leq u$ *holds modulo* $\precsim_S$ *over* $\mathbb{T}(A_\tau)$ *iff* $\hat{t} \leq \hat{u}$ *holds over* $\mathbb{T}(A)$ *modulo the simulation preorder.*

Proof. Statement 1 can be shown easily by structural induction on t. To prove statement 2, first observe that the claim can be shown to hold for inequations relating closed terms following standard lines. Using the validity of the claim over closed terms, statement 1 and Lemma 2(3), it is routine to prove that statement 2 holds for open terms too. □

3.1 Ground-completeness

Besides the equation τe previously stated in Proposition 1, there will be another important equation to consider in order to achieve an axiomatic characterization of the weak simulation preorder, namely

$$(S) \quad x \leq x + y.$$

This equation also plays an essential role in the axiomatization of the simulation preorder in the concrete case [20, 25].

Proposition 2. *The set of equations*

$$E_{S^{\leq}} = \{B_1, B_2, B_3, B_4, S, \tau e\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\precsim_S$.[4]

Proof. We limit ourselves to showing that the axiom system mentioned in the statement of the proposition is ground-complete. To this end, assume that $p, q \in \mathrm{T}(A_\tau)$ and $p \precsim_S q$. By Lemma 3, using axiom τe, we can prove the equations $p = \hat{p}$ and $q = \hat{q}$. Moreover, the inequation $\hat{p} \leq \hat{q}$ holds modulo the simulation preorder. It is well known that the axiom system $\{B_1, B_2, B_3, B_4, S\}$ is ground-complete for the simulation preorder over the language $\mathrm{T}(A)$. Therefore, the inequation $\hat{p} \leq \hat{q}$ is provable from it. By combining a proof of $\hat{p} \leq \hat{q}$, with proofs for the equations $p = \hat{p}$ and $q = \hat{q}$, one obtains a proof of $p \leq q$. □

The completeness result in Proposition 2 was announced in [45] by van Glabbeek. Since no proof is given in that paper, and for the sake of methodology, we have included here our proof.

Note that the equations W_1–W_3, even if sound for $\precsim_S$, are not needed in order to obtain a ground-complete axiomatization of $\precsim_S$ over BCCSP(A_τ). Those equations can easily be derived from the axiom system in Proposition 2.

To obtain an axiomatization for the weak simulation equivalence, we need the equation

$$(SE) \quad a(x + y) = a(x + y) + ay \qquad (a \in A).$$

This equation is well known from the setting of standard simulation equivalence, where it is known to be the key to a ground-complete axiomatization [25].

Proposition 3. *The set of equations*

$$E_{S^{=}} = \{B_1, B_2, B_3, B_4, SE, \tau e\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\approx_S$.

[4] This completeness result was announced without proof in [45] by van Glabbeek.

Proof. The algorithm *weak ready to preorder* can be applied. See [18, page 5]. A direct proof using Lemma 3 is also immediate. □

3.2 ω-completeness

Propositions 2 and 3 offer ground-complete axiomatizations for the weak simulation preorder and its kernel over BCCSP(A_τ). The inequational axiomatization of the weak simulation preorder is finite, and so is the one for its kernel if the set of actions A is finite. In the presence of an infinite collection of actions, the axiom system in Proposition 3 is finite if we consider a to be an action variable. It is natural to wonder whether the weak simulation semantics afford finite (in)equational axiomatizations that are complete over $\mathbb{T}(A_\tau)$. The following results answer this question.

Proposition 4. *If the set of actions is infinite, then the axiom system*

$$E_{S\leq} = \{B_1, \ldots, B_4, S, \tau e\}$$

is ω-complete over BCCSP(A_τ) *modulo* $\precsim_S$*.*

Proof. To prove the result we use Groote's inverted substitution technique [27]. Actually, we use a variation on that technique appearing in Chen's PhD. thesis [15], which is valid also for preorders and not only for equivalences.

In the rest of the proof, for the sake of readability, we abbreviate $E_{S\leq}$ by E. Given an inequation $t_0 \leq u_0 \in E$, let σ be the closed substitution $\sigma : V \longrightarrow \mathrm{T}(A_\tau)$ defined as follows: $\sigma(x) = a_x\mathbf{0}$, where a_x is a distinguished action for each variable $x \in V$, and every a_x does not appear in t_0, u_0. That's why we need $|A| = \infty$.

Let $\rho : \mathrm{T}(A_\tau) \longrightarrow \mathbb{T}(A_\tau)$ be structurally defined as follows.

$$\begin{array}{l}\rho(\mathbf{0}) = \mathbf{0}\\ \rho(\alpha u) = \alpha\rho(u), \text{ when } \alpha \neq a_x \text{ for each } x \in V\\ \rho(a_x u) = x\\ \rho(u+v) = \rho(u) + \rho(v)\end{array}$$

Next we check the three properties needed to apply Chen's result.

(1) $E \vdash t_0 \leq \rho(\sigma(t_0))$ and $E \vdash \rho(\sigma(u_0)) \leq u_0$.
It can be easily proved by structural induction that if an open term u does not contain action prefix operators of the form a_x, then $u = \rho(\sigma(u))$. From this (1) follows trivially.

(2) $E \vdash \rho(\sigma'(t)) \leq \rho(\sigma'(u))$, for each $t \leq u \in E$ and closed substitution σ'.
Here, as an example, we show the proof for S.

$$\begin{aligned}\rho(\sigma'(x)) &\leq \rho(\sigma'(x)) + \rho(\sigma'(y)) \\ &= \rho(\sigma'(x) + \sigma'(y)) \\ &= \rho(\sigma'(x+y))\end{aligned}$$

The first inequality is an instance of inequation S and the subsequent equalities follow from the fact that ρ and σ' are homomorphisms with respect to the choice operator $+$.

(3) $E \cup \{u_i \leq u_i', \rho(u_i) \leq \rho(u_i') \mid i = 1,2\} \vdash \rho(u_1 + u_2) \leq \rho(v_1 + v_2)$ and $E \cup \{u \leq v, \rho(u) \leq \rho(v)\} \vdash \rho(\alpha u) \leq \rho(\alpha v)$.
These are quite straightforward, so let us just prove the second claim. If $\alpha = a_x$, for some variable x, then $x \leq x$ can trivially be proved by reflexivity. Otherwise, $\rho(\alpha u) \leq \rho(\alpha v)$ becomes $\alpha\rho(u) \leq \alpha\rho(v)$, which can be proved from $E \cup \{u \leq v, \rho(u) \leq \rho(v)\}$. □

Corollary 1. *If the set of actions is infinite, then the axiom system*

$$E_{S\leq} = \{B_1, \ldots, B_4, S, \tau e\}$$

is complete over BCCSP(A_τ) *modulo* $\precsim_S$.

Proof. The axiom system $E_{S\leq}$ is both ground-complete (Proposition 2) and ω-complete (Proposition 4). It is well known that an axiom system with these properties is complete—see, for example, [6, Remark 2]. □

So the weak simulation preorder can be finitely axiomatized over $\mathbb{T}(A_\tau)$ when A is infinite. This state of affairs changes dramatically when A is a finite collection of actions of cardinality at least two.

Proposition 5. *If $1 < |A| < \infty$, then the weak simulation equivalence does not afford a finite equational axiomatization over $\mathbb{T}(A_\tau)$. In particular, no finite axiom system over $\mathbb{T}(A_\tau)$ that is sound modulo weak* *simulation equivalence can prove all the (valid) equations in the family on page 511 of [17].*

Proof. By Theorem 28 in [17], there is no finite axiom system over $\mathbb{T}(A)$ that is sound modulo simulation equivalence and can prove all the equations in the family on page 511 of [17]. We will now use the results that we have obtained so far, in combination with the reduction technique presented in [8], to lift this negative result to the setting of weak simulation equivalence over $\mathbb{T}(A_\tau)$.

By Lemma 2(2) and [8, Theorem 2], we have that the mapping $\hat{\cdot}$ preserves provability of equations. This means that, for any axiom system E over $\mathbb{T}(A_\tau)$, if E proves an equation $t = u$ then $\widehat{E}$ proves $\hat{t} = \hat{u}$, where

$$\widehat{E} = \{\widehat{t'} = \widehat{u'} \mid (t' = u') \in E\}.$$

By Lemma 2(1), $\hat{\cdot}$ reflects the family of equations in [17, page 511], since those equations relate terms that do not contain occurrences of τ. Lemma 3(2) tells us that $\hat{\cdot}$ preserves the soundness of inequations. We may therefore apply [8, Theorem 1] to infer that no finite axiom system E over $\mathbb{T}(A_\tau)$ that is sound modulo weak simulation equivalence can prove all of the equations in the family on page 511 of [17]. Therefore weak simulation equivalence affords no finite equational axiomatization over $\mathbb{T}(A_\tau)$. □

Corollary 2. *If $1 < |A| < \infty$, then the weak simulation preorder does not afford a finite inequational axiomatization over $\mathbb{T}(A_\tau)$.*

Proof. If the weak simulation preorder afforded a finite inequational axiomatization over $\mathbb{T}(A_\tau)$ then one could obtain a finite equational axiomatization for weak simulation equivalence by applying the algorithm from [18]. The existence of such an axiomatization would contradict Proposition 5. Alternatively, one could replay the proof of Theorem 28 in [17], which also applies essentially unchanged to the (weak) simulation preorder. □

Remark 1. If A is a singleton then the simulation preorder coincides with trace inclusion. In that case, the simulation preorder is finitely based over $\mathbb{T}(A)$, as is simulation equivalence —see, e.g., [6]. Those axiomatizations can be lifted to the setting of weak simulation semantics simply by adding the equation τe to any complete axiomatization of the simulation preorder or equivalence.

Weak Simulation	Ground-complete		Complete	
Finite Equations	Order	Equiv.	Order	Equiv.
$\lvert A\rvert = \infty$	$E_{S^{\leq}}$	$E_{S^{=}}$	$E_{S^{\leq}}$	$E_{S^{=}}$
$1 < \lvert A\rvert < \infty$	$E_{S^{\leq}}$	$E_{S^{=}}$	Do not exist	
$\lvert A\rvert = 1$	$E_{S_1^{\leq}}$	$E_{S_1^{=}}$	$E_{S_1^{\leq}}$	$E_{S_1^{=}}$

Table 2. Axiomatizations for the weak simulation semantics

$E_{S^{\leq}} = \{B_1\text{–}B_4, \tau e, S\}$ $E_{S^{=}} = \{B_1\text{–}B_4, \tau e, SE\}$ $E_{S_1^{\leq}} = \{B_1\text{–}B_4, \tau e, S, TE, Sg_{\leq}\}$ $E_{S_1^{=}} = \{B_1\text{–}B_4, \tau e, TE, Sg\}$	$(\tau e)\quad \tau x = x$ $(S)\quad x \leq x + y$ $(Sg_{\leq})\ x \leq ax$ $(SE)\quad a(x+y) = a(x+y) + ay$ $(TE)\quad a(x+y) = ax + ay$ $(Sg)\quad ax = ax + x$

Table 3. Axioms for the weak simulation semantics

Tables 2–3 summarize the positive and negative results on the existence of finite axiomatizations for weak simulation semantics. On Table 2, and in subsequent ones, 'Do not exist' indicates that there is no *finite* (in)equational axiomatization for the corresponding semantic relation.

3.3 Observational equivalence and simulation

For the sake of completeness, in this section we mention a rather natural behavioural relation that has an easy equational characterization. Let us consider the relation $\lesssim'_S$ defined as follows:

Definition 2. *$\lesssim'_S$ is the largest relation over closed terms in $\mathrm{T}(A_\tau)$ satisfying the following condition whenever $p \lesssim'_S q$ and $\alpha \in A_\tau$:*

- *if $p \xrightarrow{\alpha} p'$ there exists some q' such that $q \stackrel{\tau}{\Longrightarrow}\xrightarrow{\alpha}\stackrel{\tau}{\Longrightarrow} q'$ and $p' \precsim_S q'$, that is, p' is weak simulated by q'.*

So, unlike the weak simulation preorder, the relation $\lesssim'_S$ requires that initial internal steps of one process be matched by at least one internal step from the other. This is similar to the extra requirement imposed by observational congruence with respect to weak bisimilarity [39].

Proposition 6. *The relation $\lesssim'_S$ is a precongruence over $\mathrm{T}(A_\tau)$, which is finer than the weak simulation preorder. Besides, for every term t, we have that $t \lesssim'_S \tau t$, but, in general, $\tau t \not\lesssim'_S t$.*

Proof. It is clear that $p \lesssim'_S q$ implies $p \precsim_S q$ since the condition imposed by the definition of the weak simulation preorder (Definition 1) is weaker than the one in Definition 2 above. That $\lesssim'_S$ is a precongruence can be proved exactly as it was done for $\precsim_S$ in Proposition 1.

Moreover, for any term that cannot initially perform a τ action, it holds that $\tau t \precsim_S t$ but $\tau t \not\lesssim'_S t$. In particular, $\tau \mathbf{0} \not\lesssim'_S \mathbf{0}$. □

Now we present a technical lemma that will be useful in the proof of Proposition 7. This lemma establishes a simple relationship between the weak simulation preorder, $\precsim_S$, and $\lesssim'_S$.

Lemma 4. *For all $p, q \in \mathrm{T}(A_\tau)$, we have that $p \precsim_S q$ implies $p \lesssim'_S \tau q$.*

Proposition 7. *The set of equations*

$$E = BW \cup \{S\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim'_S$.

Proof. Ground-completeness is proved by induction on the depth of terms and using the fact that $p \precsim_S q$ implies $p \lesssim'_S \tau q$, which was proved in Lemma 4.

Let $p = \sum_{i=1}^{n} \alpha_i p_i$, where $n \geq 0$, and suppose that $p \lesssim'_S q$. Then, by definition of $\lesssim'_S$, we know that, for any transition $p \xrightarrow{\alpha_i} p_i$, there is some q' such that $q \stackrel{\tau}{\Longrightarrow}\xrightarrow{\alpha_i}\stackrel{\tau}{\Longrightarrow} q'$ and $p_i \precsim_S q'$. By Lemma 4, we have that $p_i \lesssim'_S \tau q'$ and, applying the induction hypothesis, also that $E \vdash p_i \leq \tau q'$. Using closure under prefixes, equation W_1 and inequation S, we infer that $E \vdash \alpha p_i \leq \alpha q' + q$. The weak derivatives of q can be absorbed into q by using the τ-laws, see Milner [39], and therefore $E \vdash \alpha p_i \leq q$. As this is true for each i, using B_3 we may conclude that $E \vdash p \leq q$, which was to be shown. □

So $\lesssim'_S$ is axiomatized precisely with the equations for observational equivalence and the simulation inequation S.

Remark 2. In fact, in the presence of at least two actions in A, the axiom system $BW \cup \{S\}$ is sound and complete for BCCSP(A_τ) modulo $\lesssim'_S$.

4 Weak Complete Simulation

We now study the notion of complete simulation preorder in a setting with τ actions. Recall that, in the setting without τ, a complete simulation is a simulation relation that is only allowed to relate a state with no outgoing transitions to states with the same property.

Definition 3. *For* $\mathrm{T}(A_\tau)$ *terms, we say that process p* must terminate *(or* is mute*), written $p\Downarrow$, iff there does not exist any $a \in A$ such that $p \stackrel{a}{\Longrightarrow}$. That is, if the set of visible initial actions of process p is empty, written $I^*(p) = \emptyset$.*

Note that p is not mute, written $p \not\Downarrow$, if, and only if, there exist $n \geq 0$ and $a \in A$ such that $p(\xrightarrow{\tau})^n \xrightarrow{a}$, where $(\xrightarrow{\tau})^n$ denotes the n-fold composition of the relation $\xrightarrow{\tau}$.

Definition 4. *The* weak complete simulation preorder, *denoted by $\precsim_{CS}$, is the largest relation over terms in $\mathrm{T}(A_\tau)$ satisfying the following conditions whenever $p \precsim_{CS} q$ and $\alpha \in A_\tau$:*

- *if $p \xrightarrow{\alpha} p'$ then there exists some term q' such that $q \xRightarrow{\alpha} q'$ and $p' \precsim_{CS} q'$, and*
- *if $p\Downarrow$ then $q\Downarrow$.*

We say that $p, q \in \mathrm{T}(A_\tau)$ are weak complete simulation equivalent, *written $p \approx_{CS} q$, iff p and q are related by the kernel of $\precsim_{CS}$, that is when both $p \precsim_{CS} q$ and $q \precsim_{CS} p$ hold.*

The following result is standard.

Lemma 5. *If $p \precsim_{CS} q$, then*

1. *$I^*(p) \subseteq I^*(q)$, and*
2. *$p\Downarrow$ if, and only if, $q\Downarrow$.*

Example 1. $\precsim_{CS}$ is not a precongruence with respect to the choice operator of BCCSP(A_τ). It is immediate to show that $\tau\mathbf{0} \precsim_{CS} \mathbf{0}$, however $\tau\mathbf{0} + a \not\precsim_{CS} \mathbf{0} + a$. If $\tau\mathbf{0} + a$ performs the τ-transition, the process evolves to $\mathbf{0}$, which satisfies $\mathbf{0}\Downarrow$; however $\mathbf{0} + a$ can only transform into itself by a $\xRightarrow{\tau}$ transition and it does not satisfy the mute predicate, $(\mathbf{0} + a)\not\Downarrow$.

Definition 5. *We denote by $\sqsubseteq_{CS}$ the largest precongruence included in $\precsim_{CS}$. Formally, $p \sqsubseteq_{CS} q$ iff*

- *$p \precsim_{CS} q$,*
- *$p \precsim_{CS} q \Rightarrow \forall \alpha \in A_\tau \quad \alpha p \precsim_{CS} \alpha q$, and*
- *$p \precsim_{CS} q \Rightarrow \forall r \in \mathrm{T}(A_\tau) \quad p + r \precsim_{CS} q + r$.*

The definition of the largest precongruence included in $\precsim_{CS}$ is purely algebraic and difficult to use to study that relation. Our aim in what follows will therefore be to obtain a behavioural characterization of $\sqsubseteq_{CS}$. In what follows, as usual, we use $(\xrightarrow{\tau})^+$ to denote the transitive closure of the relation $\xrightarrow{\tau}$.

Definition 6. *The preorder relation* $\lesssim_{CS}$ *between processes in* $\mathrm{T}(A_\tau)$ *is defined as follows:* $p \lesssim_{CS} q$ *iff*

- $p \precsim_{CS} q$*, and*
- *whenever* $p \xrightarrow{\tau} p'$ *for some* p' *such that* $p'\Downarrow$*, there exists some* q' *such that* $q(\xrightarrow{\tau})^+q'$ *and* $q'\Downarrow$*.*

We denote the kernel of $\lesssim_{CS}$ *by* $\eqsim_{CS}$*.*

Example 2. It is immediate to see that $\tau\mathbf{0} \not\lesssim_{CS} \mathbf{0}$. On the other hand, $\tau a \lesssim_{CS} a$ does hold because the second requirement in Definition 6 is vacuous. In general, $\tau p \lesssim_{CS} p + q$ holds for all p and q provided that p is *not* $\mathbf{0}$. (Recall that we consider terms up to $B1$–$B4$.)

Lemma 6. *Assume that* $p \precsim_{CS} q$ *and* p *is not mute. Then* $p \precsim_{CS} q + r$ *for each closed term* r*.*

Proof. Define the relation $\mathcal{R}$ as follows: $(p, q + r) \in \mathcal{R}$ iff

- p is not mute and
- $p \precsim_{CS} q$.

It is not hard to see that the relation $\mathcal{R} \cup \precsim_{CS}$ is a weak complete simulation. □

Proposition 8 (Behavioural characterization of $\sqsubseteq_{CS}$**).** $p \lesssim_{CS} q$ *if, and only if,* $p \sqsubseteq_{CS} q$*, for all* $p, q \in \mathrm{T}(A_\tau)$*.*

Proof. We prove the two implications separately.

For the implication from right to left, assume that $p \sqsubseteq_{CS} q$. We shall prove that $p \lesssim_{CS} q$ also holds. To this end, note first that $p \precsim_{CS} q$ because $\sqsubseteq_{CS}$ in included in $\precsim_{CS}$. Moreover, $p + a \precsim_{CS} q + a$, and we shall now prove that this yields that, whenever $p \xrightarrow{\tau} p'$ and $p'\Downarrow$, there exists some q' such that $q(\xrightarrow{\tau})^+q'$ and $q'\Downarrow$. This will complete the proof that $p \lesssim_{CS} q$. So, assume that $p \xrightarrow{\tau} p'$ and $p'\Downarrow$. Then $p + a \xrightarrow{\tau} p'$. Since $p + a \precsim_{CS} q + a$, there is some q' such that $q + a \overset{\tau}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$. Since $p'\Downarrow$ and $q + a$ is not mute, it must be the case that $q + a(\xrightarrow{\tau})^+q'$. Hence $q(\xrightarrow{\tau})^+q'$. As $p' \precsim_{CS} q'$ and $p'\Downarrow$, we have that q' is mute, and we are done.

We now prove the implication from left to right. By the definition of $\lesssim_{CS}$, we have that $\lesssim_{CS}$ is included in $\precsim_{CS}$. It therefore suffices to show that $\lesssim_{CS}$ is a congruence. To this end, assume that $p \lesssim_{CS} q$. It is easy to see that $\alpha p \lesssim_{CS} \alpha q$ for each action α. We claim that $p + r \lesssim_{CS} q + r$ also holds for each closed term r. To establish this claim we consider each of the conditions in Definition 6 in turn.

- We first prove that $p + r \precsim_{CS} q + r$. Assume, first of all, that $p + r \xrightarrow{\alpha} p'$ for some p'. The only interesting case to consider is when this transition stems from p, that is when $p \xrightarrow{\alpha} p'$. We will prove that $q + r \xRightarrow{\alpha} q'$ and $p' \precsim_{CS} q'$ for some q'. Since $p \precsim_{CS} q$ because $p \lesssim_{CS} q$, this is clear in all cases apart from when
  - $\alpha = \tau$ and
  - q is the only τ-derivative of itself for which $p' \precsim_{CS} q$.

  This means that p' is *not* mute. Indeed, if p' were mute then, as $p \lesssim_{CS} q$, there would be some q' such that $q + r(\xrightarrow{\tau})^+ q'$ and $q' \Downarrow$. For such a q', we would have that $p' \precsim_{CS} q'$. So, $p \xrightarrow{\tau} p'$ and p' is not mute as claimed. It then follows that $p' \precsim_{CS} q + r$ by Lemma 6, and we are done, since $q + r \xRightarrow{\tau} q + r$.
- We now show that if $p + r \xrightarrow{\tau} p'$ and $p' \Downarrow$, then there exists some q' such that $q + r(\xrightarrow{\tau})^+ q'$ and $q' \Downarrow$. To this end, assume that $p + r \xrightarrow{\tau} p'$ and $p' \Downarrow$. Then either $p \xrightarrow{\tau} p'$ or $r \xrightarrow{\tau} p'$. The latter case is immediate. In the former case, since $p \lesssim_{CS} q$, we have that $q(\xrightarrow{\tau})^+ q'$ and $q' \Downarrow$, for some q'. Therefore, it holds that $q + r(\xrightarrow{\tau})^+ q'$ and $q' \Downarrow$, which was to be shown.

This completes the proof. □

We shall now provide an alternative characterization of the preorder $\lesssim_{CS}$, and therefore of $\sqsubseteq_{CS}$, over $\mathrm{T}(A_\tau)$. This definition of $\lesssim_{CS}$ bears a strong resemblance to the characterization of the largest precongruence included in the weak ready simulation preorder, when A is finite and non-empty, that we shall present in Section 5.3.

Definition 7. *The preorder relation $\lesssim_{CS}^{N}$ over $\mathrm{T}(A_\tau)$ is defined as follows: $p \lesssim_{CS}^{N} q$ iff*

- *whenever* $p \xrightarrow{a} p'$*, there exists some* q' *such that* $q \stackrel{a}{\Longrightarrow} q'$ *and* $p' \precsim_{CS} q'$*;*
- *whenever* $p \xrightarrow{\tau} p'$*,*
  - *either there exists some* q' *such that* $q(\xrightarrow{\tau})^+ q'$ *and* $p' \precsim_{CS} q'$
  - *or* $p' \not\Downarrow$ *and* $p' \precsim_{CS} q$*;*
- *if* $p\Downarrow$ *then* $q\Downarrow$*.*

Proposition 9. *$p \lesssim_{CS}^{N} q$ iff $p \lesssim_{CS} q$, for all $p, q \in \mathrm{T}(A_\tau)$.*

Proof. We prove the two implications separately. First of all, note that the implication from left to right follows immediately from the definition of the relations $\lesssim_{CS}^{N}$ and $\lesssim_{CS}$.

Assume now that $p \lesssim_{CS} q$ and $p \xrightarrow{\tau} p'$ for some p'. By the definition of $\lesssim_{CS}$, we have that $p \precsim_{CS} q$. Therefore there exists some q' such that $q \stackrel{\tau}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$. Assume that $q = q'$ and q is the only state that it can reach via $\stackrel{\tau}{\Longrightarrow}$ that weakly complete simulates p'. We claim that $p' \not\Downarrow$. Indeed, if $p'\Downarrow$ then, by the definition of $\lesssim_{CS}$, there would be some q'' such that $q(\xrightarrow{\tau})^+ q''$ and $q''\Downarrow$. For that q'', it would hold that $p' \precsim_{CS} q''$, and this would contradict our assumption that q is the only state that it can reach via $\stackrel{\tau}{\Longrightarrow}$ that weakly complete simulates p'. The other two clauses in the definition of $\lesssim_{CS}^{N}$ follow immediately from the definition of $\lesssim_{CS}$. □

4.1 Ground-completeness

In order to find a set of equations that gives a ground-complete axiomatization for the largest precongruence included in the weak complete simulation preorder, it is natural to consider the following (conditional) equations.

$$
\begin{array}{ll}
(CS_\tau) & (x\Downarrow \Leftrightarrow y\Downarrow) \Rightarrow x \leq x + y \\
(CS_{\tau e}) & \tau(ax + y) = ax + y
\end{array}
$$

The first equation, CS_τ, is similar to the key axiom in the axiomatization for the complete simulation preorder in the concrete case, see e.g. [20]. However, in our setting, the mute predicate takes into account the silent steps of processes. This conditional equation restricts the applicability of inequation S, which is only sound in (weak) complete simulation semantics when the terms substituted for the variables x and y have the same 'termination status'.

The second equation, $CS_{\tau e}$, is a restricted version of equation τe, which we used for the weak simulation preorder and is unsound in weak complete simulation semantics. Intuitively, equation $CS_{\tau e}$ expresses the fact that a process of the form τp, for some term p that is not mute, is weak complete simulation equivalent to p. In fact, equation $CS_{\tau e}$ could 'equivalently' be formulated as a conditional equation thus:

$$x \not\Downarrow \Rightarrow \tau x = x.$$

Lemma 7. *For every term p such that $p \not\Downarrow$, we can prove using $CS_{\tau e}$ that $\tau p = p$.*

Proof. Assume that there exist some $n \geq 0$ and $a \in A$ such that $p(\xrightarrow{\tau})^n \xrightarrow{a}$. We prove the lemma by induction on n.

- Base case, $n = 0$. Then $p = ap_1 + p_2$, for some p_1 and p_2, and we may apply directly $CS_{\tau e}$.
- Induction step, $n > 0$. Then $p = \tau p_1 + p_2$, for some p_1 and p_2, with $p_1(\xrightarrow{\tau})^{n-1} \xrightarrow{a}$. By the induction hypothesis, we derive $p_1 = \tau p_1$ and $p = p_1 + p_2$. We again apply the induction hypothesis to derive $\tau(p_1 + p_2) = (p_1 + p_2)$, that is, $\tau p = p$. □

Proposition 10. *The set of equations*

$$E^c_{CS\leq} = BW \cup \{CS_{\tau e}, CS_\tau\},$$

where CS_τ is conditional, is sound and ground-complete for BCCSP(A_τ) *modulo $\lesssim_{CS}$.*

Proof. The soundness of the axioms is obvious. To prove ground completeness we establish that $p \lesssim_{CS} q$ implies $E^c_{CS\leq} \vdash p \leq q$, by structural induction on p. In the rest of the proof, for the sake of readability we abbreviate $E^c_{CS\leq}$ by E.

$p = \mathbf{0}$. Then $q\Downarrow$ and $E \vdash \mathbf{0} \leq q$ by application of CS_τ.

$p = \alpha p'$**.** Considering $p \xrightarrow{\alpha} p'$ we have that

1. either $q \stackrel{\tau}{\Longrightarrow}\stackrel{\alpha}{\longrightarrow}\stackrel{\tau}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$, for some q',
2. or $\alpha = \tau$, $p' \not\Downarrow$ and $p' \precsim_{CS} q$.

In the first case, using W_1–W_3 we can infer $q = q + \alpha q'$, and from $p' \precsim_{CS} q'$ we obtain $p' \lesssim_{CS} \tau q'$. By the induction hypothesis, $E \vdash p' \leq \tau q'$. Therefore, we have that $E \vdash \alpha p' \leq \alpha \tau q'$ and thus that $E \vdash \alpha p' \leq \alpha q'$. So

$$E \vdash p + q = \alpha p' + q \leq \alpha q' + q = q.$$

Finally, given that $p \Downarrow \Leftrightarrow q \Downarrow$, using CS_τ and transitivity, we obtain $E \vdash p \leq q$.

In the second case, from $p' \not\Downarrow$ and $p' \precsim_{CS} q$ we conclude that $q \not\Downarrow$. Using Lemma 7 we have $E \vdash \tau q = q$ and $E \vdash p = \tau p' = p'$. Since $p' \precsim_{CS} q$, we have that $p' \lesssim_{CS} \tau q$. By the induction hypothesis, $E \vdash p' \leq \tau q$, and we are done.

$p = p_1 + p_2$**.** In this case we have that, for $i = 1, 2$, either $p_i = \mathbf{0}$ or $p_i \lesssim_{CS} q$. The result follows immediately by applying the induction hypothesis. □

Axiom CS_τ highlights the similarities with the concrete version of complete simulation and with the theory of constrained simulations [20]. However, it is natural to wonder whether it is possible to find a finite, non-conditional and ground-complete axiomatization for $\lesssim_{CS}$ over BCCSP(A_τ). Indeed, this is possible; it is enough to substitute the conditional equation CS_τ with the following inequations.

$$(CS) \quad ax \leq ax + y$$
$$(\tau N) \quad \mathbf{0} \leq \tau \mathbf{0}$$

Proposition 11. *The set of unconditional inequations*

$$E_{CS^{\leq}} = BW \cup \{CS_{\tau e}, CS, \tau N\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim_{CS}$.

Proof. Observe, first of all, that both $ax \leq ax + y$ and $\mathbf{0} \leq \tau\mathbf{0}$ can be derived using CS_τ. Therefore, we only need to prove that any use of the conditional axiom CS_τ in a proof of an inequation $p \leq q$

can equivalently be replaced by the application of those two inequational ones and the rest of axioms in $E_{CS\leq}$.

For the case $p\not\Downarrow$ and $q\not\Downarrow$, from $p\not\Downarrow$ we infer, by possibly using W_2, that there exist some $a \in A$ and p_1 such that $p = ap_1 + p$. Next, using CS, we deduce $ap_1 \leq ap_1 + q$, and finally $p \leq p + q$.

For the case $p\Downarrow$ and $q\Downarrow$, we reason as follows. Since $p\Downarrow$, we have that either $p = \mathbf{0}$ or the equation $p = \tau\mathbf{0}$ can be proved using W_1 and W_2. Similarly, either $q = \mathbf{0}$ or the equation $q = \tau\mathbf{0}$ can be proved using W_1 and W_2. If $p = \mathbf{0}$ the inequation $p \leq q$ can be proven by possibly applying τN. If $p = \tau\mathbf{0}$ then, by the soundness of the axiom system $E^c_{CS\leq}$, we have that $q = \tau\mathbf{0}$, and we are done. □

Remark 3. It is clear that we could substitute equation τN by

$$(\tau g) \quad x \leq \tau x$$

in the axiomatization above, since the inequation τg is sound for BCCSP(A_τ) modulo $\lesssim_{CS}$ and is more general than τN.

Let us now move on to the ground-complete axiomatization of the largest congruence included in complete simulation equivalence. In order to axiomatize that congruence, it is natural to consider the following equation.

$$(CSE_\tau) \quad (x\Downarrow \Leftrightarrow y\Downarrow) \;\Rightarrow a(x+y) = a(x+y) + ax$$

This equation is essentially the same one that was used in earlier conditional axiomatizations for complete simulation equivalence in the concrete case [20]. However, we remark that the mute predicate deals with silent transitions, although we only use visible actions when describing the equation CSE_τ. Note that the conditional equation

$$(x\Downarrow \Leftrightarrow y\Downarrow) \;\Rightarrow \tau(x+y) = \tau(x+y) + \tau x$$

is sound with respect to $\eqsim_{CS}$. As the following lemma states, however, each of its closed instantiations are derivable using the axiom system $BW \cup \{CS_{\tau e}\}$. This observation will be useful in the proof of Proposition 12 to follow.

Lemma 8. *Suppose that $p\Downarrow$ if, and only if, $q\Downarrow$. Then*

$$BW \cup \{CS_{\tau e}\} \vdash \tau(p+q) = \tau(p+q) + \tau p.$$

Proof. For the case $p\not\Downarrow$ and $q\not\Downarrow$, from $p\not\Downarrow$ we infer, by Lemma 7, that

$$\tau(p+q) = \tau(p+q) + p + q = \tau(p+q) + \tau p.$$

For the case $p\Downarrow$ and $q\Downarrow$, we reason as follows. Since $p\Downarrow$, we have that either $p = \mathbf{0}$ or the equation $p = \tau\mathbf{0}$ can be proved using W_1 and W_2. Similarly, either $q = \mathbf{0}$ or the equation $q = \tau\mathbf{0}$ can be proved using W_1 and W_2. In all cases, $\tau(p+q) = \tau(p+q) + \tau p$ follows, by possibly using W_1. □

Proposition 12. *The set of conditional equations*

$$E^c_{CS^=} = BW \cup \{CS_{\tau e}, CSE_\tau\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\eqsim_{CS}$.

Proof. We prove, by induction on the depth of p, that

$$p \lesssim_{CS} q \text{ implies } E^c_{CS^=} \vdash q = q + p,$$

from which the claim follows immediately. Let $p = \sum_{i=1}^{n} \alpha_i p_i$, where $n \geq 0$, and let $i \in \{1, \ldots, n\}$. Then $p \xrightarrow{\alpha_i} p_i$. By definition of $\lesssim_{CS}$, we know that there is some q' such that $q \stackrel{\alpha_i}{\Longrightarrow} q'$ and $p_i \precsim_{CS} q'$. There are two possible cases:

1. $q \stackrel{\tau}{\Longrightarrow}\xrightarrow{\alpha_i}\stackrel{\tau}{\Longrightarrow} q'$ or
2. $\alpha_i = \tau$, $p_i\not\Downarrow$ and $p_i \precsim_{CS} q$.

We proceed with the proof by examining these two cases in turn.

1. Assume that $q \stackrel{\tau}{\Longrightarrow}\xrightarrow{\alpha_i}\stackrel{\tau}{\Longrightarrow} q'$. By possibly applying axiom W_3, we derive $q = q + \alpha_i q'$. Since $p_i \precsim_{CS} q'$, we have that $p_i \lesssim_{CS} \tau q'$. Therefore, by the induction hypothesis,

   $$E^c_{CS^=} \vdash \tau q' = \tau q' + p_i.$$

   From the above equation and axiom W_1 we may now derive

   $$\alpha_i q' = \alpha_i \tau q' = \alpha_i(\tau q' + p_i). \qquad (1)$$

Using $p_i \lesssim_{CS} \tau q'$, we infer $p_i\Downarrow$ iff $\tau q'\Downarrow$. Therefore, if $\alpha_i \in A$, we may apply the conditional equation CSE_τ to infer

$$E^c_{CS^=} \vdash \alpha_i(\tau q' + p_i) = \alpha_i(\tau q' + p_i) + \alpha_i p_i.$$

By transitivity, we may now conclude that, when $\alpha_i \in A$,

$$\alpha_i q' = \alpha_i q' + \alpha_i p_i.$$

Therefore,

$$E^c_{CS^=} \vdash q = q + \alpha_i q' = q + \alpha_i p_i.$$

If $\alpha_i = \tau$, then the above equation follows from (1) by Lemma 8.

2. Assume that $\alpha_i = \tau$, $p_i\not\Downarrow$ and $p_i \precsim_{CS} q$. Since $p_i \precsim_{CS} q$, we have that $q\not\Downarrow$ and $p_i \lesssim_{CS} \tau q$. Therefore, by the induction hypothesis,

$$E^c_{CS^=} \vdash \tau q = \tau q + p_i.$$

As $p_i\not\Downarrow$ and $q\not\Downarrow$, by Lemma 7, we have that $p_i = \tau p_i$ and $q = \tau q$. Therefore,

$$E^c_{CS^=} \vdash q = q + \tau p_i.$$

Concluding, for each $1 \leq i \leq n$,

$$E^c_{CS^=} \vdash q = q + \alpha_i p_i.$$

Therefore $E^c_{CS^=} \vdash q = q + p$, as required. □

To turn the previous axiomatization into one without conditional equations we consider the equation

$$(CSE) \quad a(bx + y + z) = a(bx + y + z) + a(bx + z) \qquad (a, b \in A).$$

This is the same equation that is used when axiomatizing complete simulation equivalence in a setting without silent moves.

Proposition 13. *The set of unconditional equations*

$$E_{CS^=} = BW \cup \{CS_{\tau e}, CSE\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\eqsim_{CS}$.

Proof. In light of the above result, we only need to show that any use of the conditional axiom CSE_τ in a proof of an equation $p = q$ can equivalently be replaced by the application of axioms in $E_{CS^=}$.

Recall that if $p \stackrel{a}{\Longrightarrow} p'$ then $p = p + ap'$ can be proved using the τ-laws. Therefore, the pattern $ax + z$ characterizes the set of processes p such that $p \not\Downarrow$. For such processes, any application of the conditional equation CSE_τ can therefore be simulated by using CSE.

The other possible case is when $p \Downarrow$ and $q \Downarrow$. In this case, both p and q are either $\mathbf{0}$ or are provably equal to $\tau\mathbf{0}$, using W_1 and W_2 in the latter case. But then

$$a(p+q) = a(p+q) + ap$$

can be proved in all cases, by possibly using W_1. □

4.2 Nonexistence of finite complete axiomatizations

We shall now prove that if A contains at least one action, then the (in)equational theory of $\lesssim_{CS}$ over BCCSP(A_τ) does not have a finite basis. (The assumption that A be nonempty is, of course, necessary for such a result. In the trivial case that A is empty, the inequation $x \leq y$ suffices to obtain a complete axiomatization.)

For the sake of clarity, we recall that we consider terms up to the least congruence generated by axioms B1–B4, that is, up to strong bisimilarity.

Our proof of the claimed nonfinite axiomatizability result for the (in)equational theory of $\lesssim_{CS}$ over BCCSP(A_τ) will be based on the following infinite family of inequations, which are sound modulo $\lesssim_{CS}$:

$$a^n x \leq a^n\mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

To see that each of the inequations in the above family is sound, it suffices to observe that if $p \approx_{CS} \mathbf{0}$ then $a^n p \lesssim_{CS} a^n\mathbf{0}$ for each $n \geq 0$, and $a^n p \lesssim_{CS} a^n(p+a)$ otherwise, if $n \geq 1$. (Note that the assumption that $n \geq 1$ is necessary for the soundness of the above type of inequation. Indeed, the inequation $x \leq \mathbf{0} + (x+a)$ is *not* sound modulo $\lesssim_{CS}$ because $\mathbf{0} \not\lesssim_{CS} \mathbf{0} + (\mathbf{0} + a)$.)

Proposition 14. *If* $|A| \geq 1$ *then the (in)equational theory of* $\lesssim_{CS}$ *over* BCCSP(A_τ) *does not have a finite (in)equational basis. In particular, the following statements hold true.*

1. *No finite set of sound inequations over* BCCSP(A_τ) *modulo* $\lesssim_{CS}$ *can prove all of the sound inequations in the family*

$$a^n x \leq a^n \mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

2. *No finite set of sound (in)equations over* BCCSP(A_τ) *modulo* $\lesssim_{CS}$ *can prove all of the sound equations in the family*

$$a^n x + a^n \mathbf{0} + a^n(x+a) = a^n \mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

Before we embark on the proof of the above result, let us point out that the families of (in)equations that lie at the heart of the negative results in Proposition 14, as well as the structure of the proof of that result, stem from [4, 17], where it is shown that, in the setting without τ, the complete simulation preorder and equivalence afford no finite inequational axiomatization. The details of our argument are based on the developments in [4], but we need to take into account the role played by the internal action τ, since it could be the case that by introducing it one could obtain the desired finite basis, even if such basis does not exist for the concrete case with no silent moves.

Proposition 14 is a corollary of the following result.

Proposition 15. *Assume that* $|A| \geq 1$*. Let* E *be a collection of inequations whose elements are sound modulo* $\lesssim_{CS}$ *and have depth smaller than* n*. Suppose furthermore that the inequation* $t \leq u$ *is derivable from* E *and that* $u \precsim_{CS} a^n \mathbf{0} + a^n(x+a)$*. Then* $t \stackrel{a^n}{\Longrightarrow} x$ *implies* $u \stackrel{a^n}{\Longrightarrow} x$*.*

Having shown the above result, statement 1 in Proposition 14 can be proved as follows. Let E be a finite inequational axiom system that is sound modulo $\lesssim_{CS}$. Pick n larger than the depth of any axiom in E. Then, by Proposition 15, E cannot prove the valid inequation

$$a^n x \leq a^n \mathbf{0} + a^n(x+a),$$

and is therefore incomplete. Indeed,

$$a^n x \stackrel{a^n}{\Longrightarrow} x.$$

On the other hand, the only terms t such that

$$a^n\mathbf{0} + a^n(x+a) \stackrel{a^n}{\Longrightarrow} t$$

holds are $\mathbf{0}$ and $x+a$. So $a^n\mathbf{0} + a^n(x+a) \stackrel{a^n}{\Longrightarrow} x$ does not hold.

Statement 2 in Proposition 14 is a corollary of Proposition 14(1). To see this, assume Proposition 14(1) and suppose, towards a contradiction, that there is a finite set of sound (in)equations over BCCSP(A_τ) modulo $\lesssim_{CS}$ that can prove all of the equations in the family

$$a^n x + a^n\mathbf{0} + a^n(x+a) = a^n\mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

Recall that we may assume that E is closed with respect to symmetry and that, under this assumption, there is no difference between the rules of inference of equational and inequational logic. Thus E can prove all the inequations

$$a^n x + a^n\mathbf{0} + a^n(x+a) \leq a^n\mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

Observe now that the sound inequation CS, namely

$$ax \leq ax + y,$$

can be used to show that

$$a^n x \leq a^n x + a^n\mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

Therefore, by transitivity, the finite set of sound inequations $E \cup \{CS\}$ can prove all of the inequations in the family

$$a^n x \leq a^n\mathbf{0} + a^n(x+a) \qquad (n \geq 1).$$

This, however, contradicts Proposition 14(1).

In order to show Proposition 15, we shall first prove that the property mentioned in that statement holds true for instantiations of sound inequations whose depth is smaller than n. Next we use this fact to argue that the stated property is preserved by arbitrary inequational derivations from a collection of inequations whose elements are sound modulo $\lesssim_{CS}$ and have depth smaller than n.

Definition 8. *We say that a term t has an occurrence of variable x reachable via a sequence s of visible actions if there is some term t' such that $t \stackrel{s}{\Longrightarrow} x + t'$.*

For example, $ax + a\mathbf{0}$ has an occurrence of x reachable via a because $ax + a\mathbf{0} \stackrel{a}{\longrightarrow} x$ and $x = x + \mathbf{0}$ can be shown using B1.

Lemma 9. *Assume that $t \lesssim_{CS} u$ and that t has an occurrence of variable x reachable via a sequence s of visible actions. Then u has an occurrence of variable x reachable via s.*

Proof. Assume that $t \lesssim_{CS} u$ and that t has an occurrence of variable x reachable via a sequence s of visible actions. Let m be larger than the depth of u. Consider the closed substitution σ mapping x to a^m and every other variable to $\mathbf{0}$. Since t has an occurrence of variable x reachable via s, it is easy to see that $\sigma(t) \stackrel{sa^m}{\Longrightarrow} \mathbf{0}$. As $\sigma(t) \lesssim_{CS} \sigma(u)$ because $t \lesssim_{CS} u$ by assumption, it must be the case that $\sigma(u) \stackrel{sa^m}{\Longrightarrow} p$ for some p such that $\mathbf{0} \precsim_{CS} p$. Such a p is mute. As the depth of u is smaller than m, σ maps all variables different from x to $\mathbf{0}$, $\sigma(u) \stackrel{sa^m}{\Longrightarrow} p$ and p is mute, it follows that $u \stackrel{s}{\Longrightarrow} x + u'$ for some u', which was to be shown. □

The following lemma is the first stepping stone towards the proof of Proposition 15. It establishes that the property mentioned in that statement holds true for instantiations of sound inequations whose depth is smaller than n.

Lemma 10. *Suppose that $t \lesssim_{CS} u$ and that n is larger than the depth of t. Then $\sigma(t) \stackrel{a^n}{\Longrightarrow} x$ implies $\sigma(u) \stackrel{a^n}{\Longrightarrow} x$.*

Proof. Assume that $\sigma(t) \stackrel{a^n}{\Longrightarrow} x$. Since n is larger than the depth of t, there are some $0 \leq i < n$ and some variable z such that t has an occurrence of variable z reachable via a^i and $\sigma(z) \stackrel{a^{n-i}}{\Longrightarrow} x$. As $t \lesssim_{CS} u$, Lemma 9 yields that u has an occurrence of variable z reachable via a^i. Therefore $\sigma(u) \stackrel{a^n}{\Longrightarrow} x$, which was to be shown. □

Lemma 11. *Let p be a closed term such that $I^*(p) = \emptyset$. Then $p \stackrel{\tau}{\Longrightarrow} \mathbf{0}$.*

Proof. By structural induction on p. □

We will now argue that the property stated in Proposition 15 is preserved by arbitrary inequational derivations from a collection of inequations whose elements are sound modulo $\lesssim_{CS}$ and have depth smaller than n. The following lemma will allow us to handle closure under action prefixing in that proof.

Lemma 12. *Assume that* $at \precsim_{CS} au \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$*, and that* $at \stackrel{a^n}{\Longrightarrow} x$*. Then,* $au \stackrel{a^n}{\Longrightarrow} x$*.*

Proof. Assume that

1. $at \precsim_{CS} au \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$, and
2. $at \stackrel{a^n}{\Longrightarrow} x$.

Since $at \stackrel{a^n}{\Longrightarrow} x$, we have that at has an occurrence of x reachable via a^n. Therefore, by Lemma 9, so does au. This means that $au \stackrel{a^n}{\Longrightarrow} x + u'$ for some u'. Observe now that $au \stackrel{a^n}{\Longrightarrow} \mathbf{0}$ cannot hold, because this would contradict $au \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$. Indeed, assume, towards a contradiction, that $au \stackrel{a^n}{\Longrightarrow} \mathbf{0}$. Consider a closed substitution σ that maps x to a. Then $\sigma(au) \stackrel{a}{\longrightarrow} \sigma(u)$. The only terms that can be reached from $\sigma(a^n\mathbf{0} + a^n(x + a))$ via $\stackrel{a}{\Longrightarrow}$ are $a^{n-1}\mathbf{0}$ and $a^{n-1}(a + a)$. However, neither $\sigma(u) \precsim_{CS} a^{n-1}\mathbf{0}$ nor $\sigma(u) \precsim_{CS} a^{n-1}(a + a)$ holds. Indeed, the former fails because

$$\sigma(u) \stackrel{a^{n-1}}{\Longrightarrow} a + \sigma(u') \not\precsim_{CS} \mathbf{0},$$

and the latter because $\sigma(u) \stackrel{a^{n-1}}{\Longrightarrow} \mathbf{0} \not\precsim_{CS} a + a$.

Consider now the closed substitution $\sigma_\mathbf{0}$ that maps all variables to $\mathbf{0}$. Then $\sigma_\mathbf{0}(at) \stackrel{a^n}{\Longrightarrow} \mathbf{0}$ because $at \stackrel{a^n}{\Longrightarrow} x$ by the proviso of the lemma. As $at \precsim_{CS} au$, we have that $\sigma_\mathbf{0}(at) \precsim_{CS} \sigma_\mathbf{0}(au)$. Therefore, $\sigma_\mathbf{0}(au) \stackrel{a^n}{\Longrightarrow} p$ for some closed term p such that $\mathbf{0} \precsim_{CS} p$. Using Lemma 11, $\sigma_\mathbf{0}(au) \stackrel{a^n}{\Longrightarrow} p \stackrel{\tau}{\Longrightarrow} \mathbf{0}$. Since, by our earlier observation, $au \stackrel{a^n}{\Longrightarrow} \mathbf{0}$ cannot hold, we have that $au \stackrel{a^n}{\Longrightarrow} u''$ for some u'' such that $u'' \neq \mathbf{0}$ and $\sigma_\mathbf{0}(u'') = \mathbf{0}$. Such a u'' can only contain occurrences of the variable x (by Lemma 9 and the assumption that $au \precsim_{CS} a^n\mathbf{0} + a^n(x + a)$). Therefore $u'' = x$ and we are done. □

We now have all the necessary ingredients to complete the proof of Proposition 15, and therefore of statement 1 in Proposition 14.

Proof. (of Proposition 15) Assume that E is a collection of inequations whose elements have depth smaller than n and are sound modulo $\lesssim_{CS}$. Suppose furthermore that

- the inequation $t \leq u$ is derivable from E,
- $u \precsim_{CS} a^n\mathbf{0} + a^n(x+a)$ and
- $t \stackrel{a^n}{\Longrightarrow} x$.

(Observe that n is positive because it is larger than the depth of E.) We shall prove that $u \stackrel{a^n}{\Longrightarrow} x$ by induction on the derivation of $t \leq u$ from E. We proceed by examining the last rule used in the proof of $t \leq u$ from E. The case of reflexivity is trivial and that of transitivity follows by applying the inductive hypothesis twice. If $t \leq u$ is proved by instantiating an inequation in E, then the claim follows by Lemma 10. We are therefore left with the congruence rules, which we examine separately below.

- Suppose that E proves $t \leq u$ because $t = \tau t'$, $u = \tau u'$ and E proves $t' \leq u'$ by a shorter inference. Observe that $t' \stackrel{a^n}{\Longrightarrow} x$, since $t = \tau t' \stackrel{a^n}{\Longrightarrow} x$. Moreover, $u' \precsim_{CS} a^n\mathbf{0} + a^n(x+a)$. The induction hypothesis yields that $u' \stackrel{a^n}{\Longrightarrow} x$. Therefore, we obtain that $u = \tau u' \stackrel{a^n}{\Longrightarrow} x$, as required.
- Suppose that E proves $t \leq u$ because $t = at'$, $u = au'$ and E proves $t' \leq u'$ by a shorter inference. By the soundness of E, the fact that $\lesssim_{CS}$ is included in $\precsim_{CS}$ and the proviso of the proposition, we have that

$$t = at' \precsim_{CS} u = au' \precsim_{CS} a^n\mathbf{0} + a^n(x+a)$$

  and $t \stackrel{a^n}{\Longrightarrow} x$. Lemma 12 now yields $u \stackrel{a^n}{\Longrightarrow} x$, as required.
- Suppose that E proves $t \leq u$ because $t = t_1 + t_2$, $u = u_1 + u_2$ and E proves $t_i \leq u_i$, $1 \leq i \leq 2$, by shorter inferences. Since $t \stackrel{a^n}{\Longrightarrow} x$ and n is positive, we may assume, without loss

of generality, that $t_1 \stackrel{a^n}{\Longrightarrow} x$. Using the soundness of E and the fact that $t_1 \not\approx_{CS} \mathbf{0}$, it is not hard to see that

$$u_1 \precsim_{CS} a^n\mathbf{0} + a^n(x + a).$$

Therefore we may apply the induction hypothesis to infer that $u_1 \stackrel{a^n}{\Longrightarrow} x$. Hence, as n is positive, $u \stackrel{a^n}{\Longrightarrow} x$, as required.

This completes the proof. □

Corollary 3. *If* $|A| \geq 1$ *then the collection of (in)equations in at most one variable that hold over* BCCSP(A_τ) *modulo* $\lesssim_{CS}$ *does not have a finite (in)equational basis. Moreover, for each n, the collection of all sound (in)equations of depth at most n cannot prove all the valid (in)equations in at most one variable that hold in weak complete simulation semantics over* BCCSP(A_τ).

Weak Complete Simulation	Ground-complete		Complete	
Finite Equations	Order	Equiv.	Order	Equiv.
$1 \leq \|A\| = \infty$	$E_{CS^{\leq}}$	$E_{CS^{=}}$	Do not exist	

Table 4. Axiomatizations for the largest (pre)congruence included in the weak complete simulation semantics

Tables 4–5 summarize the positive and negative results on the existence of finite axiomatizations for weak complete simulation semantics.

4.3 Observational equivalence and complete simulation

As we did in Section 3.3 for weak simulation semantics, we now study the 'forced to be' precongruence based on the requirements for Milner's observational congruence. Let us consider the following definition of a preorder relation between processes.

Unconditional	
$E_{CS^{\leq}} = BW \cup \{CS_{\tau e}, CS, \tau N\}$ $E_{CS^{=}} = BW \cup \{CS_{\tau e}, CSE\}$	$(CS_{\tau e})$ $\tau(ax+y) = ax+y$ (CS) $ax \leq ax+y$ (τN) $\mathbf{0} \leq \tau\mathbf{0}$ (CSE) $b(ax+y+z) =$ $b(ax+y+z)+b(ax+z)$
Conditional	
$E^{c}_{CS^{\leq}} = BW \cup \{CS_{\tau e}, CS_{\tau}\}$ $E^{c}_{CS^{=}} = BW \cup \{CS_{\tau e}, CSE_{\tau}\}$	(CS_{τ}) $(x\Downarrow \Leftrightarrow y\Downarrow) \Rightarrow x \leq x+y$ (CSE_{τ}) $(x\Downarrow \Leftrightarrow y\Downarrow) \Rightarrow$ $a(x+y) = a(x+y)+ax$

Table 5. Axioms for the largest (pre)congruence included in the weak completed simulation semantics

Definition 9. *$\lesssim'_{CS}$ is the largest relation over closed terms in $\mathrm{T}(A_\tau)$ satisfying the following condition whenever $p \lesssim'_{CS} q$ and $\alpha \in A_\tau$:*

- *if $p \xrightarrow{\alpha} p'$ then there exists some q' such that $q \stackrel{\tau}{\Longrightarrow}\xrightarrow{\alpha}\stackrel{\tau}{\Longrightarrow} q'$ and $p' \precsim_{CS} q'$, that is, p' is weak complete simulated by q'.*

The relation $\lesssim'_{CS}$ is a precongruence over $\mathrm{T}(A_\tau)$, which is finer than the largest precongruence included in the weak complete simulation preorder—that is, $\lesssim'_{CS} \subset \sqsubseteq_{CS}$. The following result is similar to Lemma 4 in Section 3.3, and is useful in finding an axiomatization for $\lesssim'_{CS}$.

Lemma 13. *We have that $p \precsim_{CS} q$ implies $p \lesssim'_{CS} \tau q$, for all $p, q \in$* $\mathrm{T}(A_\tau)$

Proposition 16. *The set of equations*

$$E = BW \cup \{CS_\tau\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo $\lesssim'_{CS}$.*

Proof. To prove ground-completeness we can proceed as in the proof of Proposition 7, observing that Lemma 13 and axiom CS_τ can be used. □

The set of axioms in Proposition 16 is similar to the one that characterizes $\lesssim_{CS}$ considered in Proposition 10. However, to axiomatize $\lesssim'_{CS}$ we do not need the equation $CS_{\tau e}$, which is unsound.

In fact, we can also provide a non-conditional axiomatization for $\lesssim'_{CS}$ in the same way we did for the relation $\lesssim_{CS}$.

Proposition 17. *The set of equations*

$$E = BW \cup \{CS, \tau N\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim'_{CS}$.

Proof. The same proof strategy we adopted in Proposition 11 to obtain a ground-complete unconditional axiomatization of $\lesssim_{CS}$ from the conditional one can be used here. Let us note that, in that proof, we did not make use at all of the axiom $CS_{\tau e}$, which is the axiom needed for $\lesssim_{CS}$, but missing in the axiomatization of $\lesssim'_{CS}$. □

We conclude our study of this variation on the weak complete simulation preorder by showing that, like $\lesssim_{CS}$, it does not afford a finite (in)equational basis.

Proposition 18. *If* $|A| \geq 1$ *then the (in)equational theory of* $\lesssim'_{CS}$ *over* BCCSP(A_τ) *does not have a finite (in)equational basis.*

Proof. Observe that the family of inequations

$$a^n x \leq a^n \mathbf{0} + a^n (x + a) \qquad (n \geq 1)$$

is sound modulo $\lesssim'_{CS}$. Since $\lesssim'_{CS}$ is included in $\lesssim_{CS}$, by statement 1 in Theorem 14 no finite axiom system that is sound modulo $\lesssim'_{CS}$ can prove all the inequations in the above family. Therefore no finite axiom system that is sound modulo $\lesssim'_{CS}$ can be complete. □

5 Weak Ready Simulation

In this section, we shall study the equational theory of the largest precongruence included in the weak ready simulation preorder. We first define the notion of weak ready simulation that will be the cornerstone in subsequent developments. We then proceed to study its induced precongruence, first in the case in which the set of actions A is infinite and then in case that A is finite.

In order to define the weak ready simulation semantics we recall the definition of function I^*, presented in Section 2, that returns the set of initial visible actions of a term.

$$I^*(t) = \{a \mid a \in A \text{ and } t \stackrel{a}{\Longrightarrow} t' \text{ for some } t'\}.$$

Definition 10. *The* weak ready simulation preorder*, which we denote by $\precsim_{RS}$, is the largest relation over terms in* $\mathrm{T}(A_\tau)$ *satisfying the following conditions whenever $p \precsim_{RS} q$ and $\alpha \in A_\tau$:*

- *if $p \stackrel{\alpha}{\longrightarrow} p'$ then there exists some term q' such that $q \stackrel{\alpha}{\Longrightarrow} q'$ and $p' \precsim_{RS} q'$, and*
- *$I^*(p) = I^*(q)$.*

We say that $p, q \in \mathrm{T}(A_\tau)$ are weak ready simulation equivalent*, written $p \approx_{RS} q$, iff p and q are related by the kernel of $\precsim_{RS}$, that is when both $p \precsim_{RS} q$ and $q \precsim_{RS} p$ hold.*

Example 3. $\precsim_{RS}$ is not a precongruence with respect to the choice operator of BCCSP(A_τ). It is immediate to show that $\tau a \precsim_{RS} a$. However, $\tau a + b \not\precsim_{RS} a + b$. Indeed, by performing a τ-transition, $\tau a + b$ evolves into a, and it is not possible for $a + b$ to transform itself in a process that is able to weak ready simulate a.

5.1 Discussion on the definition of Weak Ready Simulation

Definition 11. *We denote by $\sqsubseteq_{RS}$ the largest precongruence included in $\precsim_{RS}$. Formally, $p \sqsubseteq_{RS} q$ iff*

- *$p \precsim_{RS} q$,*
- *$p \precsim_{RS} q \Rightarrow \forall \alpha \in A_\tau \quad \alpha p \precsim_{RS} \alpha q$, and*
- *$p \precsim_{RS} q \Rightarrow \forall r \in \mathrm{T}(A_\tau) \quad p + r \precsim_{RS} q + r$.*

Once more, the definition of the relation $\sqsubseteq_{RS}$, largest precongruence included in $\precsim_{RS}$, is purely algebraic and difficult to use to study that relation. Our aim in what follows will therefore be to obtain a behavioural characterization of $\precsim_{RS}$. Unlike in the setting of complete simulation semantics, the behavioural characterization of the relation $\sqsubseteq_{RS}$ and its axiomatic properties will depend crucially on whether the set of visible actions A is finite or infinite.

5.2 Infinite alphabet of actions

We start by studying the equational theory of the precongruence relation $\sqsubseteq_{RS}$ when A is infinite. Our first aim is to provide an explicit characterization of $\sqsubseteq_{RS}$.

Definition 12. *The order relation $\lesssim_{RS}$ between processes is defined as follows: We say that $p \lesssim_{RS} q$ iff*

- *for any $\alpha \in A_\tau$ such that $p \stackrel{\alpha}{\longrightarrow} p'$, there exists some q' such that $q \stackrel{\tau}{\Longrightarrow}\stackrel{\alpha}{\longrightarrow}\stackrel{\tau}{\Longrightarrow} q'$ with $p' \precsim_{RS} q'$, and*
- *$I^*(p) = I^*(q)$.*

We denote the kernel of $\lesssim_{RS}$ by $\eqsim_{RS}$.

Proposition 19 (Behavioural characterization of $\sqsubseteq_{RS}$). *If A is infinite then $p \lesssim_{RS} q$ if, and only if, $p \sqsubseteq_{RS} q$, for all $p, q \in \mathrm{T}(A_\tau)$. Therefore, $\eqsim_{RS}$ coincides with the kernel of the preorder $\sqsubseteq_{RS}$.*

Proof. It suffices to show that $\lesssim_{RS} = \sqsubseteq_{RS}$ when A is infinite. It is routine to show that $\lesssim_{RS}$ is a precongruence included in $\precsim_{RS}$. Therefore $\lesssim_{RS}$ is included in $\sqsubseteq_{RS}$, because $\sqsubseteq_{RS}$ is the largest precongruence included in $\precsim_{RS}$. To show the converse inclusion, assume that $p \sqsubseteq_{RS} q$. Since A is infinite, there is some action $a \in A \setminus I^*(p+q)$. As $p \sqsubseteq_{RS} q$, we have that $p + a \precsim_{RS} q + a$. A standard argument now yields that $p \lesssim_{RS} q$, and we are done. □

Ground-completeness We shall now provide ground-complete (conditional) axiomatizations of the relations $\lesssim_{RS}$ and $\eqsim_{RS}$.

To axiomatize $\lesssim_{RS}$ using conditional inequations, the key axiom is

$$(RS_\tau) \quad I^*(x) = I^*(y) \Rightarrow x \leq x + y.$$

This axiom mirrors the one used in the concrete setting in [20, 25].

The following technical lemma shows the relation between the weak ready simulation preorder and its induced precongruence, by means of the operational characterization provided in Definition 12. This lemma will be useful in the proof of Proposition 20.

Lemma 14. *If $p \precsim_{RS} q$ then $p \lesssim_{RS} \tau q$.*

The following proposition provides us with an axiomatic characterization of $\lesssim_{RS}$. One of the equations is conditional and a natural extension of the one that characterizes the ready simulation in the concrete case. Later we will also prove that the axiomatic characterization can be given without conditional axioms, as also happened for the semantics without τ-transitions.

Proposition 20. *The set of equations*

$$E^c_{RS\leq} = BW \cup \{RS_\tau\},$$

in which RS_τ is conditional, is sound and ground-complete for $\lesssim_{RS}$ over the language BCCSP(A_τ).

Proof. Checking the soundness of the axioms is straightforward. Let us therefore concentrate on ground-completeness:

$$p \lesssim_{RS} q \Rightarrow E^c_{RS\leq} \vdash p \leq q.$$

In the rest of the proof we use E instead of $E^c_{RS\leq}$. We proceed by induction on the depth of process p.

If $|p| = 0$ then p is $\mathbf{0}$, and given that $I^*(p) = I^*(q)$ either q is the also $\mathbf{0}$, and trivially $E \vdash 0 \leq 0$, or q is $\tau\mathbf{0}$, and, by using RS_τ and W_2, we have $E \vdash \mathbf{0} \leq \mathbf{0} + \tau\mathbf{0} = \tau\mathbf{0}$.

Let assume $|p| = n + 1$. As $p \lesssim_{RS} q$, we know that for each $p \xrightarrow{\alpha} p'$ there exists some q' such that $q \overset{\tau}{\Longrightarrow}\xrightarrow{\alpha}\overset{\tau}{\Longrightarrow} q'$ and $p' \precsim_{RS} q'$. By Lemma 14 we know $p' \lesssim_{RS} \tau q'$ and by induction hypothesis we have $E \vdash p' \leq \tau q'$. Therefore, $E \vdash \alpha p' \leq \alpha\tau q'$ and using W_1 also $E \vdash \alpha p' \leq \alpha q'$.

That is, for every α-derivative of p we can prove using E that there exists a larger α-weak derivative of q. By congruence we get that $E \vdash \sum \alpha_i p_i \leq \sum \alpha_i q_i$, where $p = \sum \alpha_i p_i$.

On the left side we have exactly process p, but on the right side we have the sum of *some* of the weak derivatives of process q. Using congruence we can add process q on both sides getting $E \vdash p + q \leq \sum \alpha_i q_i + q$. The weak derivatives of q can be absorbed by using the τ-laws, see Milner [39], and therefore $E \vdash p + q \leq q$.

Finally, given that $p \lesssim_{RS} q$ we have that $I^*(p) = I^*(q)$ and we can use RS_τ to derive $E \vdash p \leq p + q$, and then by transitivity we conclude $E \vdash p \leq q$. □

We now give a ground-complete and unconditional axiomatization for the weak ready simulation preorder. For that we will consider the equations

$$\begin{array}{ll} (RS) & ax \leq ax + ay \\ (\tau g) & x \leq \tau x \end{array}$$

Equation RS is a well known and important one in the study of process semantics. Together with B_1–B_4, it characterizes the ready simulation preorder in the concrete case. RS also appears as a necessary condition for process semantics in many general results in process theory—see, e.g. [5, 21, 22].

As for equation τg, this is indeed a simple and natural one that is satisfied by any 'natural' precongruence on processes with silent moves.

Proposition 21. *The set of non-conditional equations defined by*

$$E_{RS\leq} = BW \cup \{RS,\ \tau g\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim_{RS}$.

Proof. The soundness of the equations in $E_{RS\leq}$ is straightforward.

To prove their ground-completeness we use Proposition 20 and show that any application of the axiom RS_τ can be mimicked using $E_{RS\leq}$. More precisely, we show that whenever $I^*(p) = I^*(q)$ we have $E_{RS\leq} \vdash p \leq p + q$.

We use structural induction on q, but we employ the weaker hypothesis $I^*(p) \supseteq I^*(q)$ from which $I^*(p) = I^*(q)$ follows obviously.

- If q is the **0** process then it is obvious that $E_{RS\leq} \vdash p \leq p + q$.

– If $q = aq'$, then $a \in I^*(p)$ and with the τ-laws we can derive $p = p + ap'$ for some $p' \in \mathrm{T}(A_\tau)$. Applying RS we get that $E_{RS\leq} \vdash ap' \leq ap' + aq'$ and therefore $E_{RS\leq} \vdash p + ap' \leq p + ap' + aq'$ from which $E_{RS\leq} \vdash p \leq p + q$ follows by using the τ-laws.
– If $q = \tau q'$ then $I^*(q') = I^*(q)$, and therefore $I^*(q') \subseteq I^*(p)$, by induction hypothesis $E_{RS\leq} \vdash p \leq p + q'$ and by using the axiom τg we have $E_{RS\leq} \vdash p \leq p + q$ as desired.
– If $q = q_1 + q_2$, then $I^*(q_1) \subseteq I^*(p)$ and $I^*(q_2) \subseteq I^*(p + q_1)$ by induction we obtain $E_{RS\leq} \vdash p \leq p + q_1$ and $E_{RS\leq} \vdash p + q_1 \leq p + q_1 + q_2$ from which $E_{RS\leq} \vdash p \leq p + q$ follows by transitivity. □

To obtain a ground-complete axiomatization of the largest congruence included in weak ready simulation equivalence, it would be desirable to use a general 'ready-to-preorder result' [5, 22] as the one we have for the concrete case. There is indeed a similar result for weak semantics by Chen, Fokkink and van Glabbeek, see [18], but unfortunately it is not general enough to cover the case of the weak ready simulation congruence in Definition 12.

We provide a direct proof of a ground-completeness result in which a key role is played by the equation

$$(RSE_\tau) \quad I^*(x) = I^*(y) \Rightarrow \alpha(x + y) = \alpha(x + y) + \alpha y,$$

which is quite similar to the equation needed for the concrete case.

Proposition 22. *The set of equations*

$$E^c_{RS=} = BW \cup \{RSE_\tau\},$$

in which RSE_τ is conditional, is sound and ground-complete for $\eqsim_{RS}$ over the language BCCSP(A_τ).

Proof. By definition $p \eqsim_{RS} q$ iff $p \lesssim_{RS} q$ and $q \lesssim_{RS} p$. We want to prove that

$$p \eqsim_{RS} q \Leftrightarrow E^c_{RS=} \vdash p = q.$$

To prove the soundness of the equations in $E^c_{RS=}$, the only non-trivial case is to show that $\alpha(x + y) + \alpha y \lesssim_{RS} \alpha(x + y)$ assuming

$I^*(x) = I^*(y)$. But, if $I^*(p) = I^*(q)$, for any p and q, we have that the transition $\alpha(p+q) + \alpha q \xrightarrow{\alpha} q$ can be simulated by $\alpha(p+q) \overset{\alpha}{\Longrightarrow} p+q$ with $q \lesssim_{RS} p+q$, because $I^*(p) = I^*(q)$.

To prove ground-completeness we show by induction on the depth of p that $p \lesssim_{RS} q \Rightarrow E^c_{RS=} \vdash q = q+p$. Therefore, by symmetry $q \lesssim_{RS} p \Rightarrow E^c_{RS=} \vdash p = q+p$, and by the rules of equational logic $p \eqsim_{RS} q \Rightarrow E^c_{RS=} \vdash p = q$.

Let's then complete the proof by showing by induction on the depth of p that indeed $p \lesssim_{RS} q$ implies $E^c_{RS=} \vdash q = q+p$.

- The base case is trivial, $p = \mathbf{0}$ then $q = \mathbf{0}$ and $E^c_{RS=} \vdash \mathbf{0} = \mathbf{0} + \mathbf{0}$.
- For the inductive case, let assume that $p \lesssim_{RS} q$ and therefore for every α such that $p \xrightarrow{\alpha} p'$, $q \overset{\alpha}{\Longrightarrow} q'$ and $p' \precsim_{RS} q'$. By using Lemma 14 we know that $p' \lesssim_{RS} \tau q'$. Applying the induction hypothesis, we can assume that

  $$E^c_{RS=} \vdash \tau q' = \tau q' + p'.$$

  As we know that $I^*(p') = I^*(\tau q')$, we have that $I^*(p') = I^*(q')$ and we can use the equation RSE_τ to get

  $$E^c_{RS=} \vdash \alpha(\tau q' + p') = \alpha(\tau q' + p') + \alpha p'.$$

  Since $\tau q' = \tau q' + p'$, we can simplify

  $$E^c_{RS=} \vdash \alpha(\tau q') = \alpha(\tau q') + \alpha p'.$$

  By using equation W_1 we have $E^c_{RS=} \vdash \alpha q' = \alpha q' + \alpha p'$, and we can add q on both sides to get

  $$E^c_{RS=} \vdash q + \alpha q' = q + \alpha q' + \alpha p'.$$

  Finally the τ-laws allow for the absorption of α-derivatives and we can conclude that $E^c_{RS=} \vdash q = q + \alpha p'$ for every α and p' such that $p \xrightarrow{\alpha} p'$.
  Adding up every possible α-derivative of process p, we get $E^c_{RS=} \vdash q = q + p$ as desired.

□

In order to give an unconditional axiomatization of $\eqsim_{RS}$, we consider the following equations:

$$
\begin{array}{ll}
(RSE) & \alpha(bx + z + by) = \alpha(bx + z + by) + \alpha(bx + z) \\
(RSE_{\tau e}) & \alpha(x + \tau y) = \alpha(x + \tau y) + \alpha(x + y)
\end{array}
$$

Proposition 23. *The set of equations*

$$E_{RS^=} = BW \cup \{RSE, RSE_{\tau e}\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\eqsim_{RS}$*.*

Proof. Soundness can be proved by noticing that both RSE and $RSE_{\tau e}$ are particular instances of the conditional equation RSE_τ.

To prove ground-completeness we use the same ideas as in proof of Proposition 21. We show that whenever $I^*(p) = I^*(q)$ we have $E_{RS^=} \vdash \alpha(p + q) = \alpha(p + q) + \alpha p$. Actually we establish the slightly more general result that says that this is the case even if we only have $I^*(p) \supseteq I^*(q)$. The proof proceed by structural induction on q and uses the same case analysis adopted in the proof of Proposition 21. □

An (ω-)complete axiomatization We shall now provide an axiomatization for the relation $\lesssim_{RS}$ that is (ω-)complete.

Proposition 24. *If the set of actions A is infinite, then the axiom system*

$$E_{RS^\leq} = BW \cup \{RS, \tau g\}$$

is ω-complete for BCCSP(A_τ) *modulo* $\lesssim_{RS}$*.*

Proof. The proof is analogous to that of Proposition 4. We therefore omit the details. □

Reasoning as in the proof of Corollary 1, we obtain the following result.

Corollary 4. *If the set of actions is infinite, then the axiom system*

$$E_{RS^\leq} = BW \cup \{RS, \tau g\}$$

is complete for BCCSP(A_τ) *modulo* $\lesssim_{RS}$*.*

5.3 Axiomatizing $\sqsubseteq_{RS}$ when A is finite

Proposition 19 gives an explicit characterization of the largest precongruence included in the weak ready simulation preorder when the collection of actions is infinite. In this section, we shall study the (in)equational theory of $\sqsubseteq_{RS}$ when the set of observable actions A is finite and non-empty.

First of all, note that, if A is finite then the relation $\lesssim_{RS}$ defined in Definition 12 is *not* the largest precongruence included in the weak ready simulation preorder. To see this, consider the terms

$$p = \tau \sum_{a \in A} a \text{ and } q = \sum_{a \in A} a. \tag{2}$$

Observe that, for each $r \in \mathrm{BCCSP}(A_\tau)$, the following statements hold:

1. $p \precsim_{RS} q + r$ and
2. $p + r \precsim_{RS} q + r$.

(Both the above relations hold because $I^*(r) \subseteq I^*(q) = I^*(p) = A$.) It follows that $p \sqsubseteq_{RS} q$. On the other hand, $p \not\lesssim_{RS} q$ because q cannot initially perform a τ-labelled transition, unlike p.

Definition 13. *The relation $\lesssim^F_{RS}$ between processes is defined as follows: We say that $p \lesssim^F_{RS} q$ iff*

- *for each $a \in A$ and p' such that $p \xrightarrow{a} p'$, there exists some q' such that $q \xRightarrow{a} q'$ with $p' \precsim_{RS} q'$;*
- *for each p' such that $p \xrightarrow{\tau} p'$,*
  - *either there exists some q' such that $q(\xrightarrow{\tau})^+ q'$ with $p' \precsim_{RS} q'$,*
  - *or $I^*(p') = A$ and $p' \precsim_{RS} q$; and*
- *$I^*(q) \subseteq I^*(p)$.*

Note that $p \lesssim^F_{RS} q$, for the processes p and q defined in (2). Indeed, since $I^*(q) = A$, process q can match the initial τ-labelled transition from p by remaining idle.

Proposition 25 (Behavioural characterization of $\sqsubseteq_{RS}$). *If A is finite then $p \lesssim^F_{RS} q$ if, and only if, $p \sqsubseteq_{RS} q$, for all $p, q \in \mathrm{T}(A_\tau)$.*

Proof. To establish the 'if' implication, it suffices only to show that, for all $p, q \in \mathrm{BCCSP}(A_\tau)$,

1. $p \precsim_{RS} q$ and
2. $p + \sum_{a \in A} a \precsim_{RS} q + \sum_{a \in A} a$

imply $p \precsim_{RS}^F q$. In order to prove this claim, in light of the assumption that $p \precsim_{RS} q$, we only need to prove that if $p \xrightarrow{\tau} p'$ and $I^*(p') \neq A$ then there exists some q' such that $q(\xrightarrow{\tau})^+ q'$ with $p' \precsim_{RS} q'$. However, this is an immediate consequence of the assumption that $p + \sum_{a \in A} a \precsim_{RS} q + \sum_{a \in A} a$ because $I^*(q + \sum_{a \in A} a) = A$.

To establish the 'only if' implication, since $\precsim_{RS}^F$ is included in $\precsim_{RS}$, it suffices to prove that $\precsim_{RS}^F$ is a precongruence. It is clear that $\precsim_{RS}^F$ is preserved by action prefixing. We shall therefore focus on showing that $\precsim_{RS}^F$ is preserved by $+$. To this end, assume that $p \precsim_{RS}^F q$ and let r be a closed term. We shall now prove that $p + r \precsim_{RS}^F q + r$, and focus on the only interesting case of the argument.

Suppose that $p + r \xrightarrow{\tau} p'$ because $p \xrightarrow{\tau} p'$. Since $p \precsim_{RS}^F q$, we have that

– either there exists some q' such that $q(\xrightarrow{\tau})^+ q'$ with $p' \precsim_{RS} q'$,
– or $I^*(p') = A$ and $p' \precsim_{RS} q$.

In the former case, $q + r(\xrightarrow{\tau})^+ q'$ also holds, and we are done. In the latter case, we claim that $p' \precsim_{RS} q + r$ also holds, and we are done. To see that our claim does hold, observe that the relation

$$\mathcal{R} = \{(p_1, q_1 + r) \mid p_1 \precsim_{RS} q_1 \text{ and } I^*(p_1) = A\} \cup \precsim_{RS}$$

is a weak ready simulation. Indeed, suppose that $p_1 \mathcal{R} q_1 + r$ and $p_1 \xrightarrow{\tau} p_1'$. If $I^*(p_1') = A$ then $p_1' \mathcal{R} q_1 + r$, and we are done. Otherwise, it must be the case that there exists some q_1' such that $q_1(\xrightarrow{\tau})^+ q_1'$ and $p_1' \precsim_{RS} q_1'$. This follows because, since $p_1 \precsim_{RS} q_1$ and $I^*(p_1) = A$ yield that $I^*(q_1) = A$, it cannot be the case that $p_1' \precsim_{RS} q_1$. Checking that every observable transition from p_1 can be matched by $q_1 + r$ in the sense of Definition 10 is immediate. □

We collect below some observations on the relationships between $\precsim_{RS}$ and $\lesssim_{RS}^{F}$.

Proposition 26. *For all p, q, the following statements hold.*

1. *If $p \lesssim_{RS}^{F} \tau q$ then $p \precsim_{RS} q$.*
2. *Assume that $p \lesssim_{RS}^{F} q$, $p \xrightarrow{\tau} p'$, $I^*(p') = A$ and $p' \precsim_{RS} q$. Then $p' \lesssim_{RS}^{F} q$.*
3. *$p \precsim_{RS} q$ iff $p \lesssim_{RS}^{F} q$ or $p \lesssim_{RS}^{F} \tau q$.*

Proof. The first and the second claims are immediate from the definitions. The implication from right to left in the third claim holds by statement 1 in the lemma and Proposition 25. To establish the implication from left to right, assume that $p \precsim_{RS} q$ and $p \not\lesssim_{RS}^{F} q$. Then there is some p' such that $p \xrightarrow{\tau} p'$, $I^*(p') \neq A$ and q is the only term q' such that $q \overset{\tau}{\Longrightarrow} q'$ and $p' \precsim_{RS} q'$. In this case, it is not hard to see that $p \lesssim_{RS}^{F} \tau q$ holds. □

Ground-completeness In order to give a ground-complete axiomatization of the relation $\lesssim_{RS}^{F}$, we consider the equation

$$(RS_\Sigma) \quad \tau(\sum_{a \in A} a x_a + y) = \sum_{a \in A} a x_a + y$$

Proposition 27. *The set of equations*

$$E_{RS\leq}^{Fc} = BW \cup \{RS_\tau, RS_\Sigma\},$$

in which RS_τ is conditional, is sound and ground-complete for $\lesssim_{RS}^{F}$ over the language BCCSP(A_τ).

Proof. To prove soundness, we only need to check the validity of axiom RS_Σ, and more exactly that $\tau(\sum_A a p_a + q) \lesssim_{RS}^{F} \sum_A a p_a + q$, for all p_a ($a \in A$) and q, which is immediate since $\tau(\sum_A a p_a + q) \xrightarrow{\tau} \sum_A a p_a + q$ can be mimicked by the process $\sum_A a p_a + q$ by simply staying idle, because we have $I^*(\sum_A a p_a + q) = A$.

To prove the ground-completeness of the proposed axiomatization, we follow exactly the same procedure that we used in the proof of Proposition 20. The only difference appears when we consider a transition $p \xrightarrow{\tau} p'$ with $I^*(p') = A$, so that q can mimic that move by remaining idle, because we have $p' \precsim_{RS} q$. By statement 2 in Proposition 26, $p' \lesssim^F_{RS} q$. Then by applying the induction hypothesis we have $E^{Fc}_{RS\leq} \vdash p' \leq q$. But since $I^*(p') = A$, by using the τ-laws we can obtain

$$E^{Fc}_{RS\leq} \vdash p' = p' + \sum_A a p'_a$$

taking any a-derivative p'_a for each $a \in A$. And applying RS_Σ we obtain $\tau p' = p'$. Finally, we put everything together to conclude $E^{Fc}_{RS\leq} \vdash \tau p' \leq q$. □

Remark 4. Since the axiomatization $E^{Fc}_{RS\leq}$ is conditional, we could substitute in it the axiom RS_Σ by its conditional form

$$(RS^c_\Sigma) \quad (I^*(x) = A) \Rightarrow \tau p = p,$$

where we immediately recognize the restricted form of axiom τe, as it was the case for the axiom $CS_{\tau e}$, for the weak complete simulation preorder.

Proposition 28. *The set of equations*

$$E^F_{RS\leq} = BW \cup \{RS, \tau g, RS_\Sigma\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\lesssim^F_{RS}$.

Proof. Since RS_Σ is an (unconditional) equation, we can simply replay here the arguments at the proof of Proposition 21. □

We now proceed to offer (un)conditional axiomatizations of $\eqsim^F_{RS}$, the kernel of the preorder $\lesssim^F_{RS}$.

Proposition 29. *The set of equations, in which RSE_τ is conditional*

$$E^{Fc}_{RS=} = BW \cup \{RSE_\tau, RS_\Sigma\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\eqsim^F_{RS}$.

Proof. Similar to the proof of Proposition 27. □

Proposition 30. *The set of equations*

$$E^F_{RS^=} = BW \cup \{RSE, RS_{\tau e}, RS_\Sigma\}$$

is sound and ground-complete for BCCSP(A_τ) *modulo* $\eqsim^F_{RS}$.

Proof. The proof is identical to that of Proposition 23. □

Remark 5. Since, in the case $|A| < \infty$, the preorder $\lesssim^F_{RS}$ is the largest precongruence included in $\precsim_{RS}$, all the axiomatizations above are also sound and ground-complete for $\sqsubseteq_{RS}$ and its kernel, in this case. Note that the situation is similar to those for both the weak simulation and weak complete simulation preorders, where the preorders $\lesssim'_S$ and $\lesssim'_{CS}$ were finer than the corresponding largest precongruences. As in those cases, the corresponding restricted version of the axiom τe shows the difference with the finer preorders, which in this case obviously coincides with the relation $\lesssim_{RS}$, that was the largest precongruence contained in $\precsim_{RS}$ when the alphabet is infinite.

Nonexistence of finite complete axiomatizations We shall now prove that, if the set of actions A is finite, then neither $\lesssim^F_{RS}$ nor its kernel afford a finite (in)equational axiomatization. The following proposition was shown in [17]—see page 516 in that reference.

Proposition 31. *For each* $n \geq 0$, *the equation*

$$a^n x + a^n \mathbf{0} + \sum_{b \in A} a^n (x + b) = a^n \mathbf{0} + \sum_{b \in A} a^n (x + b) \qquad (3)$$

is sound modulo ready simulation equivalence, and therefore modulo the kernel of $\lesssim^F_{RS}$.

The family of equations (3) plays a crucial role in the proof of Theorem 36 in [17], to the effect that the equational theory of ready simulation equivalence is not finitely based over BCCSP(A_τ) when the set of actions is finite and contains at least two distinct actions. (In fact, as we showed in [4], ready simulation semantics is not finitely based, even when the set of actions is a singleton.) In what follows, we will follow the strategy underlying the proof of Proposition 14 to show the following result.

Proposition 32. *If $|A| \geq 1$ then the (in)equational theory of $\lesssim^F_{RS}$ over* BCCSP(A_τ) *does not have a finite (in)equational basis. In particular, the following statements hold true.*

1. *No finite set of sound inequations over* BCCSP(A_τ) *modulo $\lesssim^F_{RS}$ can prove all of the sound inequations in the family*

$$a^n x \leq a^n \mathbf{0} + \sum_{b \in A} a^n(x+b) \qquad (n \geq 1).$$

2. *No finite set of sound (in)equations over* BCCSP(A_τ) *modulo $\lesssim^F_{RS}$ can prove all of the sound equations in the family (3).*

Proposition 32 is a corollary of the following result. As usual, we will consider processes up to strong bisimilarity.

Proposition 33. *Assume that $|A| \geq 1$. Let E be a collection of inequations whose elements are sound modulo $\lesssim^F_{RS}$ and have depth smaller than n. Suppose furthermore that the inequation $t \leq u$ is derivable from E and that $u \lesssim^F_{RS} a^n \mathbf{0} + \sum_{b \in A} a^n(x+b)$. Then $t \overset{a^n}{\Longrightarrow} x$ implies $u \overset{a^n}{\Longrightarrow} x$.*

Having shown the above result, Proposition 32 (statement 1) can be proved following the same reasoning described on page 32 after Proposition 15.

Statement 2 in Proposition 32 is a corollary of statement 1 in Proposition 32. To see this, assume Proposition 32(1) and suppose, towards a contradiction, that there is a finite set of sound (in)equations over BCCSP(A_τ) modulo $\lesssim^F_{RS}$ that can prove all of the equations in the family (3). Recall that we may assume that E is closed with respect to symmetry and that, under this assumption, there is no difference between the rules of inference of equational and inequational logic. Thus E can prove all the inequations

$$a^n x + a^n \mathbf{0} + \sum_{b \in A} a^n(x+b) \leq a^n \mathbf{0} + \sum_{b \in A} a^n(x+b) \qquad (n \geq 1).$$

Observe now that the sound inequation RS, that is

$$ax \leq ax + ay,$$

can be used to show that

$$a^n x \leq a^n x + a^n \mathbf{0} + \sum_{b \in A} a^n (x + b) \qquad (n \geq 1).$$

Therefore, by transitivity, the finite set of sound inequations $E \cup \{RS\}$ can prove all of the inequations in the family

$$a^n x \leq a^n \mathbf{0} + \sum_{b \in A} a^n (x + b) \qquad (n \geq 1).$$

This, however, contradicts Proposition 32(1).

In order to show Proposition 33, we shall follow the strategy we used in the proof of Proposition 15. The crux of the proof is again to argue that the stated property is preserved by arbitrary inequational derivations from a collection of inequations whose elements are sound modulo $\lesssim_{RS}^{F}$ and have depth smaller than n.

The following lemma can be shown by mimicking the proof of Lemma 12.

Lemma 15. *Assume that at $\precsim_{RS}$ au $\precsim_{RS}$ $a^n \mathbf{0} + \sum_{b \in A} a^n (x + b)$, and that at $\stackrel{a^n}{\Longrightarrow}$ x. Then au $\stackrel{a^n}{\Longrightarrow}$ x.*

We now have all the necessary ingredients to complete the proof of Proposition 33, and therefore of statement 1 in Proposition 32.

Proof. (of Proposition 33) Assume that E is a collection of inequations whose elements have depth smaller than n and are sound modulo $\lesssim_{RS}^{F}$. Suppose furthermore that

- the inequation $t \leq u$ is derivable from E,
- $u \precsim_{RS} a^n \mathbf{0} + \sum_{b \in A} a^n (x + b)$ and
- $t \stackrel{a^n}{\Longrightarrow} x$.

We shall prove that $u \stackrel{a^n}{\Longrightarrow} x$ by induction on the derivation of $t \leq u$ from E. We proceed by examining the last rule used in the proof of $t \leq u$ from E. The case of reflexivity is trivial and that of transitivity follows by applying the induction hypothesis twice. If $t \leq u$ is proved by instantiating an inequation in E, then the claim follows by Lemma 10 because $\lesssim_{RS}^{F}$ is included in $\lesssim_{CS}$. We are therefore left with the congruence rules, which we examine separately below.

– Suppose that E proves $t \leq u$ because $t = \tau t'$, $u = \tau u'$ and that E proves $t' \leq u'$ by a shorter inference. Observe that $t' \stackrel{a^n}{\Longrightarrow} x$, since $t = \tau t' \stackrel{a^n}{\Longrightarrow} x$. Moreover,

$$u' \precsim_{RS} a^n\mathbf{0} + \sum_{b \in A} a^n(x + b).$$

The induction hypothesis yields that $u' \stackrel{a^n}{\Longrightarrow} x$. Therefore $u = \tau u' \stackrel{a^n}{\Longrightarrow} x$, as required.

– Suppose that E proves $t \leq u$ because $t = at'$, $u = au'$ and E proves $t' \leq u'$ by a shorter inference. By the soundness of E, the fact that $\precsim_{RS}^F$ is included in $\precsim_{RS}$ and the proviso of the proposition, we have that

$$t = at' \precsim_{RS} u = au' \precsim_{RS} a^n\mathbf{0} + \sum_{b \in A} a^n(x + b).$$

and $t \stackrel{a^n}{\Longrightarrow} x$. Lemma 15 now yields $u \stackrel{a^n}{\Longrightarrow} x$, as required.

– Suppose that E proves $t \leq u$ because $t = t_1 + t_2$, $u = u_1 + u_2$ and E proves $t_i \leq u_i$, $1 \leq i \leq 2$, by shorter inferences. Since $t \stackrel{a^n}{\Longrightarrow} x$ and n is positive, we may assume, without loss of generality, that $t_1 \stackrel{a^n}{\Longrightarrow} x$. This means that $I^*(t_1) = \{a\}$. (Indeed, $I^*(t) = \{a\}$ because $t \precsim_{RS} a^n\mathbf{0} + \sum_{b \in A} a^n(x + b)$.) Therefore, by the soundness of E, $I^*(u_1) = \{a\}$ also holds. It is now not hard to see that

$$u_1 \precsim_{RS} a^n\mathbf{0} + \sum_{b \in A} a^n(x + b).$$

Therefore we may apply the induction hypothesis to infer that $u_1 \stackrel{a^n}{\Longrightarrow} x$. Hence, as n is positive, $u \stackrel{a^n}{\Longrightarrow} x$, as required.

This completes the proof. □

Corollary 5. *If $1 \leq |A| < \infty$ then the collection of (in)equations in at most one variable that hold over* BCCSP(A_τ) *modulo $\precsim_{RS}^F$ does not have a finite (in)equational basis. Moreover, for each n, the collection of all sound (in)equations of depth at most n cannot prove all the valid (in)equations in at most one variable that hold in weak ready simulation semantics over* BCCSP(A_τ).

Weak Ready Simulation	Ground-complete		Complete	
Finite Equations	Order	Equiv.	Order	Equiv.
$\lvert A\rvert = \infty$	$E_{RS\leq}$	$E_{RS=}$	$E_{RS\leq}$	$E_{RS=}$
$1 \leq \lvert A\rvert < \infty$	$E^F_{RS\leq}$	$E^F_{RS=}$	Do not exist	

Table 6. Axiomatizations for the largest (pre)congruence included in the weak ready simulation semantics

Unconditional	
$E_{RS\leq} = BW \cup \{RS, \tau g\}$ $E_{RS=} = BW \cup \{RSE, RS_{\tau e}\}$ $E^F_{RS\leq} = BW \cup \{RS, \tau g, RS_\Sigma\}$ $E^F_{RS=} = BW \cup$ $\{RSE, RS_{\tau e}, RS_\Sigma\}$	(RS) $ax \leq ax + ay$ (τg) $x \leq \tau x$ (RSE) $\alpha(bx + z + by) = \alpha(bx + z + by) + \alpha(bx + z)$ $(RS_{\tau e})$ $\alpha(x + \tau y) = \alpha(x + \tau y) + \alpha(x + y)$ (RS_Σ) $\tau(\sum_A ax_a + y) = \sum_A ax_a + y$
Conditional	
$E^c_{RS\leq} = BW \cup \{RS_\tau\}$ $E^c_{CS=} = BW \cup \{RSE_\tau\}$	(RS_τ) $(I^*(x) \Leftrightarrow I^*(y)) \Rightarrow x \leq x + y$ (RSE_τ) $(I^*(x) \Leftrightarrow I^*(y)) \Rightarrow \alpha(x + y) = \alpha(x + y) + \alpha x$

Table 7. Axioms for the largest (pre)congruence included in the weak ready simulation semantics

Tables 6–7 summarize the positive and negative results on the existence of finite axiomatizations for weak ready simulation semantics.

5.4 Alternative Weak Ready Simulation Notions

Certainly, there are many possible ways to define a weak ready simulation semantics. The enormous collection of weak semantics deployed in [24] gives us an idea, but there are even more (reasonable, why not?) possibilities. If we concentrate on the non-determinism aspect of the question, leaving aside divergence and related facts, the main reason that causes this multiplicity of proposals is the double essence of invisible actions, usually represented by τ when we give the operational description of the processes. These invisible actions are either produced by the abstraction decision that hides the execution of some actions, or just represent non-deterministic choices, that therefore should have better a 'static' meaning, reflecting the 'specification level' at which these non-deterministic choices find their sense.

Our definition looks for the simplest generalization of the original one (without τ's), which uses I as main ingredient. It seems clear that the consideration of I^* instead of I is the easiest way to obtain a constraint [20] that generalizes that for the strong case, capturing the internal invisible character of τ's in an adequate way. We expected, and as we will see below, this is indeed the case, that in this way all the algebraic (good) properties of the strong semantics could be transferred in a (more or less) easy way to the weak case: indeed, a 'symbolic' substitution of I by I^* in the axiom (RS), together with the addition of the axioms for weak bisimulation WB, produce the desired axiomatization of our weak ready simulation semantics.

There are, however, several objections that could be posed to our proposal. It is true, that it does not 'weaken' the strong ready simulation if τ is included in the set of (observable) actions's) processes. In such a case we should immediately have $\tau x \leq \tau x + \tau y$, since $I(\tau x) = I(\tau x + \tau y) = \{\tau\}$, but under our definition we could have $I^*(x) \not\subset I^*(y)$, and then $I^*(\tau x) \neq I^*(\tau x + \tau y$ x).

It is true that weak bisimulation is (and was expected to be) coarser than strong bisimulation, and guided by this fact one could assume that to preserve this situation is a must when considering any other semantics. Obviously there are also some practical reasons supporting this purpose. For instance, in order to prove that two processes are weakly bisimar is enough to prove that they are are strongly bisimilar, when this is indeed the case. However, there are reasons that justify that the strong semantics that 'sees' the τ's will not be finer than the corresponding weak semantics that 'hides' their execution. When these τ's come indeed from the abstraction of some operational details that we want to hide to the external observer, then the expected relation is perfectly justified, but if they just represent non-deterministic choices, then the expected meaning of these τ's is absolutely 'static', and then any consequence of the fact of giving to them an operational meaning, as it is done when we consider the strong semantics, is definitely arguable. It would be great that our semantics would be consistent with hiding abstraction, but is not catastrophic that this is not the case, since this hiding operator has no sense if τ's are expected to represent internal choices and not abstracted internal actions.

One could also argue that if we are interested in a simple algebraic characterization of our semantics, we should have started by considering the axiom $(RS'_\tau)\alpha x \leq \alpha x + \alpha y$. We tried indeed to follow this path, but unfortunately, it seems not possible to obtain an atractive operational characterization of a weak ready simulation semantics that satisfies that axiom. In particular, if we consider the semantics that is algebraically defined by adding either (RS'_τ) or its slightly stronger version $(RS^+_\tau)\alpha x \leq \alpha x + y$, to the set of axioms WB, then the obtained semantics need ad-hoc 'up-to' mechanisms to take into account the syntactic presentation of the processes when defining (operationally) the simulation semantics.

T.Chen et al. present indeed (RS'_τ) (by the way, they simply denote it by (RS) in [18] as 'the weak ready simulation axiom' that guides their 'ready to preorder' mechanism for the weak case. This mechanism translate to the weak case that previously developped in [5, 22]. The results in that paper are technically sound, and therefore can be applied to any semantics that satisfy the axiom (RS'_τ). Unfortunately, as said above, it seems that there are not many such semantics that can be operationally defined in an appealling way. Certainly, the weak failures semantics is an exception. T.Chen et al. also studied it in [16], obtaining an ω-complete axiomatization that includes (RS'_τ), and certainly is quite close to that for the testing semantics in [23, 29], since these two semantics coincide indeed, once the differences between the syntax used in both presentations are adequately taken into account.

But failures semantics is not a simulation semantics, but instead the coarsest linear semantics attached to ready simualtion, as one can see at the extended ltbt-spectrum [19]. As a consequence, when we consider its weak version, the fact that it satisfies (RS'_τ) does not (neccesarily) mean that weak failures semantics 'comes from' a weak ready simulation semantics satisfying this axiom. As a matter of fact, we have obtain the same weak failures semantics when we have looked for the coarsest linear semantics attached to our weak ready simulation semantics. Therefore, the fact that weak failures semantics satisfies (RS'_τ) could be dued to the particular character of failures, and does not imply that the attached simulation semantics must satisfy (RS'_τ).

Another weak ready simulation semantics that also appeared in [24] has been recently used in a collection of papers [33, 35, 34], that investigate the use of disjunction in the specification of constraints to limit the behaviour of the desired implementations. This is *stable ready simulation*, that only takes into account the offers made at stable (those in which τ) cannot be executed) states. This is equivalent to give absolute priority to the execution of the τ's, with respect to the observable actions. As a consequence, when studying this semantics we can restrict ourselves to the use of pure non-deterministic choices, that become associative, so that at the end we can consider an alternate model where external and internal choices interleaves, but never are mixed. Moreover, any action prefixing an internal choice can be distributed over the choice producing an external choice between several branches that start with the same action. In this way, all the internal choices, but those at the root of the process, dissappear and this means that this stable ready simulation can be 'encoded' into the strong ready simulation, just representing a minor variant of it, that in particular can be easily axiomatized by means of the axioms that claim priority of τ's, associativity of internal choices and distributivity of prefix over internal choices.

As a conclusion, we do not claim at all that our weak ready simulation is 'the good one', but after a thorough study of the question we postulate that it is the simplest weak ready simulation notion that has at the same time a simple operational definition and good algebraic properties, as we have shown in the results of this section.

6 Conclusion

In this paper, we have offered a detailed study of the axiomatizability properties of the largest (pre)congruences over the language BCCSP induced by the 'weak' versions of the classic simulation, complete simulation and ready simulation preorders and equivalences. For each of these notions of behavioural semantics, we have presented results related to the (non)existence of finite (ground-)complete (in)equational axiomatizations. As in [17], the finite axiomatizability of the studied notions of semantics depends crucially on the cardinality of the set of observable actions. Following [19], we have also discussed ground-complete axiomatizations of those semantics using conditional (in)equations in some detail. In particular, we have shown how to prove ground-completeness results for (in)equational axiom systems from similar results for conditional axiomatizations in a fairly systematic fashion.

The results presented in this article paint a rather complete picture of the axiomatic properties of the above-mentioned weak simulation semantics over BCCSP. However, in the cases in which the studied notions of semantics do not afford finite complete axiomatizations, it would be interesting to obtain infinite, but finitely described, complete axiomatizations. This is a topic that we leave for future research.

The results presented in this study complement those offered in, e.g., [9, 44, 47], where notions of divergence-sensitive preorders based on variations on prebisimilarity [28, 38] or on the refusal simulation preorder have been given ground-complete inequational axiomatizations. They are just a first step in the study of the equational logic for notions of behavioural semantics in the extension of van Glabbeek's spectrum to behavioural semantics that abstract from internal steps in computation [24]. A natural avenue for future research is to investigate the equational logic of weak versions of semantics in van Glabbeek's spectrum that are based on notions of decorated traces. We have already started working on this topic and we plan to report on our results in a forthcoming article.

Following the developments in [1, 12, 44], it would also be interesting to study rule formats for operational semantics that provide congruence formats for the semantics considered in this paper, and to give procedures for generating ground-complete axiomatizations for them for process languages in the given formats.

In [18], Chen, Fokkink and van Glabbeek have provided an extension to weak process semantics of the 'ready to preorder' procedure for generating axiomatizations of process equivalences from those of their underlying preorders, first studied in [5, 22]. It would be worthwhile to study whether the scope of the algorithm presented in [18] can be extended to cover the case of the weak ready simulation congruence in Definition 12 and related semantics. The doctoral dissertation [15] also presents an algorithm to turn an axiomatization of a semantics for concrete processes into one for 'its induced weak semantics'. An extension of the scope of applicability of that algorithm would also be a significant advance on the state of the art in the study of axiomatizability results for process semantics over process algebras.

References

1. L. Aceto, B. Bloom, and F. W. Vaandrager. Turning sos rules into equations. *Information and Compututation*, 111(1):1–52, 1994.
2. L. Aceto, S. Capobianco, A. Ingolfsdottir, and B. Luttik. The equational theory of prebisimilarity over basic CCS with divergence. *Inf. Process. Lett.*, 108(5):284–289, 2008.
3. L. Aceto, T. Chen, W. Fokkink, and A. Ingolfsdottir. On the axiomatisability of priority. *Mathematical Structures in Computer Science*, 18(1):5–28, 2008.
4. L. Aceto, D. de Frutos Escrig, C. Gregorio-Rodríguez, and A. Ingólfsdóttir. Complete and ready simulation semantics are not finitely based over BCCSP, even with a singleton alphabet. *Information Processing Letters*, 111(9):408–413, 2011.

5. L. Aceto, W. Fokkink, and A. Ingólfsdóttir. Ready to preorder: get your BCCSP axiomatization for free! In *Algebra and Coalgebra in Computer Science, Second International Conference, CALCO 2007*, volume 4624 of *Lecture Notes in Computer Science*, pages 65–79. Springer, 2007.
6. L. Aceto, W. Fokkink, A. Ingólfsdóttir, and B. Luttik. Finite equational bases in process algebra: Results and open questions. In *Processes, Terms and Cycles*, volume 3838 of *Lecture Notes in Computer Science*, pages 338–367. Springer, 2005.
7. L. Aceto, W. Fokkink, A. Ingolfsdottir, and B. Luttik. A finite equational base for CCS with left merge and communication merge. *ACM Trans. Comput. Log.*, 10(1), 2009.
8. L. Aceto, W. Fokkink, A. Ingólfsdóttir, and M. R. Mousavi. Lifting non-finite axiomatizability results to extensions of process algebras. *Acta Informatica*, 47(3):147–177, 2010.
9. L. Aceto and M. Hennessy. Termination, deadlock and divergence. *Journal of the ACM*, 39(1):147–187, 1992.
10. L. Aceto, A. Ingolfsdottir, B. Luttik, and P. van Tilburg. Finite equational bases for fragments of CCS with restriction and relabelling. In *5th IFIP International Conference on Theoretical Computer Science*, volume 273 of *IFIP*, pages 317–332. Springer, 2008.
11. J. Baeten, T. Basten, and M. Reniers. *Process Algebra: Equational Theories of Communicating Processes*, volume 50 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, November 2009.
12. J. C. M. Baeten and E. P. de Vink. Axiomatizing gsos with termination. *Journal of Logic and Algebraic Programming*, 60-61:323–351, 2004.
13. J. A. Bergstra and J. W. Klop. Process algebra for synchronous communication. *Information and Control*, 60(1-3):109–137, 1984.
14. B. Bloom, S. Istrail, and A. R. Meyer. Bisimulation can't be traced. *Journal of the ACM*, 42(1):232–268, 1995.
15. T. Chen. *Clocks, Dice and Processes*. PhD thesis, Centrum voor Wiskunde en Informatica (CWI), 2009.
16. T. Chen and W. Fokkink. On the axiomatizability of impossible futures: Preorder versus equivalence. In *LICS*, pages 156–165. IEEE Computer Society, 2008.
17. T. Chen, W. Fokkink, B. Luttik, and S. Nain. On finite alphabets and infinite bases. *Information and Computation*, 206(5):492–519, 2008.
18. T. Chen, W. Fokkink, and R. J. van Glabbeek. Ready to preorder: The case of weak process semantics. *Information Processing Letters*, 109(2):104–111, 2008.
19. D. de Frutos-Escrig, C. Gregorio-Rodríguez, and M. Palomino. On the unification of process semantics: Equational semantics. *Electronic Notes in Theoretical Computer Science*, 249:243–267, 2009.
20. D. de Frutos-Escrig and C. Gregorio-Rodríguez. Universal coinductive characterizations of process semantics. In *5th IFIP International Conference on Theoretical Computer Science*, volume 273 of *IFIP*, pages 397–412. Springer, 2008.
21. D. de Frutos-Escrig and C. Gregorio-Rodríguez. (Bi)simulations up-to characterise process semantics. *Information and Computation*, 207(2):146–170, 2009.
22. D. de Frutos-Escrig, C. Gregorio-Rodríguez, and M. Palomino. Ready to preorder: an algebraic and general proof. *Journal of Logic and Algebraic Programming*, 78(7):539–551, 2009.
23. R. De Nicola and M. Hennessy. Testing equivalences for processes. *Theoretical Computer Science*, 34:83–133, 1984.
24. R. v. Glabbeek. The linear time - branching time spectrum II. In *CONCUR '93 - Concurrency Theory, 5th International Conference*, volume 715 of *Lecture Notes in Computer Science*, pages 66–81. Springer, 1993.

25. R. v. Glabbeek. The linear time – branching time spectrum I; the semantics of concrete, sequential processes. In *Handbook of Process Algebra*, chapter 1, pages 3–99. Elsevier, 2001.
26. R. v. Glabbeek and P. Weijland. Branching time and abstraction in bisimulation semantics. *Journal of the ACM*, 43(3):555–600, 1996.
27. J. F. Groote. A new strategy for proving omega-completeness applied to process algebra. In *CONCUR '90, Theories of Concurrency: Unification and Extension*, volume 458 of *Lecture Notes in Computer Science*, pages 314–331. Springer, 1990.
28. M. Hennessy. A term model for synchronous processes. *Informationa and Control*, 51(1):58–75, 1981.
29. M. Hennessy. *Algebraic Theory of Processes*. MIT Press, 1988.
30. M. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32:137–161, 1985.
31. C. Hoare. *Communicating Sequential Processes*. Prentice Hall, 1985.
32. K. G. Larsen and A. Skou. Bisimulation through probabilistic testing. *Information and Computation*, 94(1):1–28, 1991.
33. G. Lüttgen and W. Vogler. Ready simulation for concurrency: It's logical! In *Automata, Languages and Programming, 34th International Colloquium, ICALP 2007*, volume 4596 of *Lecture Notes in Computer Science*, pages 752–763. Springer, 2007.
34. G. Lüttgen and W. Vogler. Safe reasoning with logic lts. In *SOFSEM 2009: Theory and Practice of Computer Science, 35th Conference on Current Trends in Theory and Practice of Computer Science*, volume 5404 of *Lecture Notes in Computer Science*, pages 376–387. Springer, 2009.
35. G. Lüttgen and W. Vogler. Ready simulation for concurrency: It's logical! *Inf. Comput.*, 208(7):845–867, 2010.
36. R. Milner. An algebraic definition of simulation between programs. In *Proceedings 2nd Joint Conference on Artificial Intelligence*, pages 481–489. BCS, 1971. Also available as Report No. CS-205, Computer Science Department, Stanford University.
37. R. Milner. *A Calculus of Communicating Systems*. LNCS 92. Springer, 1980.
38. R. Milner. A modal characterisation of observable machine behaviour. In E. Astesiano and C. Böhm, editors, *CAAP '81: Trees in Algebra and Programming, 6th Colloquium*, volume 112 of *Lecture Notes in Computer Science*, pages 25–34. Springer, 1981.
39. R. Milner. *Communication and Concurrency*. Prentice Hall, 1989.
40. R. Milner. A complete axiomatisation for observational congruence of finite-state behaviors. *Information and Computation*, 81(2):227–247, 1989.
41. F. Moller. *Axioms for Concurrency*. PhD thesis, Report CST-59-89, Department of Computer Science, University of Edinburgh, 1989.
42. D. M. Park. Concurrency and automata on infinite sequences. In *Theoretical Computer Science, 5th GI-Conference*, volume 104 of *Lecture Notes in Computer Science*, pages 167–183. Springer, 1981.
43. G. D. Plotkin. A structural approach to operational semantics. *Journal of Logic and Algebraic Programming*, 60–61:17–139, 2004.
44. I. Ulidowski. Axiomatisations of weak equivalences for de simone languages. In *CONCUR '95: Concurrency Theory, 6th International Conference*, volume 962 of *Lecture Notes in Computer Science*, pages 219–233. Springer, 1995.
45. R. J. van Glabbeek. A characterisation of weak bisimulation congruence. In *Processes, Terms and Cycles: Steps on the Road to Infinity*, volume 3838 of *Lecture Notes in Computer Science*, pages 26–39. Springer, 2005.
46. M. Voorhoeve and S. Mauw. Impossible futures and determinism. *Information Processing Letters*, 80(1):51–58, 2001.

47. D. Walker. Bisimulation and divergence. *Information and Computation*, 85(2):202–241, 1990.