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A Complete Classification of the Expressiveness of Interval Logics of Allen's Relations

The General and the Dense Cases

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Abstract Interval temporal logics take time intervals, instead of time instants, as their primitive temporal entities. One of the most studied interval temporal logics is Halpern and Shoham's modal logic of time intervals HS, which associates a modal operator with each binary relation between intervals over a linear order (the so-called Allen's interval relations). In this paper, we compare and classify the expressiveness of all fragments of HS on the class of all linear orders and on the subclass of all dense linear orders. For each of these classes, we identify a complete set of definabilities between HS modalities, valid in that class, thus obtaining a complete classification of the family of all 4096 fragments of HS with respect to their expressiveness. We show that on the class of all linear orders there are exactly 1347 expressively different fragments of HS, while on the class of dense linear orders there are exactly 966 such expressively different fragments.

Keywords Interval temporal logic · Expressiveness · Bisimulation

Short preliminary versions of parts of this paper appeared in [17] (the general case) and [2] (the dense case).

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1 Introduction

Interval reasoning naturally arises in various fields of computer science and artificial intelligence, ranging from hardware and real-time system verification to natural language processing, from constraint satisfaction to planning [3,4,18,25,26,29]. Interval temporal logics make it possible to reason about interval structures over (linearly) ordered domains, where time intervals, rather than time instants, are the primitive ontological entities. The distinctive features of interval temporal logics turn out to be useful in various application domains [9,15,24,25,29]. For instance, they allow one to model *telic statements*, that is, statements that express goals or accomplishments, e.g., the statement: 'The airplane flew from Venice to Toronto' [24]. Moreover, when we restrict ourselves to discrete linear orders, some interval temporal logics are expressive enough to constrain the length of intervals, thus allowing one to specify safety properties involving quantitative conditions [24]. This is the case, for instance, with the well-known 'gas-burner' example [29]. Temporal logics with interval-based semantics have also been proposed as suitable formalisms for the specification and verification of hardware [25] and of real-time systems [29].

The variety of binary relations between intervals in a linear order was first studied by Allen [3], who investigated their use in systems for time management and planning. In [20], Halpern and Shoham introduced and systematically analyzed the (full) logic of Allen's relations, called HS in this paper, that features one modality for each Allen relation. In particular, they showed that HS is highly undecidable over most classes of linear orders. This result motivated the search for (syntactic) HS fragments offering a good balance between expressiveness and decidability/complexity [7,8,12, 13,14,22,23,24].

The problem of identifying expressive enough, yet decidable, fragments of HS that are suitable for specific classes of applications is a major research problem in the area. It requires a comparative analysis of the expressiveness of the variety of such fragments. This amounts to systematically studying mutual definabilities among the HS modalities. As an example, Bresolin et al. [10,11] identify all decidable HS fragments, and classify them in terms of both their expressive power and their complexity, with respect to the class of finite linear orders [10] and the class of strongly discrete linear orders [11].

A comparative analysis of the expressive power of the variety of HS fragments is far from being trivial, because some HS modalities are definable in terms of others, and thus syntactically different fragments may turn out to be equally expressive. To complicate matters, the definability of a specific modality by a given subset of HS modalities may depend on the class of linear orders over which the logic is interpreted. Thus, such classifications cannot, in general, be easily transferred from one class of linear orders to another: while definability does transfer from a class to all its proper sub-classes, proving a non-definability result amounts to providing a counterexample based on concrete linear orders from the considered class. As a matter of fact, different assumptions on the underlying linear orders give rise, in general, to different sets of definability equations.

Many classes of linear orders are of practical interest, including the class of all linear orders and the class of all dense (resp., discrete, finite) linear orders, as well as the particular linear order on $\mathbb{R}$ (resp., $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$). In this paper, we give a complete classification of the expressiveness of HS fragments in two of the most important cases, namely, the *general* case (i.e., over the class of all linear orders), and the *dense* case (i.e., over the class of all dense linear orders). Most of the arguments that we use to classify the expressive power of HS fragments over the class of all linear orders directly apply also to the class of all dense linear orders. Nevertheless, some extra effort is needed to obtain the dense classification from the general one, since more definability equations hold in the dense case.

We identify a complete set of valid definability equations among HS modalities for both the considered classes of linear orders (the class of all linear orders Lin and the class of all dense linear orders Den). While undefinability results in the dense case are essentially based on counterexamples referring to the linear order on $\mathbb{R}$, the proposed constructions can be easily modified to deal with other specific subclasses of the class of all dense linear orders, e.g., the linear order on $\mathbb{Q}$. This means that the results presented in this paper yield complete classifications not only with respect to the two classes mentioned above, but also with respect to each of the linear orders on $\mathbb{R}$ and $\mathbb{Q}$. Eventually, we show that there are exactly 1347 expressively different HS fragments in the general case, and 966 ones in the dense case, out of 4096 syntactically distinct subsets of HS modalities.

The rest of the paper is organized as follows. In Section 2, we define the syntax and the semantics of the interval temporal logic HS, and we introduce the basic notions of definability and expressiveness. In Section 3, we give a short account of the main results of the paper. Section 4 and Section 5 are devoted to the proofs of soundness and completeness of the proposed set of definability equations, respectively. The completeness proof turns out to be much harder than that of soundness, and thus it does not come as a surprise that Section 5 is much longer than Section 4. In the final section, we summarize in Theorem 1 the import of the collection of results shown in the previous sections, provide an assessment of the work done and outline future research directions.

2 Preliminaries

We denote the sets of natural numbers, integers, rationals, irrationals, and reals, as well as the linear orders based on them, respectively by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\overline{\mathbb{Q}}$, and $\mathbb{R}$.

Let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[a, b]$, where $a, b \in D$ and $a \leq b$. An interval is called a *point interval* if $a = b$ and a *strict interval* if $a < b$. In this paper, we assume the *strict semantics*, that is, we exclude point intervals and only consider strict intervals. The adoption of the strict semantics, excluding point intervals, instead of the *non-strict semantics*, which includes them, conforms to the definition of interval adopted by Allen in [3], but differs from the one given by Halpern and Shoham in [20]. It has at least two strong motivations: first, a number of representation paradoxes arise when the non-strict semantics is adopted, due to the presence of point intervals, as pointed out in [3]; second, when point intervals are included there seems to be no intuitive semantics for interval relations that makes them both pairwise disjoint and jointly exhaustive.

If we exclude the identity relation, there are 12 different relations between two strict intervals in a linear order, often called *Allen's relations* [3]: the six relations R_A (adjacent to), R_L (later than), R_B (begins), R_E (ends), R_D (during), and R_O (overlaps), depicted in Figure 1, and their inverses, that is, $R_{\overline{X}} = (R_X)^{-1}$, for each $X \in \{A, L, B, E, D, O\}$.

HS **modalities**	**Allen's relations**
$\langle A \rangle$	$[a,b]R_A[c,d] \Leftrightarrow b = c$
$\langle L \rangle$	$[a,b]R_L[c,d] \Leftrightarrow b < c$
$\langle B \rangle$	$[a,b]R_B[c,d] \Leftrightarrow a = c, d < b$
$\langle E \rangle$	$[a,b]R_E[c,d] \Leftrightarrow b = d, a < c$
$\langle D \rangle$	$[a,b]R_D[c,d] \Leftrightarrow a < c, d < b$
$\langle O \rangle$	$[a,b]R_O[c,d] \Leftrightarrow a < c < b < d$

Graphical representation

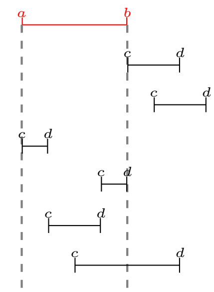

Fig. 1 Allen's interval relations and the corresponding HS modalities.

We interpret interval structures as Kripke structures, with Allen's relations playing the role of the accessibility relations. Thus, we associate a modality $\langle X \rangle$ with each Allen relation R_X. For each $X \in \{A, L, B, E, D, O\}$, the *transpose* of modality $\langle X \rangle$ is modality $\langle \overline{X} \rangle$, corresponding to the inverse relation $R_{\overline{X}}$ of R_X.

2.1 Syntax and semantics

Halpern and Shoham's logic HS [20] is a multi-modal logic with formulae built from a finite, non-empty set $\mathcal{AP}$ of atomic propositions (also referred to as proposition letters), the propositional connectives $\vee$ and $\neg$, and a modality for each Allen relation. With every subset $\{R_{X_1}, \ldots, R_{X_k}\}$ of these relations, we associate the fragment $\mathsf{X_1X_2\ldots X_k}$ of HS, whose formulae are defined by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1 \rangle\varphi \mid \ldots \mid \langle X_k \rangle\varphi,$$

where $p \in \mathcal{AP}$. The other propositional connectives and constants (e.g., $\wedge$, $\rightarrow$, and $\top$), as well as the dual modalities (e.g., $[A]\varphi \equiv \neg\langle A \rangle\neg\varphi$), can be derived in the standard way. For a fragment $\mathcal{F} = \mathsf{X_1X_2\ldots X_k}$ and a modality $\langle X \rangle$, we write $\langle X \rangle \in \mathcal{F}$ if $X \in \{X_1, \ldots, X_k\}$. Given two fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X \rangle \in \mathcal{F}_1$ implies $\langle X \rangle \in \mathcal{F}_2$, for every modality $\langle X \rangle$. Finally, for a fragment $\mathcal{F} = \mathsf{X_1X_2\ldots X_k}$ and a formula φ, we write $\varphi \in \mathcal{F}$ or, equivalently, we say that φ is an $\mathcal{F}$-formula, meaning that φ belongs to the language of $\mathcal{F}$.

The (strict) semantics of HS is given in terms of *interval models* $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $\mathbb{D}$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all (strict) intervals over $\mathbb{D}$, and V is a *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$, which assigns to every atomic proposition $p \in \mathcal{AP}$ the set of intervals $V(p)$ on which p holds. The *truth* of a formula on a given interval $[a, b]$ in an interval model M is defined by structural induction on formulae as follows:

- $M, [a, b] \Vdash p$ if and only if $[a, b] \in V(p)$, for each $p \in \mathcal{AP}$;
- $M, [a, b] \Vdash \neg\psi$ if and only if it is not the case that $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \varphi \vee \psi$ if and only if $M, [a, b] \Vdash \varphi$ or $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \langle X \rangle\psi$ if and only if there exists an interval $[c, d]$ such that $[a, b]R_X[c, d]$ and $M, [c, d] \Vdash \psi$, for each modality $\langle X \rangle$.

Formulae of HS can be interpreted over a given class of interval models. For the sake of brevity and with a benign abuse of notation, for a given class of linear orders $\mathcal{C}$, we identify the class of interval models over linear orders in $\mathcal{C}$ with the class $\mathcal{C}$ itself. Thus, we will use, for example, the expression 'formulae of HS are interpreted over the class $\mathcal{C}$ of linear orders' instead of the extended one 'formulae of HS are interpreted over the class of interval models over linear orders in $\mathcal{C}$'. Among others, we mention the following important classes of linear orders: *(i)* the class of *all* linear orders Lin; *(ii)* the class of (all) *dense* linear orders Den, that is, those in which for every pair of distinct points there exists at least one point in between them — e.g., $\mathbb{Q}$ and $\mathbb{R}$; *(iii)* the class of (all) *discrete* linear orders, that is, those in which every element, apart from the greatest element, if it exists, has an immediate successor, and every element, other than the least element, if it exists, has an immediate predecessor — e.g., $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}+\mathbb{Z}$[1] ; *(iv)* the class of (all) *finite* linear orders, that is, those having only finitely many points. All the classes of linear orders we consider in this paper are (left/right) *symmetric*, namely, if a class $\mathcal{C}$ contains a linear order $\mathbb{D} = \langle D, \prec \rangle$, then it also contains (a linear order isomorphic to) its dual linear order $\mathbb{D}^d = \langle D, \succ \rangle$, where $\succ$ is the inverse of $\prec$.

A formula ϕ of HS is *valid* over a class $\mathcal{C}$ of linear orders, denoted by $\Vdash_{\mathcal{C}} \phi$, if it is true on every interval in every interval model belonging to $\mathcal{C}$. Two formulae ϕ and ψ are *equivalent* relative to the class $\mathcal{C}$ of linear orders, denoted by $\phi \equiv_{\mathcal{C}} \psi$, if $\Vdash_{\mathcal{C}} \phi \leftrightarrow \psi$.

2.2 Definability and expressiveness

The following definition formalizes the notion of definability of modalities in terms of others.

Definition 1 (definability) A modality $\langle X \rangle$ of HS is *definable* in an HS fragment $\mathcal{F}$ relative to a class $\mathcal{C}$ of linear orders, denoted $\langle X \rangle \lhd_{\mathcal{C}} \mathcal{F}$, if $\langle X \rangle p \equiv_{\mathcal{C}} \psi$ for some $\mathcal{F}$-formula ψ over the atomic proposition p, for any $p \in \mathcal{AP}$. Then, the equivalence $\langle X \rangle p \equiv_{\mathcal{C}} \psi$ is called a *definability equation* for $\langle X \rangle$ in $\mathcal{F}$ relative to $\mathcal{C}$. We write $\langle X \rangle \ntriangleleft_{\mathcal{C}} \mathcal{F}$ if $\langle X \rangle$ is not definable in $\mathcal{F}$ relative to $\mathcal{C}$.

As we already noticed, smaller classes of linear orders inherit the definabilities holding for larger classes. Formally, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are classes of linear orders such that $\mathcal{C}_1 \subset \mathcal{C}_2$, then all definabilities holding for $\mathcal{C}_2$ are also valid for $\mathcal{C}_1$. However, more definabilities can possibly hold for $\mathcal{C}_1$. On the other hand, undefinability results for $\mathcal{C}_1$ hold also for $\mathcal{C}_2$. In the rest of the paper, we will omit the class of linear orders when it is clear from the context (e.g., we will simply write $\langle X \rangle p \equiv \psi$ and $\langle X \rangle \lhd \mathcal{F}$ instead of $\langle X \rangle p \equiv_{\mathcal{C}} \psi$ and $\langle X \rangle \lhd_{\mathcal{C}} \mathcal{F}$, respectively).

It is known from [20] that, in the strict semantics, all HS modalities are definable in the fragment containing modalities $\langle A \rangle$, $\langle B \rangle$, and $\langle E \rangle$, and their transposes $\langle \overline{A} \rangle$, $\langle \overline{B} \rangle$, and $\langle \overline{E} \rangle$. (In the non-strict semantics, the four modalities $\langle B \rangle$, $\langle E \rangle$, $\langle \overline{B} \rangle$, and $\langle \overline{E} \rangle$ suffice, as shown in [28].) In this paper, we compare and classify the expressiveness of all HS fragments with respect to the class of all linear orders and to the class of

[1] In the literature, these are sometimes called *weakly discrete* linear orders, to distinguish them from the so-called *strongly discrete* ones, where, for every pair of distinct points, there are only finitely many points in between them — e.g., $\mathbb{N}$, $\mathbb{Z}$, but not $\mathbb{Z}+\mathbb{Z}$.

all dense linear orders. Formally, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be any pair of such fragments. For a given class $\mathcal{C}$ of linear orders, we say that:

- $\mathcal{F}_2$ is *at least as expressive as* $\mathcal{F}_1$, denoted by $\mathcal{F}_1 \preceq \mathcal{F}_2$, if each modality $\langle X \rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$;
- $\mathcal{F}_1$ is *strictly less expressive* than $\mathcal{F}_2$ (or, equivalently, $\mathcal{F}_2$ is *strictly more expressive* than $\mathcal{F}_1$), denoted by $\mathcal{F}_1 \prec \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ holds, but $\mathcal{F}_2 \preceq \mathcal{F}_1$ does not hold;
- $\mathcal{F}_1$ and $\mathcal{F}_2$ are *equally expressive* (or *expressively equivalent*), denoted by $\mathcal{F}_1 \equiv \mathcal{F}_2$, if both $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_2 \preceq \mathcal{F}_1$ hold;
- $\mathcal{F}_1$ and $\mathcal{F}_2$ are *expressively incomparable*, denoted by $\mathcal{F}_1 \bowtie \mathcal{F}_2$, if neither $\mathcal{F}_1 \preceq \mathcal{F}_2$ nor $\mathcal{F}_2 \preceq \mathcal{F}_1$ hold.

Now, we define the notion of optimal definability, relative to a class $\mathcal{C}$ of linear orders, as follows.

Definition 2 (optimal definability) A definability $\langle X \rangle \lhd \mathcal{F}$ is *optimal* if $\langle X \rangle \not\lhd \mathcal{F}'$ for each fragment $\mathcal{F}'$ such that $\mathcal{F}' \prec \mathcal{F}$.

In order to show non-definability of a given modality in an HS fragment, we use a standard technique in modal logic, based on the notion of *bisimulation* and the invariance of modal formulae with respect to bisimulations (see, e.g., [6,21]). Let $\mathcal{F}$ be an HS fragment. An $\mathcal{F}$-bisimulation between two interval models $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V' \rangle$ over $\mathcal{AP}$ is a relation $Z \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ satisfying the following properties:

- *local condition*: Z-related intervals satisfy the same atomic propositions in $\mathcal{AP}$;
- *forward condition*: if $[a,b]Z[a',b']$ and $[a,b]R_X[c,d]$ for some $\langle X \rangle \in \mathcal{F}$, then there exists some $[c',d']$ such that $[a',b']R_X[c',d']$ and $[c,d]Z[c',d']$;
- *backward condition*: if $[a,b]Z[a',b']$ and $[a',b']R_X\ [c',d']$ for some $\langle X \rangle \in \mathcal{F}$, then there exists some $[c,d]$ such that $[a,b]R_X[c,d]$ and $[c,d]Z[c',d']$.

The important property of bisimulations used here is that any $\mathcal{F}$-bisimulation preserves the truth of *all* $\mathcal{F}$-formulae, that is, if $([a,b],[a',b']) \in Z$ and Z is an $\mathcal{F}$-bisimulation, then $[a,b]$ and $[a',b']$ satisfy exactly the same $\mathcal{F}$-formulae. Thus, in order to prove that a modality $\langle X \rangle$ is *not* definable in $\mathcal{F}$, it suffices to construct a pair of interval models $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V' \rangle$, and an $\mathcal{F}$-bisimulation Z between them, relating a pair of intervals $[a,b] \in \mathbb{I}(\mathbb{D})$ and $[a',b'] \in \mathbb{I}(\mathbb{D}')$, such that $M,[a,b] \Vdash \langle X \rangle p$ and $M',[a',b'] \nVdash \langle X \rangle p$. In this case, we say that Z *violates* $\langle X \rangle$. It is worth pointing out that non-definability results obtained using bisimulations are not restricted to the finitary logics we consider in this paper, but also apply to extensions with infinite disjunctions and with fixed-point operators.

3 A summary of the results

As we have already pointed out, every subset of the set of the 12 modalities corresponding to Allen's relations gives rise to a fragment of HS. There are 2^{12} (the cardinality of the powerset of the set of HS modalities) such fragments. Due to possible definabilities of some of these modalities in terms of others, not all these fragments are expressively different. We consider here the problem of obtaining a complete classification of all HS fragments with respect to their expressive power over the

Bibliography	Equations	Definabilities	Linear orders [# of fragments]
[20]	$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$	$\langle L\rangle \lhd \mathsf{A}$	Lin [1347] (and thus Den)
	$\langle \overline{L}\rangle p \equiv \langle \overline{A}\rangle\langle \overline{A}\rangle p$	$\langle \overline{L}\rangle \lhd \overline{\mathsf{A}}$	
	$\langle O\rangle p \equiv \langle E\rangle\langle \overline{B}\rangle p$	$\langle O\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$	
	$\langle \overline{O}\rangle p \equiv \langle B\rangle\langle \overline{E}\rangle p$	$\langle \overline{O}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$	
	$\langle D\rangle p \equiv \langle E\rangle\langle B\rangle p$	$\langle D\rangle \lhd \mathsf{BE}$	
	$\langle \overline{D}\rangle p \equiv \langle \overline{E}\rangle\langle \overline{B}\rangle p$	$\langle \overline{D}\rangle \lhd \overline{\mathsf{BE}}$	
[this paper]	$\langle L\rangle p \equiv \langle \overline{B}\rangle [E] \langle \overline{B}\rangle\langle E\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$	
	$\langle \overline{L}\rangle p \equiv \langle \overline{E}\rangle [B] \langle \overline{E}\rangle\langle B\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$	
	$\langle L\rangle p \equiv \langle O\rangle(\langle O\rangle\top \wedge [O]\langle D\rangle\langle O\rangle p)$	$\langle L\rangle \lhd \mathsf{DO}$	Den [966] (but not Lin)
	$\langle \overline{L}\rangle p \equiv \langle \overline{O}\rangle(\langle \overline{O}\rangle\top \wedge [\overline{O}]\langle D\rangle\langle \overline{O}\rangle p)$	$\langle \overline{L}\rangle \lhd \mathsf{D}\overline{\mathsf{O}}$	
	$\langle L\rangle p \equiv \langle \overline{B}\rangle [D] \langle \overline{B}\rangle\langle D\rangle\langle \overline{B}\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{D}$	
	$\langle \overline{L}\rangle p \equiv \langle \overline{E}\rangle [D] \langle \overline{E}\rangle\langle D\rangle\langle \overline{E}\rangle p$	$\langle \overline{L}\rangle \lhd \overline{\mathsf{E}}\mathsf{D}$	
	$\langle L\rangle p \equiv \langle O\rangle [E] \langle O\rangle\langle O\rangle p$	$\langle L\rangle \lhd \mathsf{EO}$	
	$\langle \overline{L}\rangle p \equiv \langle \overline{O}\rangle [B] \langle \overline{O}\rangle\langle \overline{O}\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{O}}$	
	$\langle L\rangle p \equiv \langle O\rangle(\langle O\rangle\top \wedge [O]\langle B\rangle\langle O\rangle\langle O\rangle p)$	$\langle L\rangle \lhd \mathsf{BO}$	
	$\langle \overline{L}\rangle p \equiv \langle \overline{O}\rangle(\langle \overline{O}\rangle\top \wedge [\overline{O}]\langle E\rangle\langle \overline{O}\rangle\langle \overline{O}\rangle p)$	$\langle \overline{L}\rangle \lhd \mathsf{E}\overline{\mathsf{O}}$	
	$\langle L\rangle p \equiv \langle O\rangle(\langle O\rangle\top \wedge [O][\overline{L}]\langle O\rangle\langle O\rangle p)$	$\langle L\rangle \lhd \overline{\mathsf{L}}\mathsf{O}$	
	$\langle \overline{L}\rangle p \equiv \langle \overline{O}\rangle(\langle \overline{O}\rangle\top \wedge [\overline{O}][L]\langle \overline{O}\rangle\langle \overline{O}\rangle p)$	$\langle \overline{L}\rangle \lhd \mathsf{L}\overline{\mathsf{O}}$	

Table 1 Complete set of optimal definabilities.

considered classes of linear orders. In other words, for any two HS fragments, we want to determine how they relate to each other with respect to expressiveness, that is, whether one is strictly less expressive than the other, or they are expressively equivalent, or incomparable.

In order to obtain such a classification, all we need to do is to provide a provably complete set of optimal definabilities between HS modalities. Indeed, having such a set, it is immediate to decide how any two given fragments relate with respect to their expressiveness. Table 1 presents such a complete set of optimal definabilities, partitioned in three groups (top, middle, and bottom). Some of them were already known from [20] to hold with respect to the class of all linear orders Lin (group on the top) and, consequently, with respect to the class of all dense linear orders Den; the rest (group in the middle and group at the bottom) are the subject of the present work: the definabilities in the group in the middle hold for both classes Lin and Den; the ones in the group at the bottom only hold for the class Den.

This paper is devoted to proving that Table 1 does present a complete set of optimal definabilities for all HS operators. This means that, for each operator $\langle X\rangle$, there are no more optimal definabilities of $\langle X\rangle$ in any HS fragment, apart from those indicated in Table 1.

To this end, as a first step, we need to identify for each operator $\langle X\rangle$ all maximal HS fragments not containing $\langle X\rangle$ as definable, according to the definabilities of Table 1. We call this task the MaxUndef problem. For those HS operators that are definable by means of only few definabilities, e.g., $\langle D\rangle$ and $\langle O\rangle$, or for those that are not definable at all in terms of the others, e.g., $\langle A\rangle$, $\langle B\rangle$, and $\langle E\rangle$, such a task is trivial and can be carried out by hand. However, in general solving MaxUndef turns out to be quite time-consuming when the operator under consideration has a large number of definabilities (this is the case, for instance, with the HS operator $\langle L\rangle$ and the operators of the logic studied in [5]). To solve the MaxUndef problem for the modalities $\langle L\rangle$ and $\langle \overline{L}\rangle$, we have used the automated procedure designed and implemented in [1].

Operators	Lin	Den	$\mathcal{M}(X)$: maximal $\mathcal{F}$ s.t. $\langle X\rangle \ntriangleleft \mathcal{F}$	$\mu(X)$: minimal $\mathcal{F}'$ s.t. $\mathcal{F}' \equiv \mathcal{F}$
$\langle L\rangle$/ $\langle \overline{L}\rangle$	•		$\mathsf{BEDO\overline{ALEDO}}$ / $\mathsf{\overline{BEDO}ALEDO}$	$\mathsf{BEO\overline{AED}}$ / $\mathsf{\overline{BEO}AED}$
	•		$\mathsf{BDO\overline{ALBEDO}}$ / $\mathsf{\overline{BDO}ALBEDO}$	$\mathsf{BDO\overline{ABE}}$ / $\mathsf{\overline{BDO}ABE}$
$\langle L\rangle$/ $\langle \overline{L}\rangle$		•	$\mathsf{O\overline{BEDO}}$ / $\mathsf{O\overline{BEDO}}$	$\mathsf{O\overline{BEO}}$ / $\mathsf{O\overline{BEO}}$
		•	$\mathsf{BED\overline{ALEDO}}$ / $\mathsf{BEDAL\overline{BD}O}$	$\mathsf{BE\overline{AED}}$ / $\mathsf{BEA\overline{BD}}$
		•	$\mathsf{B\overline{ALBEDO}}$ / $\mathsf{EAL\overline{BEDO}}$	$\mathsf{B\overline{ABE}}$ / $\mathsf{EA\overline{BE}}$
$\langle E\rangle$/ $\langle \overline{E}\rangle$	•	•	$\mathsf{ALBDO\overline{ALBEDO}}$ / $\mathsf{\overline{ALBDO}ALBEDO}$	$\mathsf{ABDO\overline{ABE}}$ / $\mathsf{\overline{ABDO}ABE}$
$\langle B\rangle$/ $\langle \overline{B}\rangle$	•	•	$\mathsf{ALEDO\overline{ALBEDO}}$ / $\mathsf{\overline{ALEDO}ALBEDO}$	$\mathsf{AED\overline{ABEO}}$ / $\mathsf{\overline{AED}ABEO}$
$\langle A\rangle$/ $\langle \overline{A}\rangle$	•	•	$\mathsf{LBEDO\overline{ALBEDO}}$ / $\mathsf{\overline{LBEDO}ALBEDO}$	$\mathsf{BE\overline{ABE}}$ / $\mathsf{\overline{BE}ABE}$
$\langle D\rangle$/ $\langle \overline{D}\rangle$	•	•	$\mathsf{ALBO\overline{ALBEDO}}$ / $\mathsf{\overline{ALBO}ALBEDO}$	$\mathsf{ABO\overline{ABE}}$ / $\mathsf{\overline{ABO}ABE}$
	•	•	$\mathsf{ALEO\overline{ALBEDO}}$ / $\mathsf{\overline{ALEO}ALBEDO}$	$\mathsf{AE\overline{ABEO}}$ / $\mathsf{\overline{AE}ABEO}$
$\langle O\rangle$/ $\langle \overline{O}\rangle$	•	•	$\mathsf{ALBED\overline{ALEDO}}$ / $\mathsf{\overline{ALBED}ALEDO}$	$\mathsf{ABE\overline{AED}}$ / $\mathsf{\overline{ABE}AED}$
	•	•	$\mathsf{ALBD\overline{ALBEDO}}$ / $\mathsf{\overline{ALBD}ALBEDO}$	$\mathsf{ABD\overline{ABE}}$ / $\mathsf{\overline{ABD}ABE}$

Table 2 Maximal fragments that do not define $\langle X\rangle$ according to definabilities in Table 1.

It is worth pointing out that the MaxUndef problem is interesting in its own right, thanks to its connections, established in [1], with other well-known classic problems in different areas of computer science, such as the the problem of finding all the maximal models of a given Horn theory (which has been shown to be polynomially equivalent to MaxUndef), or the problem of enumerating all the hitting sets of a given hyper-graph (which can be seen as a restriction of MaxUndef to a specific, well-defined class of instances — see [1] for a detailed account).

Table 2 shows the outcome of this preliminary step. Building on it, it is possible to disprove the existence of more definabilities using the notion of bisimulation as described at the end of Section 2.

In what follows, we first prove the validity of the new definabilities given in this paper, that is, the ones that appear in the middle and bottom groups in Table 1; then, following the above-described pattern, we prove that Table 1 contains a complete set of optimal definabilities relative to each of the classes Lin and Den. While proving soundness of the given sets of definability equations is quite straightforward, proving their completeness is a non-trivial task, which requires a deep understanding of the expressive power of a fragment of HS and the, often very delicate, construction of bisimulations relating carefully constructed interval models. We note that, even though all the definabilities for all the operators but $\langle L\rangle$ and $\langle \overline{L}\rangle$ were known since [20], no proof of their completeness was available so far.

4 Soundness

We only need to prove the soundness of the set of definability equations listed in the second and third groups of Table 1. Since all classes of linear orders considered here are (left/right) symmetric, we can restrict our attention to the equations for $\langle L\rangle$ (the soundness proofs for those for $\langle \overline{L}\rangle$ are symmetric).

Lemma 1 (soundness for Lin) *The set of definability equations given in Table 1 for the class* Lin *of all linear orders is sound.*

Proof As we have already pointed out above, we only have to prove that the equivalence $\langle L\rangle p \equiv \langle\overline{B}\rangle[E]\langle\overline{B}\rangle\langle E\rangle p$ holds over Lin (the validity of $\langle\overline{L}\rangle p \equiv \langle\overline{E}\rangle[B]\langle\overline{E}\rangle\langle B\rangle p$ follows by symmetry). First, we prove the left-to-right direction. To this end, suppose that $M, [a, b] \Vdash \langle L\rangle p$ for some model M and interval $[a, b]$. This means that there exists an interval $[c, d]$ such that $b < c$ and $M, [c, d] \Vdash p$. We exhibit an interval $[a, y]$, with $y > b$ such that, for every x (strictly) in between a and y, the interval $[x, y]$ is such that there exist two points y' and x' such that $y' > y$, $x < x' < y'$, and $[x', y']$ satisfies p. Let y be equal to c. The interval $[a, c]$, which is started by $[a, b]$, is such that for any of its ending intervals, that is, for any interval of the form $[x, c]$, with $a < x$, we have that $x < c < d$ and $M, [c, d] \Vdash p$. As for the other direction, we must show that $\langle\overline{B}\rangle[E]\langle\overline{B}\rangle\langle E\rangle p$ implies $\langle L\rangle p$. To this end, suppose that $M, [a, b] \Vdash \langle\overline{B}\rangle[E]\langle\overline{B}\rangle\langle E\rangle p$ for a model M and an interval $[a, b]$. Then, there exists an interval $[a, c]$, for some $c > b$, such that $[E]\langle\overline{B}\rangle\langle E\rangle p$ is true on $[a, c]$. As a consequence, the interval $[b, c]$ must satisfy $\langle\overline{B}\rangle\langle E\rangle p$, that means that there are two points x and y such that $y > c$, $b < x < y$, and $[x, y]$ satisfies p. Since $x > b$, it follows that $M, [a, b] \Vdash \langle L\rangle p$. □

Lemma 2 (soundness for Den**)** *The set of definability equations given in Table 1 for the class* Den *of all dense linear orders is sound.*

Proof Consider the equivalence $\underline{\langle L\rangle p \equiv \langle O\rangle(\langle O\rangle\top \wedge [O]\langle D\rangle\langle O\rangle p)}$ interpreted over the class Den. First, suppose that $M, [a, b] \Vdash \langle L\rangle p$ for an interval $[a, b]$ in a model M. We want to prove that $M, [a, b] \Vdash \langle O\rangle(\langle O\rangle\top \wedge [O]\langle D\rangle\langle O\rangle p)$ holds as well. By $M, [a, b] \Vdash \langle L\rangle p$, it follows that there exists an interval $[c, d]$ in M such that $b < c$ and $M, [c, d] \Vdash p$. Consider an interval $[a', c]$, with $a < a' < b$ (the existence of such a point a' is guaranteed by the density of the linear order). It is such that $[a, b]R_O[a', c]$ and it satisfies:

- $\langle O\rangle\top$, as $[a', c]R_O[b, d]$, and
- $[O]\langle D\rangle\langle O\rangle p$, as every interval $[e, f]$, with $[a', c]R_O[e, f]$, is such that $e < c < f$, and thus, by density, there exists an interval $[e', f']$ such that $[e, f]R_D[e', f']$ and $[e', f']R_O[c, d]$, which implies $M, [e, f] \Vdash \langle D\rangle\langle O\rangle p$, which, in turn, implies $M, [a', c] \Vdash [O]\langle D\rangle\langle O\rangle p$.

Hence, $M, [a', c] \Vdash \langle O\rangle\top \wedge [O]\langle D\rangle\langle O\rangle p$ and $M, [a, b] \Vdash \langle O\rangle(\langle O\rangle\top \wedge [O]\langle D\rangle\langle O\rangle p)$. As for the opposite direction, let us assume that $M, [a, b] \Vdash \langle O\rangle(\langle O\rangle\top \wedge [O]\langle D\rangle\langle O\rangle p)$ for an interval $[a, b]$ in a model M. That means that there exists an interval $[c, d]$, with $[a, b]R_O[c, d]$, such that:

- $M, [c, d] \Vdash \langle O\rangle\top$, and thus there exists a point $e > d$, and
- $M, [c, d] \Vdash [O]\langle D\rangle\langle O\rangle p$.

The interval $[b, e]$ is such that $[c, d]R_O[b, e]$, and thus, by the second condition above, it satisfies $\langle D\rangle\langle O\rangle p$. Therefore, there exist an interval $[f, g]$ such that $[b, e]R_D[f, g]$, and an interval $[h, i]$ such that $[f, g]R_O[h, i]$ and p holds over $[h, i]$. Since $h > b$, we conclude that $M, [a, b] \Vdash \langle L\rangle p$.

Now, consider $\underline{\langle L\rangle p \equiv \langle\overline{B}\rangle[D]\langle\overline{B}\rangle\langle D\rangle\langle\overline{B}\rangle p}$. Suppose that $M, [a, b] \Vdash \langle L\rangle p$ for an interval $[a, b]$ in a model M. Thus, as before, there exists an interval $[c, d]$ in M such that $b < c$ and $M, [c, d] \Vdash p$. By definition of $R_{\overline{B}}$, it holds that $[a, b]R_{\overline{B}}[a, c]$. We now show that $[a, c]$ satisfies $[D]\langle\overline{B}\rangle\langle D\rangle\langle\overline{B}\rangle p$. First, every interval $[e, f]$, with $[a, c]R_D[e, f]$, is such that $e < c$. We claim that $M, [e, f] \Vdash \langle\overline{B}\rangle\langle D\rangle\langle\overline{B}\rangle p$. To see this, let us consider the interval $[e, d]$. We observe that $[e, f]R_{\overline{B}}[e, d]$ holds. Moreover, by the density of M, there exists a point d', with $c < d' < d$, such that $[e, d]R_D[c, d']$ holds and $[c, d']$ satisfies $\langle\overline{B}\rangle p$, because p holds over $[c, d]$ and $[c, d']R_{\overline{B}}[c, d]$. Thus, $M, [e, f] \Vdash \langle\overline{B}\rangle\langle D\rangle\langle\overline{B}\rangle p$, as claimed. As for the opposite direction, suppose that

$M, [a, b] \Vdash \langle\overline{B}\rangle[D]\langle\overline{B}\rangle\langle D\rangle\langle\overline{B}\rangle p$ for an interval $[a, b]$ in a model M. That means that there exists a point $c > b$ such that the interval $[a, c]$ satisfies $[D]\langle\overline{B}\rangle\langle D\rangle\langle\overline{B}\rangle p$. As a particular instance of the latter formula, every interval $[e, f]$ such that $b < e < f < c$ (the existence of such an interval $[e, f]$ is guaranteed by the density of M) must satisfy $\langle\overline{B}\rangle\langle D\rangle\langle\overline{B}\rangle p$ which means that there exists a point $g > f$ such that $M, [e, g] \Vdash \langle D\rangle\langle\overline{B}\rangle p$, which implies, in turn, the existence of two points h, i, with $e < h < i$, such that $M, [h, i] \Vdash p$. Since $h > b$, we have that $M, [a, b] \Vdash \langle L\rangle p$.

Next, let us focus on $\underline{\langle L\rangle p \equiv \langle O\rangle[E]\langle O\rangle\langle O\rangle p}$. Suppose that $M, [a, b] \Vdash \langle L\rangle p$ for an interval $[a, b]$ in a model M. Once again, this means that there exists an interval $[c, d]$ in M such that $b < c$ and $M, [c, d] \Vdash p$. Consider an interval $[a', c]$, with $a < a' < b$ (the existence of such a point a' is guaranteed by the density of M). It holds that $[a, b]R_O[a', c]$. We prove that $M, [a', c] \Vdash [E]\langle O\rangle\langle O\rangle p$. Indeed, for every interval $[e, c]$, with $[a', c]R_E[e, c]$, by the density of M, there exist a point f, with $e < f < c$, and a point g, with $c < g < d$, such that the interval $[f, g]$ satisfies $\langle O\rangle p$ as $[f, g]R_O[c, d]$. Thus, $M, [e, c] \Vdash \langle O\rangle\langle O\rangle p$, $M, [a', c] \Vdash [E]\langle O\rangle\langle O\rangle p$, and $M, [a, b] \Vdash \langle O\rangle[E]\langle O\rangle\langle O\rangle p$. In order to prove the converse direction, suppose that $M, [a, b] \Vdash \langle O\rangle[E]\langle O\rangle\langle O\rangle p$ for an interval $[a, b]$ in a model M. That means that there exists an interval $[c, d]$ such that $[a, b]R_O[c, d]$ and $M, [c, d] \Vdash [E]\langle O\rangle\langle O\rangle p$. As a particular instance, the interval $[e, d]$, for some e such that $b < e < d$ (the existence of such a point e is guaranteed by the density of M), satisfies $\langle O\rangle\langle O\rangle p$, that implies the existence of an interval $[f, g]$, with $f > e$ $(> b)$, satisfying p. It immediately follows that $M, [a, b] \Vdash \langle L\rangle p$.

Consider now $\underline{\langle L\rangle p \equiv \langle O\rangle(\langle O\rangle\top \wedge [O]\langle B\rangle\langle O\rangle\langle O\rangle p)}$. Suppose that $M, [a, b] \Vdash \langle L\rangle p$ for an interval $[a, b]$ in a model M, which implies, as ever, that there exists an interval $[c, d]$ in M such that $b < c$ and $M, [c, d] \Vdash p$. Consider an interval $[a', c]$, with $a < a' < b$ (the existence of such a point a' is guaranteed by the density of M). This interval is such that $[a, b]R_O[a', c]$ and it satisfies:

- $\langle O\rangle\top$, as $[a', c]R_O[b, d]$, and
- $[O]\langle B\rangle\langle O\rangle\langle O\rangle p$, as every interval $[e, f]$, with $[a', c]R_O[e, f]$, is such that $e < c < f$; thus, the interval $[e, c]$ is such that $[e, f]R_B[e, c]$, and, by the density of M, there exists an interval $[g, h]$ such that $[e, c]R_O[g, h]$ and $[g, h]R_O[c, d]$, and this implies $M, [e, c] \Vdash \langle O\rangle\langle O\rangle p$, which, in turn, implies $M, [a', c] \Vdash [O]\langle B\rangle\langle O\rangle\langle O\rangle p$.

Hence, $M, [a', c] \Vdash \langle O\rangle\top \wedge [O]\langle B\rangle\langle O\rangle\langle O\rangle p$, and thus $M, [a, b] \Vdash \langle O\rangle(\langle O\rangle\top \wedge [O]\langle B\rangle\langle O\rangle\langle O\rangle p)$. Conversely, suppose that $M, [a, b] \Vdash \langle O\rangle(\langle O\rangle\top \wedge [O]\langle B\rangle\langle O\rangle\langle O\rangle p)$ for an interval $[a, b]$ in a model M. That means that there exists an interval $[c, d]$ such that:

- $[a, b]R_O[c, d]$,
- $M, [c, d] \Vdash \langle O\rangle\top$, and thus there exists a point $f > d$, and
- $M, [c, d] \Vdash [O]\langle B\rangle\langle O\rangle\langle O\rangle p$.

By the density of M, there exists a point e, with $b < e < d$. The interval $[e, f]$ is such that $[c, d]R_O[e, f]$, and thus, by the third condition above, it satisfies $\langle B\rangle\langle O\rangle\langle O\rangle p$, which implies the existence of an interval $[g, h]$, with $g > e(> b)$, satisfying p. It immediately follows that $M, [a, b] \Vdash \langle L\rangle p$.

Finally, consider $\underline{\langle L\rangle p \equiv \langle O\rangle(\langle O\rangle\top \wedge [O][\overline{L}]\langle O\rangle\langle O\rangle p)}$. Suppose that $M, [a,\ b] \Vdash \langle L\rangle p$ for an interval $[a, b]$ in a model M. Thus, there exists an interval $[c, d]$ in M such that $b < c$ and $M, [c, d] \Vdash p$. Consider an interval $[a', c]$, with $a < a' < b$ (the existence of such a point a' is guaranteed by the density of M). This interval is such that $[a,b]R_O[a',c]$ and it satisfies both $\langle O\rangle\top$, as $[a',c]R_O[b,d]$, and $[O][\overline{L}]\langle O\rangle\langle O\rangle p$, thanks to the following argument. Every interval $[e,f]$, with $[a',c]R_O[e,f]$, is such that $e<c$. Thus, every interval $[g,h]$, with $[e,f]R_{\overline{L}}[g,h]$, satisfies $\langle O\rangle\langle O\rangle p$ (by the density of M, there exist $g<i<h$ and $c<j<d$ such that both $[g,h]R_O[i,j]$ and $[i,j]R_O[c,d]$ hold). Therefore, we have that $M,[a',c]\Vdash[O][\overline{L}]\langle O\rangle\langle O\rangle p$, which implies $M,[a,b]\Vdash\langle O\rangle(\langle O\rangle\top\wedge[O][\overline{L}]\langle O\rangle\langle O\rangle p)$. As for the other direction, suppose that $M,[a,b]\Vdash\langle O\rangle(\langle O\rangle\top\wedge[O][\overline{L}]\langle O\rangle\langle O\rangle p)$ for an interval $[a,b]$ in a model M. That means that there exists an interval $[c,d]$ such that $[a,b]R_O[c,d]$, $M,[c,d]\Vdash\langle O\rangle\top$ (and thus, there exists a point $f>d$), and that $M,[c,d]\Vdash[O][\overline{L}]\langle O\rangle\langle O\rangle p$. As a specific instance, consider the interval $[e,f]$, for some e such that $b<e<d$ (the existence of such a point e is guaranteed by the density of M). Since $[c,d]R_O[e,f]$, then we have $M,[e,f]\Vdash[\overline{L}]\langle O\rangle\langle O\rangle p$, which, in turn, together with the density assumption, implies the existence of an interval $[g,h]$, with $b<g<h<e$, that satisfies $\langle O\rangle\langle O\rangle p$. Thus, there exists an interval $[i,j]$, with $i>g(>b)$, which satisfies p. It immediately follows that $M,[a,b]\Vdash\langle L\rangle p$. □

5 Completeness

As we have already pointed out, proving completeness of the set of definabilities is the most difficult task in obtaining the expressiveness classification we seek. Following the general pattern described in Section 3, we first compute, for each operator $\langle X\rangle$, the set $\mathcal{M}(X)$ (4th column of Table 2), containing all the maximal fragments $\mathcal{F}$ not containing $\langle X\rangle$ as definable, according to the definabilities of Table 1 (i.e., $\langle X\rangle\not\triangleleft\mathcal{F}$ for each $\mathcal{F}\in\mathcal{M}(X)$). Then, for each operator $\langle X\rangle$ and each $\mathcal{F}\in\mathcal{M}(X)$, we compute the minimal fragment $\mathcal{F}'$ such that $\mathcal{F}'\equiv\mathcal{F}$, according to the definabilities of Table 1 (note that there exists exactly one such a fragment $\mathcal{F}'$ for each operator $\langle X\rangle$ and each $\mathcal{F}\in\mathcal{M}(X)$). We collect such fragments in the set $\mu(X)=\{\mathcal{F}'\mid\mathcal{F}\in\mathcal{M}(X)$ and $\mathcal{F}'$ is the minimal fragment such that $\mathcal{F}'\equiv\mathcal{F}\}$ (fifth column of Table 2). Finally, we provide an $\mathcal{F}'$-bisimulation that violates $\langle X\rangle$, for each operator $\langle X\rangle$ and each $\mathcal{F}'\in\mu(X)$.

In fact, we will see that, due to the symmetry between HS modalities, we do not have to actually produce a new bisimulation for every pair $\langle X\rangle$, $\mathcal{F}'$, with $\mathcal{F}'\in\mu(X)$. Thus, before proceeding further, we formalize here the concepts of symmetric HS operators and symmetric HS fragments, which will be helpful to prove our results. We say that two HS operators $\langle X\rangle$ and $\langle Y\rangle$ are *symmetric* if and only if $(\langle X\rangle,\langle Y\rangle)\in S$, where S is the relation defined as $S=\{(\langle A\rangle,\langle\overline{A}\rangle),(\langle\overline{A}\rangle,\langle A\rangle),(\langle L\rangle,\langle\overline{L}\rangle),(\langle\overline{L}\rangle,\langle L\rangle),(\langle B\rangle,\langle E\rangle),(\langle\overline{B}\rangle,\langle\overline{E}\rangle),(\langle E\rangle,\langle B\rangle),(\langle\overline{E}\rangle,\langle\overline{B}\rangle),(\langle D\rangle,\langle D\rangle),(\langle\overline{D}\rangle,\langle\overline{D}\rangle),(\langle O\rangle,\langle\overline{O}\rangle),(\langle\overline{O}\rangle,\langle O\rangle)\}$. To define the notion of symmetric fragments, we lift the relation S to a relation between fragments, denoted by $\hat{S}$ and defined as $\hat{S}=\{(\mathcal{F}_1,\mathcal{F}_2)\mid\forall\langle X\rangle\in\mathcal{F}_1\ \exists\langle Y\rangle\in\mathcal{F}_2.(\langle X\rangle,\langle Y\rangle)\in S$ and $\forall\langle Y\rangle\in\mathcal{F}_2\ \exists\langle X\rangle\in\mathcal{F}_1.(\langle Y\rangle,\langle X\rangle)\in S\}$. We say that two fragments $\mathcal{F}_1$ and $\mathcal{F}_2$ are *symmetric* if and only if $(\mathcal{F}_1,\mathcal{F}_2)\in\hat{S}$. Not surprisingly, both relations S and $\hat{S}$ are symmetric. In addition, notice that they are, in fact, functions. Therefore, we may denote by $S(\langle X\rangle)$ (resp., $\hat{S}(\mathcal{F}_1)$) the unique $\langle Y\rangle$ (resp., $\mathcal{F}_2$) such that $(\langle X\rangle,\langle Y\rangle)\in S$ (resp., $(\mathcal{F}_1,\mathcal{F}_2)\in\hat{S}$).

5.1 Completeness for $\langle L\rangle/\langle\overline{L}\rangle$: Case Lin

Lemma 3 *Table 1 presents a complete set of optimal definabilities for $\langle L\rangle$ and $\langle\overline{L}\rangle$ relative to the class* Lin.

Proof We first prove the claim for $\langle L\rangle$, and then we show how the claim for $\langle\overline{L}\rangle$ follows from a simple argument based on symmetry of operators and fragments. According to Table 2, in order to deal with the operator $\langle L\rangle$, we need to provide two bisimulations, namely a $\mathsf{BEO\overline{AED}}$-bisimulation and a $\mathsf{BDO\overline{ABE}}$-bisimulation, that violate $\langle L\rangle$.

Case $\mathsf{BEO\overline{AED}}$. Let $M_1 = \langle\mathbb{I}(\mathbb{N}), V_1\rangle$ and $M_2 = \langle\mathbb{I}(\mathbb{N}), V_2\rangle$ be two models and let V_1 and V_2 be such that $V_1(p) = \{[2, 3]\}$ and $V_2(p) = \emptyset$, where p is the only proposition letter of the language. Moreover, let Z be a relation between (intervals of) M_1 and M_2 defined as:

$$Z = \{([0, 1], [0, 1])\}.$$

It can be easily shown that Z is a $\mathsf{BEO\overline{AED}}$-bisimulation. The local property is trivially satisfied, since all Z-related intervals satisfy $\neg p$. As for the forward and backward conditions, it suffices to notice that, starting from the interval $[0, 1]$, it is not possible to reach any other interval using any of the modal operators of the fragment. At the same time, Z violates $\langle L\rangle$. Indeed, $([0, 1], [0, 1]) \in Z$ and $M_1, [0, 1] \Vdash \langle L\rangle p$, but $M_2, [0, 1] \Vdash \neg\langle L\rangle p$.

Case $\mathsf{BDO\overline{ABE}}$. Let $M_1 = \langle\mathbb{I}(\mathbb{Z}^-), V_1\rangle$ and $M_2 = \langle\mathbb{I}(\mathbb{Z}^-), V_2\rangle$ be two models based on the set $\mathbb{Z}^- = \{\ldots, -2, -1\}$ of the negative integers, and let V_1 and V_2 be such that $V_1(p) = \{[-2, -1]\}$ and $V_2(p) = \emptyset$, where p is the only proposition letter of the language. Moreover, let Z be the relation between (intervals of) M_1 and M_2 defined as follows:

$$([x, y], [w, z]) \in Z \Leftrightarrow [x, y] = [w, z] \text{ and } [x, y] \neq [-2, -1].$$

We prove that Z is a $\mathsf{BDO\overline{ABE}}$-bisimulation. First, the local property is trivially satisfied, since all Z-related intervals satisfy $\neg p$. Moreover, starting from any interval, the only interval that satisfies p, viz., $[-2, -1]$, cannot be reached using the set of modal operators featured by our fragment. At the same time, Z violates $\langle L\rangle$, as $([-4, -3], [-4, -3]) \in Z$ and $M_1, [-4, -3] \Vdash \langle L\rangle p$, but $M_2, [-4, -3] \Vdash \neg\langle L\rangle p$. Thus, we can conclude that there are no more definabilities for $\langle L\rangle$ over Lin, apart from those listed in Table 1.

Now suppose, towards a contradiction, that a new definability for $\langle\overline{L}\rangle$ in some fragment $\mathcal{F}$ exists, due to the definability equation ξ. It is easy to see that, since $\langle L\rangle$ and $\langle\overline{L}\rangle$ are symmetric, there must also be a new definability for $\langle L\rangle$ in $\hat{S}(\mathcal{F})$ (the corresponding equation would be obtained by replacing every operator $\langle X\rangle$ occurring in ξ with its symmetric one $S(\langle X\rangle)$). But this contradicts the fact that there are no more definabilities for $\langle L\rangle$, as we just proved. Thus, the claim holds for $\langle\overline{L}\rangle$ as well. □

5.2 Completeness for $\langle L\rangle/\langle\overline{L}\rangle$: Case Den

The case Den is more complicate than the case Lin. The bisimulations of this section, one for each of the three fragments indicated in Table 2, namely $\mathsf{O\overline{BEO}}$,

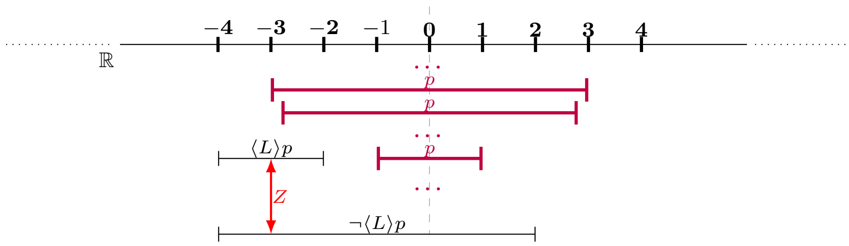

Fig. 2 $\mathsf{O\overline{BEO}}$-bisimulation that violates $\langle L\rangle$, relative to Den.

$\mathsf{BE\overline{AED}}$, and $\mathsf{B\overline{ABE}}$, make use of the following observation. If $\mathbb{D}$ is a dense linear order without least and greatest elements, then for each $[a,b] \in \mathbb{I}(\mathbb{D})$ and $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$ there exists an interval $[c,d] \in \mathbb{I}(\mathbb{D})$ such that $[a,b]R_X[c,d]$. Moreover, we prove here a (rather straightforward) technical result that will be used in the second construction in the proof of Lemma 5 (case $\mathsf{BE\overline{AED}}$). Consider the function $f : \mathbb{R} \to \{x \in \mathbb{R} \mid x < 1\}$, defined as follows:

$$f(x) = \begin{cases} x - 1 & \text{if } x \leq 1 \\ 1 - \frac{1}{x} & \text{if } x > 1 \end{cases}$$

Lemma 4 *The function f is a monotonically increasing bijection from $\mathbb{R}$ to $(-\infty, 1)$ such that $f(x) < x$ for every $x \in \mathbb{R}$.*

Proof Let $f' : \{x \in \mathbb{R} \mid x \leq 1\} \to \{x \in \mathbb{R} \mid x \leq 0\}$ and $f'' : \{x \in \mathbb{R} \mid x > 1\} \to \{x \in \mathbb{R} \mid 0 < x < 1\}$ be defined as $f'(x) = x - 1$ and $f''(x) = 1 - \frac{1}{x}$, respectively. Clearly, f' and f'' are bijective functions. Moreover, it is easy to verify that f' and f'' are such that (i) they are monotonically increasing and (ii) $f'(x) < x$ (resp., $f''(x) < x$) for every $x \in \mathtt{dom}f'$ (resp., $x \in \mathtt{dom}f''$). Observe that $\mathtt{dom}f'$ and $\mathtt{dom}f''$ (resp., $\mathtt{codom}f'$ and $\mathtt{codom}f''$) partition $\mathtt{dom}f$ (resp, $\mathtt{codom}f$). Now, it is easy to observe that f is well defined. To verify that it is an injection, consider $x, x' \in \mathbb{R}$, with $x \neq x'$. If $x, x' \leq 1$ (resp., $x, x' > 1$), it holds $f(x) = f'(x) \neq f'(x') = f(x')$ (resp., $f(x) = f''(x) \neq f''(x') = f(x')$), as f' (resp., f'') is an injection; if $x \leq 1$ and $x' > 1$, then it holds $f(x) = f'(x) \neq f''(x) = f(x')$, as the codomains of f' and f'' are disjoint sets. Surjectivity of f follows from the surjectivity of f' and f''. Thus, f is a bijection. To prove that it is monotonically increasing, consider $x, x' \in \mathbb{R}$, with $x < x'$. If $x, x' \leq 1$ (resp., $x, x' > 1$), it holds that $f(x) < f(x')$, as f' (resp., f'') is monotonically increasing; if $x \leq 1$ and $x' > 1$, then it holds that $f(x) < f(x')$, as every element in the image of f' is less than every element in the image of f''. Finally, from the fact that $f'(x) < x$ for every $x \in \mathbb{R}$, with $x \leq 1$, and that $f''(x) < x$ for every $x \in \mathbb{R}$, with $x > 1$, it follows that $f(x) < x$ for every $x \in \mathbb{R}$. □

Lemma 5 *Table 1 presents a complete set of optimal definabilities for $\langle L\rangle$ and $\langle\overline{L}\rangle$ relative to the class* Den.

Proof We give an explicit proof only for the operator $\langle L\rangle$. The same argument used at the end of the proof of Lemma 3, based on symmetry, can be applied to show that the claim holds for $\langle\overline{L}\rangle$ as well. According to Table 2, we need to provide a bisimulation that violates $\langle L\rangle$ for three fragments of HS.

<u>Case $\mathsf{O\overline{BEO}}$</u>. Consider the two interval models M and M', defined as $M = M' = \langle \mathbb{I}(\mathbb{R}), V\rangle$, where $V(p) = \{[-a, a] \mid a \in \mathbb{R}\}$ (observe that no interval $[c, d]$, with $c \geq 0$, satisfies p). Moreover, let $Z = \{([a, b], [a', b']) \mid -a \sim b \text{ and } -a' \sim b' \text{ for some } \sim\in \{<, =, >\}\}$ (see Fig. 2). It is immediate to check that $[-4, -2]Z[-4, 2]$, that $M, [-4, -2] \Vdash \langle L\rangle p$ (as $M, [-1, 1] \Vdash p$) and that $M', [-4, 2] \Vdash \neg\langle L\rangle p$ (as no interval $[c, d]$, with $c > 0$, satisfies p). In order to complete the proof for this fragment, we now proceed to show that Z is an $\mathsf{O\overline{BEO}}$-bisimulation. To this end, consider a pair $([a, b], [a', b'])$ of Z-related intervals. The following chain of equivalences hold:

$$M, [a, b] \Vdash p \Leftrightarrow -a = b \Leftrightarrow -a' = b' \Leftrightarrow M, [a', b'] \Vdash p.$$

This implies that the local condition is satisfied. As for the forward condition, consider three intervals $[a, b]$, $[a', b']$, and $[c, d]$ such that $[a, b]Z[a', b']$ and $[a, b]R_X[c, d]$ for some $X \in \{O, \overline{B}, \overline{E}, \overline{O}\}$. We need to exhibit an interval $[c', d']$ such that $[a', b']R_X[c', d']$ and $[c, d]Z[c', d']$. We distinguish three cases.

- If $-a > b$ and $-a' > b'$, then, as a preliminary step, we show that the following facts hold: (i) $a < 0$ and $a' < 0$; (ii) $|a| > |b|$ and $|a'| > |b'|$. We only show the proofs for $a < 0$ and $|a| > |b|$ and we omit the ones for $a' < 0$ and $|a'| > |b|$, which are analogous. As for the former claim above, it is enough to observe that, if $a \geq 0$, then $a \geq 0 \geq -a > b$, which implies $b < a$, leading to a contradiction with the fact that $[a, b]$ is an interval (thus $a < b$). Notice that, as an immediate consequence, we have that $|a| = -a$ holds. As for the latter claim above, firstly we suppose, by contradiction, that $|a| = |b|$ holds. Then, $-a = |a| = |b|$ holds and this implies either $b = -a$, contradicting the hypothesis that $-a > b$, or $b = a$, contradicting the fact that $[a, b]$ is an interval. Secondly, we suppose, again by contradiction, that $|a| < |b|$ holds. Then, by the former claim, we have that $0 < -a = |a| < |b|$ holds, which implies $b \neq 0$. Now, we show that both $b < 0$ and $b > 0$ lead to a contradiction. If $b < 0$, then $|b| = -b$, and thus it holds $-a < -b$, which amounts to $a > b$, contradicting the fact that $[a, b]$ is an interval. If $b > 0$, then $|b| = b$, and thus it holds $-a < b$, which contradicts the hypothesis that $-a > b$. This proves the two claims above. Now, we distinguish the following sub-cases.
  - If $X = O$, then $[c, d]$ is such that $a < c < b < d$. We distinguish the following cases.
    - If $-c > d$, then take some c' such that $a' < c' < -|b'| < 0$ (notice also that $c' < -|b'| \leq b'$ trivially holds), and d' such that $b' < d' < |c'| = -c'$ (the existence of such points c', d' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_O[c', d']$ and $[c, d]Z[c', d']$.
    - If $-c = d$, then take some c' such that $a' < c' < -|b'| < 0$, and $d' = -c'$ (the existence of such a point c' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_O[c', d']$ and $[c, d]Z[c', d']$.
    - If $-c < d$, then take c' such that $a' < c' < -|b'| < 0$, and any $d' > -c'$ (the existence of such a point c' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_O[c', d']$ and $[c, d]Z[c', d']$.
  - If $X = \overline{B}$, then $[c, d]$ is such that $a = c < b < d$. We distinguish the cases below.
    - If $-c > d$, then take $c' = a'$ and d' such that $b' < d' < -a' = -c'$ (the existence of such a point d' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_{\overline{B}}[c', d']$ and $[c, d]Z[c', d']$.

  - If $-c = d$, then take $c' = a'$ and $d' = -c'(= -a' > b')$. The interval $[c', d']$ is such that $[a', b']R_{\overline{B}}[c', d']$ and $[c, d]Z[c', d']$.
  - If $-c < d$, then take $c' = a'$ and any $d' > -c'(= -a' > b')$. The interval $[c', d']$ is such that $[a', b']R_{\overline{B}}[c', d']$ and $[c, d]Z[c', d']$.
- If $X = \overline{E}$, then $[c, d]$ is such that $c < a < b = d$. Notice that $|c| = -c > -a = |a|$ holds, because $c < a < 0$. Thus $-c > -a > b = d$ also holds. Then, take $d' = b'$ and any $c' < a'$. We have that $-c' > -a' > b' = d'$. The interval $[c', d']$ is therefore such that $[a', b']R_{\overline{E}}[c', d']$ and $[c, d]Z[c', d']$.
- If $X = \overline{O}$, then $[c, d]$ is such that $c < a < d < b$. Notice that $|c| = -c > -a = |a|$ holds, because $c < a < 0$. Thus $-c > -a > b > d$ also holds. Then, take some d' such that $a' < d' < b'$ and any $c' < a'$ (the existence of such a point d' is guaranteed by the density of $\mathbb{R}$). Thus, it holds $-c' > -a' > b' > d'$. The interval $[c', d']$ is therefore such that $[a', b']R_{\overline{O}}[c', d']$ and $[c, d]Z[c', d']$.

- If $-a = b$ and $-a' = b'$, then we have that $a < 0$ (resp., $a' < 0$) and $b > 0$ (resp., $b' > 0$). Indeed, if $a \geq 0$ held, then $b = -a \leq 0 \leq a$ would also hold, contradicting the fact that $[a, b]$ is an interval (and thus $b > a$). From $a < 0$ and $-a = b$, it immediately follows that $b > 0$. The facts that $a' < 0$ and $b' > 0$ can be shown analogously. Notice also that, from $-a = b$ and $-a' = b'$, it follows that $|a| = |b|$ and $|a'| = |b'|$. Now, we distinguish the following sub-cases.
  - If $X = O$, then $[c, d]$ is such that $a < c < b < d$. Notice that $-c \leq |c| < |a| = |b| = b < d$ holds. Then, take $c' = 0$ and any $d' > b'(> 0)$. We have that $-c' < d'$. The interval $[c', d']$ is such that $[a', b']R_O[c', d']$ and $[c, d]Z[c', d']$.
  - If $X = \overline{B}$, then $[c, d]$ is such that $a = c < b < d$. Notice that $-c = -a = b < d$ holds. Then, take $c' = a'$ and any $d' > b'$. We have that $-c' = -a' = b' < d'$. The interval $[c', d']$ is such that $[a', b']R_{\overline{B}}[c', d']$ and $[c, d]Z[c', d']$.
  - If $X = \overline{E}$, then $[c, d]$ is such that $c < a < b = d$. Notice that $|c| = -c > -a = |a|$ holds, because $c < a < 0$. Thus $-c > -a = b = d$ also holds. Then, take $d' = b'$ and any $c' < a'$. We have that $-c' > -a' = b' = d'$. The interval $[c', d']$ is such that $[a', b']R_{\overline{E}}[c', d']$ and $[c, d]Z[c', d']$.
  - If $X = \overline{O}$, then $[c, d]$ is such that $c < a < d < b$. Notice that $|c| = -c > -a = |a|$ holds, because $c < a < 0$. Thus $-c > -a = b > d$ also holds. Then, take $d' = 0$ and any $c' < a'(< 0)$. We have that $-c' > d'$. The interval $[c', d']$ is such that $[a', b']R_{\overline{O}}[c', d']$ and $[c, d]Z[c', d']$.
- If $-a < b$ and $-a' < b'$, then the following facts hold: (i) $b > 0$ (otherwise, $-a < b \leq 0$ would hold, which implies $a > 0 \geq b$, contradicting the fact that $[a, b]$ is an interval), (ii) $|b| = b$ (this follows directly from $b > 0$), and (iii) $|a| < |b|$ (otherwise, $|a| \geq |b| = b$ would hold, which implies either $a \geq b$, contradicting the fact that $[a, b]$ is an interval, or $-a \geq b$, contradicting the hypothesis that $-a < b$). Now, we distinguish the following sub-cases.
  - If $X = \overline{O}$, then $[c, d]$ is such that $c < a < d < b$. We distinguish the cases below.
    - If $-c < d$, then take some d' and c' such that $|a'| < d' < |b'| = b'$ and $-d' < c' < |a'| = -c$ (the existence of points c', d' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_{\overline{O}}[c', d']$ and $[c, d]Z[c', d']$.
    - If $-c = d$, then take some d' such that $|a'| < d' < |b'| = b'$ and $c' = -d'$ (the existence of such a point d' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_{\overline{O}}[c', d']$ and $[c, d]Z[c', d']$.

    - If $-c > d$, then take some d' and c' such that $|a'| < d' < |b'| = b'$ and $c' < -d'$ (the existence of points c', d' is guaranteed by the left-unboundedness and the density of $\mathbb{R}$, respectively). The interval $[c', d']$ is such that $[a', b']R_{\overline{O}}[c', d']$ and $[c, d]Z[c', d']$.
- If $X = \overline{E}$, then $[c, d]$ is such that $c < a < b = d$. We distinguish the following cases.
    - If $-c < d$, then take $d' = b'$ and some c' such that $-d' < c' < a'$ (the existence of such a point c' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_{\overline{E}}[c', d']$ and $[c, d]Z[c', d']$.
    - If $-c = d$, then take $d' = b'$ and $c' = -d'(= -b' < a')$. The interval $[c', d']$ is such that $[a', b']R_{\overline{E}}[c', d']$ and $[c, d]Z[c', d']$.
    - If $-c > d$, then take $d' = b'$ and any $c' < -d'(= -b' < a')$. The interval $[c', d']$ is such that $[a', b']R_{\overline{E}}[c', d']$ and $[c, d]Z[c', d']$.
- If $X = \overline{B}$, then $[c, d]$ is such that $a = c < b < d$. Notice that $-d < -b < a = c$. Then, take $c' = a'$ and any $d' > b'$. It holds that $c' = a' > -b' > -d'$. The interval $[c', d']$ is such that $[a', b']R_{\overline{B}}[c', d']$ and $[c, d]Z[c', d']$.
- If $X = O$, then $[c, d]$ is such that $a < c < b < d$. Notice that $-d < -b < a < c$. Then, take some c' such that $a' < c' < b'$ (the existence of such a point c' is guaranteed by the density of $\mathbb{R}$) and any $d' > b'$. It holds that $c' > a' > -b' > -d'$. The interval $[c', d']$ is such that $[a', b']R_O[c', d']$ and $[c, d]Z[c', d']$.

Since the relation Z is symmetric, the forward condition implies the backward condition, as follows. Consider a pair $([a, b], [a', b'])$ of Z-related intervals and an interval $[c', d']$ such that $[a', b']R_X[c', d']$, for some $X \in \{O, \overline{B}, \overline{E}, \overline{O}\}$. We need to find an interval $[c, d]$ such that $[a, b]R_X[c, d]$ and $[c, d]Z[c', d']$. By symmetry, $([a', b'], [a, b]) \in Z$, as well. By the forward condition, we know that for every interval $[c', d']$ such that $[a', b']R_X[c', d']$, for some $X \in \{O, \overline{B}, \overline{E}, \overline{O}\}$, there exists an interval $[c, d]$ such that $[a, b]R_X[c, d]$ and $[c', d']Z[c, d]$. By symmetry $[c, d]Z[c', d']$ also holds, hence the backward condition is fulfilled, too. Therefore, Z is an $\mathsf{O\overline{BEO}}$-bisimulation that violates $\langle L \rangle$.

<u>Case $\mathsf{BE\overline{AED}}$</u>. Consider two interval models M and M', defined as $M = M' = \langle \mathbb{I}(\mathbb{R}), V \rangle$, where $V(p) = \{[a, b] \mid a = f(b)\}$ and where f is the function defined at the beginning of Section 5.2. In addition, let $Z = \{([a, b], [a', b']) \mid a \sim f(b), a' \sim f(b')$ where $\sim \in \{<, =, >\}\}$ (see Fig. 3). It is immediate to check that $[-1, 0]Z[0, 1]$ (as $f(0) = -1$ and $f(1) = 0$), that $M, [-1, 0] \Vdash \langle L \rangle p$ (as $M, [0.5, 2] \Vdash p$ because $f(2) = 0.5$) and that $M', [0, 1] \Vdash \neg\langle L \rangle p$ (as no interval $[c, d]$, with $c > 1$, satisfies p because c is not in the image of f for each $c > 1$). Now, in order to show that Z is a $\mathsf{BE\overline{AED}}$-bisimulation, consider a pair $([a, b], [a', b'])$ of Z-related intervals. The following chain of double implications holds:

$$M, [a, b] \Vdash p \Leftrightarrow a = f(b) \Leftrightarrow a' = f(b') \Leftrightarrow M', [a', b'] \Vdash p.$$

This implies that the local condition holds. As for the forward condition, consider three intervals $[a, b]$, $[a', b']$, and $[c, d]$ such that $[a, b]Z[a', b']$ and $[a, b]R_X[c, d]$ for some $X \in \{B, E, \overline{A}, \overline{E}, \overline{D}\}$. We need to exhibit an interval $[c', d']$ such that $[a', b']R_X[c', d']$ and $[c, d]Z[c', d']$. We distinguish three cases.

- If $a > f(b)$ and $a' > f(b')$, then we distinguish the following sub-cases.

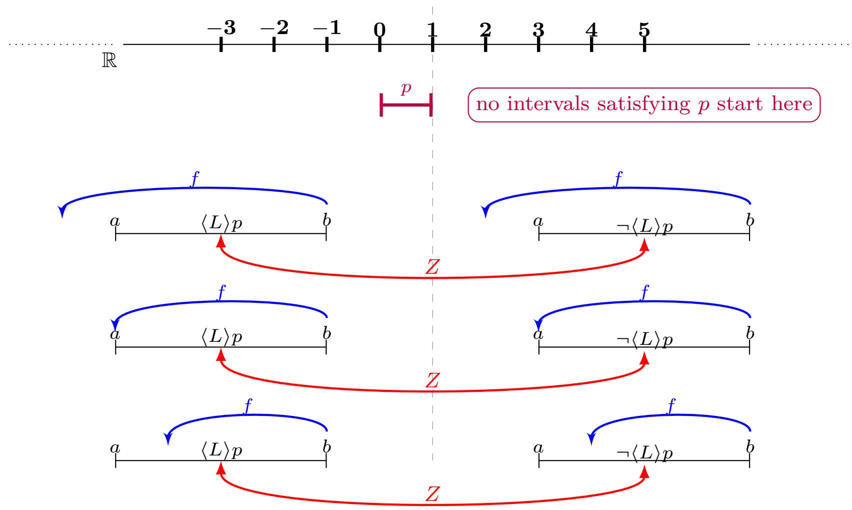

Fig. 3 $\mathsf{BE}\overline{\mathsf{AED}}$-bisimulation that violates $\langle L\rangle$, relative to Den.

  - If $X = B$, then $[c,d]$ is such that $a = c < d < b$. By the monotonicity of f, we have that $f(d) < f(b) < a = c$. Moreover, by the monotonicity of f, for every interval $[c',d']$, with $[a',b']R_B[c',d']$, $f(d') < c'$ holds, and thus $[c,d]Z[c',d']$.
  - If $X = E$, then $[c,d]$ is such that $a < c < b = d$. Thus, $f(d) = f(b) < a < c$. For every interval $[c',d']$, with $[a',b']R_E[c',d']$, $f(d') < c'$ holds, and thus $[c,d]Z[c',d']$.
  - If $X = \overline{A}$, then $[c,d]$ is such that $c < d = a$. Now, if $c < f(d) = f(a)$, then, by the definition of f and Lemma 4, there exists a point c' such that $c' < f(a') < a'$. Thus, the interval $[c',d']$, with $d' = a'$, is such that $[a',b']R_{\overline{A}}[c',d']$ and $[c,d]Z[c',d']$. If $c = f(d) = f(a)$, then take $c' = f(a') < a'$. The interval $[c',d']$, with $d' = a'$, is such that $[a',b']R_{\overline{A}}[c',d']$ and $[c,d]Z[c',d']$. If $c > f(d) = f(a)$, then, by the density of $\mathbb{R}$, the definition of f, and Lemma 4, there exists a point c' such that $f(a') < c' < a'$. The interval $[c',d']$, with $d' = a'$, is such that $[a',b']R_{\overline{A}}[c',d']$ and $[c,d]Z[c',d']$.
  - If $X = \overline{E}$, then $[c,d]$ is such that $c < a < b = d$. There are three possibilities. If $c < f(d)$, then, by the definition of f, there exists a point c' such that $c' < f(b') < a'$. Thus, the interval $[c',d']$, with $d' = b'$, is such that $[a',b']R_{\overline{E}}[c',d']$ and $[c,d]Z[c',d']$. If $c = f(d)$, then the interval $[c',d']$, with $d' = b'$ and $c' = f(d')$, is such that $[a',b']R_{\overline{E}}[c',d']$ and $[c,d]Z[c',d']$. If $c > f(d)$, then, by the density of $\mathbb{R}$, there exists a point c' such that $f(b') < c' < a'$, and the interval $[c',d']$, with $d' = b'$, is such that $[a',b']R_{\overline{E}}[c',d']$ and $[c,d]Z[c',d']$.
  - If $X = \overline{D}$, then $[c,d]$ is such that $c < a < b < d$. If $c < f(d)$, then, take $c' = f(a')$ and any $d' > b'$. The interval $[c',d']$ is such that $[a',b']R_{\overline{D}}[c',d']$ and $[c,d]Z[c',d']$. If $c = f(d)$ (resp., $c > f(d)$), then, by the density of $\mathbb{R}$ and the monotonicity and the surjectivity of f, there exist two points c',d' such that $c' < a' < b' < d'$ and $c' = f(d')$ (resp., $c' > f(d')$). Thus, the interval $[c',d']$ is such that $[a',b']R_{\overline{D}}[c',d']$ and $[c,d]Z[c',d']$.
- If $a < f(b)$ and $a' < f(b')$, then we distinguish the following sub-cases.

- If $X = B$, then $[c, d]$ is such that $a = c < d < b$. Now, if $c < f(d)$ (resp., $c = f(d)$, $c > f(d)$), then, by the density of $\mathbb{R}$ and by the monotonicity and the surjectivity of f, there exists a point d' such that $a' < d' < b'$ and $a' < f(d')$ (resp., $a' = f(d')$, $a' > f(d')$). Thus, the interval $[c', d']$, with $c' = a'$, is such that $[a', b']R_B[c', d']$ and $[c, d]Z[c', d']$.
- If $X = E$, then $[c, d]$ is such that $a < c < b = d$. Now, if $c < f(d)$ (resp., $c = f(d)$, $c > f(d)$), then, by the density of $\mathbb{R}$, there exists a point c' such that $a' < c' < b'$ and $c' < f(b')$ (resp., $c' = f(b')$, $c' > f(b')$). Thus, the interval $[c', d']$, with $d' = b'$, is such that $[a', b']R_E[c', d']$ and $[c, d]Z[c', d']$.
- If $X = \overline{A}$, then the same argument of the case when $a > f(b)$ and $a' > f(b')$ (and $X = \overline{A}$) applies.
- If $X = \overline{E}$, then $[c, d]$ is such that $c < a < b = d$. Thus, $c < a < f(b) = f(d)$. For every interval $[c', d']$, with $[a', b']R_{\overline{E}}[c', d']$, it holds $c' < f(d')$, and thus $[c, d]Z[c', d']$.
- If $X = \overline{D}$, then $[c, d]$ is such that $c < a < b < d$. Thus, by the monotonicity of f, it holds that $c < a < f(b) < f(d)$. For every interval $[c', d']$, with $[a', b']R_{\overline{D}}[c', d']$, it holds, by the monotonicity of f, that $c' < f(d')$, and thus $[c, d]Z[c', d']$.

- If $a = f(b)$ and $a' = f(b')$, then we distinguish the following sub-cases.
  - If $X = B$, then $[c, d]$ is such that $a = c < d < b$. Thus, by the monotonicity of f, it holds that $f(d) < f(b) = a = c$. For every interval $[c', d']$, with $[a', b']R_B[c', d']$, by the monotonicity of f, we have that $f(d') < c'$, and thus $[c, d]Z[c', d']$.
  - If $X = E$, then $[c, d]$ is such that $a < c < b = d$. Thus, $c > a = f(b) = f(d)$ holds. For every interval $[c', d']$, with $[a', b']R_E[c', d']$, we have that $c' > f(d')$, and thus $[c, d]Z[c', d']$.
  - If $X = \overline{A}$, then the same argument of the case when $a > f(b)$ and $a' > f(b')$ (and $X = \overline{A}$) applies.
  - If $X = \overline{E}$, then $[c, d]$ is such that $c < a < b = d$. Thus, $c < a = f(b) = f(d)$. For every interval $[c', d']$, with $[a', b']R_{\overline{E}}[c', d']$, $c' < f(d')$ holds, and thus $[c, d]Z[c', d']$.
  - If $X = \overline{D}$, then $[c, d]$ is such that $c < a < b < d$. Thus, by the monotonicity of f, it holds that $c < a = f(b) < f(d)$. For every interval $[c', d']$, with $[a', b']R_{\overline{D}}[c', d']$, by the monotonicity of f, we have that $c' < f(d')$, and thus $[c, d]Z[c', d']$.

The backward condition can be immediately verified by observing that the forward condition is satisfied and that Z is a symmetric relation. Therefore, Z is a $\mathsf{BE\overline{AED}}$-bisimulation that violates $\langle L \rangle$.

Case $\mathsf{B\overline{ABE}}$. Consider the two interval models M and M', defined as $M = \langle \mathbb{I}(\mathbb{R}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbb{R}), V' \rangle$, respectively, where $V(p) = \{[a, b] \mid a, b \in \mathbb{Q} \text{ or } a, b \in \mathbb{R} \setminus \mathbb{Q}\}$ and $V'(p) = \{[a', b'] \mid a' \leq 0 \text{ and } (a', b' \in \mathbb{Q} \text{ or } a', b' \in \mathbb{R} \setminus \mathbb{Q})\}$. Moreover, let $Z = \{([a, b], [a', b']) \mid a' \leq -1 \text{ and } M, [a, b] \Vdash p \text{ iff } M', [a', b'] \Vdash p\}$. The fact that the local condition is respected follows immediately from the definition. As for the forward condition, consider a pair $([a, b], [a', b'])$ of Z-related intervals. By definition of Z, it holds that $a' \leq -1$ (and thus $a' \leq 0$). Let $X \in \{B, \overline{A}, \overline{B}, \overline{E}\}$. For every interval $[c', d']$, with $[a', b']R_X[c', d']$, it holds that $c' \leq -1$ (and thus $c' \leq 0$). Recall that $\overline{\mathbb{Q}} = \mathbb{R} \setminus \mathbb{Q}$. Since $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ are both dense and unbounded, there exist (i) an interval $[c'', d'']$, such that $[a', b']R_X[c'', d'']$, with $c'', d'' \in \mathbb{Q}$ or $c'', d'' \in \overline{\mathbb{Q}}$, and (ii) an interval

$[c''', d''']$, such that $[a', b']R_X[c''', d''']$, with $c''' \in \mathbb{S}$, $d''' \in \mathbb{S}'$ for some $\mathbb{S}, \mathbb{S}' \in \{\mathbb{Q}, \overline{\mathbb{Q}}\}$, with $\mathbb{S} \neq \mathbb{S}'$. Therefore, for every $[c, d]$ such that $[a, b]R_X[c, d]$, there exists $[c', d']$ such that $[a', b']R_X[c', d']$ and $[c, d]Z[c', d']$. The backward condition can be checked with an analogous argument. It is now immediate to check that $[-1, 0]Z[-1, 0]$, that $M, [-1, 0] \Vdash \langle L\rangle p$ (as $M, [1, 2] \Vdash p$) and that $M', [-1, 0] \Vdash \neg\langle L\rangle p$ (as no interval $[c, d]$, with $c > 0$, satisfies p in M'). Thus, Z is a $\mathsf{B\overline{A}\overline{B}\overline{E}}$-bisimulation that violates $\langle L\rangle$. □

5.3 Completeness for $\langle E\rangle/\langle\overline{E}\rangle/\langle B\rangle/\langle\overline{B}\rangle$: Cases Lin and Den

Lemma 6 *Table 1 presents a complete set of optimal definabilities for $\langle E\rangle$, $\langle\overline{E}\rangle$, $\langle B\rangle$, and $\langle\overline{B}\rangle$ relative to both classes* Lin *and* Den*.*

Proof We only give the bisimulations relative to the operators $\langle E\rangle$ and $\langle\overline{E}\rangle$, thus proving the claim for these two modalities. As usual, since $\langle E\rangle$ and $\langle B\rangle$ (resp., $\langle\overline{E}\rangle$ and $\langle\overline{B}\rangle$) are symmetric, the claim for $\langle B\rangle$ and $\langle\overline{B}\rangle$ follows by symmetry. According to Table 2, in order to deal with $\langle E\rangle$, we need to provide an $\mathsf{ABDO\overline{A}\overline{B}\overline{E}}$-bisimulation that violates $\langle E\rangle$.

<u>Case $\mathsf{ABDO\overline{A}\overline{B}\overline{E}}$</u>. Let $M_1 = \langle \mathbb{I}(\mathbb{R}), V_1\rangle$ and $M_2 = \langle \mathbb{I}(\mathbb{R}), V_2\rangle$, where

- p is the only proposition letter of the language,
- the valuation function $V_1 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ is defined as: $[x, y] \in V_1(p) \Leftrightarrow x \in \mathbb{Q}$ if and only if $y \in \mathbb{Q}$, and
- the valuation function $V_2 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ is given by: $[w, z] \in V_2(p) \overset{def}{\Leftrightarrow} w \in \mathbb{Q}$ if and only if $z \in \mathbb{Q}$, and $([0, 3], [w, z]) \notin R_E$.

Moreover, let Z be a relation between (intervals of) M_1 and M_2 defined as follows: $([x, y], [w, z]) \in Z \Leftrightarrow [x, y] \in V_1(p)$ if and only if $[w, z] \in V_2(p)$. It is easy to verify that that $([0, 3], [0, 3]) \in Z$, $M_1, [0, 3] \Vdash \langle E\rangle p$, but $M_2, [0, 3] \Vdash \neg\langle E\rangle p$. We show now that Z is an $\mathsf{ABDO\overline{A}\overline{B}\overline{E}}$-bisimulation between M_1 and M_2. The local condition immediately follows from the definition. As for the forward condition, it can be checked as follows. Let $[x, y]$ and $[w, z]$ be two Z-related intervals, and let us assume that $[x, y]R_X[x', y']$ holds for some $X \in \{A, B, D, O, \overline{A}, \overline{B}, \overline{E}\}$. We have to exhibit an interval $[w', z']$ such that $[x', y']$ and $[w', z']$ are Z-related, and $[w, z]$ and $[w', z']$ are R_X-related. We proceed by considering each case in turn.

- If $X = A$, then $y = x'$. We can always find a point z' such that $z' > \max\{3, z\}$ and $z' \in \mathbb{Q}$ if and only if $y' \in \mathbb{Q}$ (since both $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ are right-unbounded). This implies that $[x', y']$ and $[z, z']$ are Z-related. Since $[w, z]$ and $[z, z']$ are obviously R_A-related, we have the thesis.
- If $X = B$, the argument is similar to the previous one, but, in this case, the density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ plays a major role. We choose a point z' such that $w < z' < z$, $z' \neq 3$, and $z' \in \mathbb{Q}$ if and only if $y' \in \mathbb{Q}$. The interval $[w, z']$ is such that $[x', y']$ and $[w, z']$ are Z-related, and $[w, z]$ and $[w, z']$ are R_B-related.
- If $X = D$, it suffices to choose two points w' and z' such that $w < w' < z' < z$, $z' \neq 3$, w' belongs to $\mathbb{Q}$ if and only if x' does, and z' belongs to $\mathbb{Q}$ if and only if y' does. The existence of such points is guaranteed by the density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$. The interval $[w', z']$ is such that $[w, z]R_D[w', z']$ and $[x', y']Z[w', z']$.

– If $X = O$, then w' and z' are required to be such that $w < w' < z < z'$, and both density and right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ must be exploited in order to choose a point w' such that $w < w' < z$ and $w' \in \mathbb{Q}$ if and only if x' does, and a point z' such that $z' > \max\{3, z\}$ and z' belongs to $\mathbb{Q}$ if and only if y' does. The interval $[w', z']$ is such that $[w, z]R_O[w', z']$ and $[x', y']Z[w', z']$.
– If $X = \overline{A}$, then there exists a point w'' such that $w'' < \min\{0, w\}$ and $w'' \in \mathbb{Q}$ if and only if w does (and thus $M', [w'', w] \Vdash p$) and there exists a point w''' such that $w''' < w$ and $w''' \in \mathbb{Q}$ if and only if $w \in \overline{\mathbb{Q}}$ (and thus $M', [w''', w] \Vdash \neg p$). We choose $w' = w''$ if $M, [x', y'] \models p$, otherwise we choose $w' = w'''$. The interval $[w', w]$ is such that $[w, z]R_{\overline{A}}[w', w]$ and $[x', y']Z[w', w]$.
– If $X = \overline{B}$, then there exists a point z'' such that $z'' > \max\{3, z\}$ and $z'' \in \mathbb{Q}$ if and only if w does (and thus $M', [w, z''] \Vdash p$) and there exists a point z''' such that $z''' > z$ and $z''' \in \mathbb{Q}$ if and only if $w \in \overline{\mathbb{Q}}$ (and thus $M', [w, z'''] \Vdash \neg p$). We choose $z' = z''$ if $M, [x', y'] \models p$, otherwise we choose $z' = z'''$. The interval $[w, z']$ is such that $[w, z]R_{\overline{B}}[w, z']$ and $[x', y']Z[w, z']$.
– If $X = \overline{E}$, then there exists a point w'' such that $w'' < \min\{0, w\}$ and $w'' \in \mathbb{Q}$ if and only if z does (and thus $M', [w'', z] \Vdash p$) and there exists a point w''' such that $w''' < w$ and $w''' \in \mathbb{Q}$ if and only if $z \in \overline{\mathbb{Q}}$ (and thus $M', [w''', z] \Vdash \neg p$). We choose $w' = w''$ if $M, [x', y'] \models p$, otherwise we choose $w' = w'''$. The interval $[w', z]$ is such that $[w, z]R_{\overline{E}}[w', z]$ and $[x', y']Z[w', z]$.

The backward condition can be verified in a very similar way and thus we omit the details. Therefore, Z is an $\mathsf{ABDO}\overline{\mathsf{ABE}}$-bisimulation that violates $\langle E \rangle$.

Now, we deal with the operator $\langle \overline{E} \rangle$. According to Table 2, we need to provide a $\overline{\mathsf{ABDO}}\mathsf{ABE}$-bisimulation that violates $\langle \overline{E} \rangle$. Such a bisimulation is very similar to the previous one, and it is defined as follows. Let $M_1 = \langle \mathbb{I}(\mathbb{R}), V_1 \rangle$ and $M_2 = \langle \mathbb{I}(\mathbb{R}), V_2 \rangle$, where

– p is the only proposition letter of the language,
– the valuation function $V_1 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ is defined as: $[x, y] \in V_1(p) \Leftrightarrow x \in \mathbb{Q}$ if and only if $y \in \mathbb{Q}$, and
– the valuation function $V_2 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ is given by: $[w, z] \in V_2(p) \overset{def}{\Leftrightarrow} w \in \mathbb{Q}$ if and only if $z \in \mathbb{Q}$, and $([0, 3], [w, z]) \notin R_{\overline{E}}$.

The relation Z is defined exactly as before: $([x, y], [w, z]) \in Z \Leftrightarrow [x, y] \in V_1(p)$ if and only if $[w, z] \in V_2(p)$. Notice that the only difference between the previous bisimulation for $\langle E \rangle$ and the new one for $\langle \overline{E} \rangle$ is in the definition of the valuation function V_2: in the former bisimulation, an interval $[w, z]$ satisfies $\neg p$ if it is a suffix of $[0, 3]$ (i.e. $[0, 3]R_E[w, z]$); in the latter one, $[w, z]$ satisfies $\neg p$ if $[0, 3]$ is a suffix of it (i.e. $[0, 3]R_{\overline{E}}[w, z]$). Following the lines of the proof given above, it is not difficult to verify that the newly-defined relation Z is an $\overline{\mathsf{ABDO}}\mathsf{ABE}$-bisimulation that violates $\langle \overline{E} \rangle$. □

5.4 Completeness for $\langle A \rangle / \langle \overline{A} \rangle$: Cases Lin and Den

The following property of the set of real numbers $\mathbb{R}$ is needed here and in the next subsection: $\mathbb{R}$ can be partitioned into any finite or countably infinite number of pairwise disjoint subsets, each one of which is dense in $\mathbb{R}$. To convince oneself of the validity of such a claim, see, e.g., [27, Thm 7.11], where the property has been proved for $\mathbb{Q}$; likewise, it holds for $\overline{\mathbb{Q}}$ and, consequently, for $\mathbb{R}$. More formally, the claim is that there are countably many nonempty sets $\mathbb{R}_i$ (resp., $\mathbb{Q}_i$, $\overline{\mathbb{Q}}_i$), with $i \in \mathbb{N}$, such that, for each $i \in \mathbb{N}$, $\mathbb{R}_i$ (resp., $\mathbb{Q}_i$, $\overline{\mathbb{Q}}_i$) is dense in $\mathbb{R}$, $\mathbb{R} = \bigcup_{i\in\mathbb{N}} \mathbb{R}_i$ (resp., $\mathbb{Q} = \bigcup_{i\in\mathbb{N}} \mathbb{Q}_i$, $\overline{\mathbb{Q}} = \bigcup_{i\in\mathbb{N}} \overline{\mathbb{Q}}_i$), and $\mathbb{R}_i \cap \mathbb{R}_j = \emptyset$, (resp., $\mathbb{Q}_i \cap \mathbb{Q}_j = \emptyset$, $\overline{\mathbb{Q}}_i \cap \overline{\mathbb{Q}}_j = \emptyset$), for each $i, j \in \mathbb{N}$ with $i \neq j$.

Lemma 7 *Table 1 presents a complete set of optimal definabilities for* $\langle A\rangle$ *and* $\langle \overline{A}\rangle$ *relative to both classes* Lin *and* Den*.*

Proof We only give the bisimulation for the operator $\langle A\rangle$. As usual, since $\langle A\rangle$ and $\langle \overline{A}\rangle$ are symmetric, the claim holds also for $\langle \overline{A}\rangle$. According to Table 2, in order to deal with $\langle A\rangle$, we need to provide a $\mathsf{BE}\overline{\mathsf{ABE}}$-bisimulation that violates $\langle A\rangle$.

Case $\mathsf{BE}\overline{\mathsf{ABE}}$. Let $M_1 = \langle \mathbb{I}(\mathbb{R}), V_1\rangle$ and $M_2 = \langle \mathbb{I}(\mathbb{R}), V_2\rangle$ be two models built on the only proposition letter p. In order to define the valuation functions V_1 and V_2, we make use of two partitions of the set $\mathbb{R}$, one for M_1 and the other one for M_2, each of them consisting of exactly four sets that are dense in $\mathbb{R}$. Formally, for $j = 1, 2$ and $i = 1, \ldots, 4$, let $\mathbb{R}^i_j$ be dense in $\mathbb{R}$. Moreover, for $j = 1, 2$, let $\mathbb{R} = \bigcup_{i=1}^4 \mathbb{R}^i_j$ and $\mathbb{R}^i_j \cap \mathbb{R}^{i'}_j = \emptyset$ for each $i, i' \in \{1, 2, 3, 4\}$ with $i \neq i'$. For the sake of the simplicity, we impose the two partitions to be equal (i.e., $\mathbb{R}^i_1 = \mathbb{R}^i_2$ for each $i, i' \in \{1, 2, 3, 4\}$). Thanks to this condition, the bisimulation relation Z, that we define below, is symmetric. For $j = 1, 2$, we force points in $\mathbb{R}^1_j$ (resp., $\mathbb{R}^2_j$, $\mathbb{R}^3_j$, $\mathbb{R}^4_j$) to behave in the same way with respect to the truth of $p/\neg p$ over the intervals they initiate and terminate by imposing the following constraints:

$$
\begin{aligned}
&\forall x, y\ (\text{if } x \in \mathbb{R}^1_j,\ \text{then } M_j, [x, y] \Vdash \neg p);\\
&\forall x, y\ (\text{if } x \in \mathbb{R}^2_j,\ \text{then } M_j, [x, y] \Vdash \neg p);\\
&\forall x, y\ (\text{if } x \in \mathbb{R}^3_j,\ \text{then } (M_j, [x, y] \Vdash p \text{ iff } y \in \mathbb{R}^1_j \cup \mathbb{R}^3_j));\\
&\forall x, y\ (\text{if } x \in \mathbb{R}^4_j,\ \text{then } (M_j, [x, y] \Vdash p \text{ iff } y \in \mathbb{R}^2_j \cup \mathbb{R}^4_j)).
\end{aligned}
$$

It can be easily shown that, from the given constraints, it immediately follows that:

$$
\begin{aligned}
&\forall x, y\ (\text{if } y \in \mathbb{R}^1_j,\ \text{then } (M_j, [x, y] \Vdash p \text{ iff } x \in \mathbb{R}^3_j));\\
&\forall x, y\ (\text{if } y \in \mathbb{R}^2_j,\ \text{then } (M_j, [x, y] \Vdash p \text{ iff } x \in \mathbb{R}^4_j));\\
&\forall x, y\ (\text{if } y \in \mathbb{R}^3_j,\ \text{then } (M_j, [x, y] \Vdash p \text{ iff } x \in \mathbb{R}^3_j));\\
&\forall x, y\ (\text{if } y \in \mathbb{R}^4_j,\ \text{then } (M_j, [x, y] \Vdash p \text{ iff } x \in \mathbb{R}^4_j)).
\end{aligned}
$$

The above constraints together induce the following definition of the valuation functions $V_j(p) : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$:

$$[x, y] \in V_j(p) \Leftrightarrow (x \in \mathbb{R}^3_j \wedge y \in \mathbb{R}^1_j \cup \mathbb{R}^3_j) \vee (x \in \mathbb{R}^4_j \wedge y \in \mathbb{R}^2_j \cup \mathbb{R}^4_j).$$

Now, let Z be the relation between (intervals of) M_1 and M_2 defined as follows. Two intervals $[x, y]$ and $[w, z]$ are Z-related if and only if at least one of the following conditions holds:

1. $x \in \mathbb{R}^1_1 \cup \mathbb{R}^2_1$ and $w \in \mathbb{R}^1_2 \cup \mathbb{R}^2_2$;
2. $x \in \mathbb{R}^3_1$, $w \in \mathbb{R}^3_2$, and ($y \in \mathbb{R}^1_1 \cup \mathbb{R}^3_1$ iff $z \in \mathbb{R}^1_2 \cup \mathbb{R}^3_2$);
3. $x \in \mathbb{R}^3_1$, $w \in \mathbb{R}^4_2$, and ($y \in \mathbb{R}^1_1 \cup \mathbb{R}^3_1$ iff $z \in \mathbb{R}^2_2 \cup \mathbb{R}^4_2$);
4. $x \in \mathbb{R}^4_1$, $w \in \mathbb{R}^3_2$, and ($y \in \mathbb{R}^2_1 \cup \mathbb{R}^4_1$ iff $z \in \mathbb{R}^1_2 \cup \mathbb{R}^3_2$);
5. $x \in \mathbb{R}^4_1$, $w \in \mathbb{R}^4_2$, and ($y \in \mathbb{R}^2_1 \cup \mathbb{R}^4_1$ iff $z \in \mathbb{R}^2_2 \cup \mathbb{R}^4_2$).

In order to provide the reader with an intuitive idea, we would like to remark that two intervals $[x, y]$ and $[w, z]$ that are Z-related are such that if, for instance, x and w occur in the sets $\mathbb{R}_1^3$ and $\mathbb{R}_2^3$, respectively (second clause), then either y and z both occur in odd-numbered partitions or they both occur in even-numbered partitions. Notice also that, since $\mathbb{R}_1^i = \mathbb{R}_2^i$ for every $i \in \{1, 2, 3, 4\}$, Z is symmetric. Let us consider now two intervals $[x, y]$ and $[w, z]$ such that $x \in \mathbb{R}_1^1$, $w \in \mathbb{R}_2^1$, $y \in \mathbb{R}_1^3$, and $z \in \mathbb{R}_2^1$. By definition of Z, $[x, y]$ and $[w, z]$ are Z-related, and by definition of V_1 and V_2, there exists $y' > y$ such that $M_1, [y, y'] \Vdash p$, but there is no $z' > z$ such that $M_2, [z, z'] \Vdash p$. Thus, it holds that $M_1, [x, y] \Vdash \langle A \rangle p$ and $M_2, [w, z] \Vdash \neg \langle A \rangle p$. In order to complete the proof, we show that the relation Z is a $\mathsf{BE\overline{A}\overline{B}\overline{E}}$-bisimulation. It can be easily checked that every pair $([x, y], [w, z])$ of Z-related intervals is such that either $[x, y] \in V_1(p)$ and $[w, z] \in V_2(p)$ or $[x, y] \notin V_1(p)$ and $[w, z] \notin V_2(p)$. In order to verify the forward condition, let $[x, y]$ and $[w, z]$ be two Z-related intervals. For each modal operator $\langle X \rangle$ of the language and each interval $[x', y']$ such that $[x, y] R_X [x', y']$, we have to exhibit an interval $[w', z']$ such that $[x', y']$ and $[w', z']$ are Z-related, and $[w, z]$ and $[w', z']$ are R_X-related. We proceed by considering each case in turn.

- Let $X = B$. If $x \in \mathbb{R}_1^1 \cup \mathbb{R}_1^2$ and $w \in \mathbb{R}_2^1 \cup \mathbb{R}_2^2$, then for any z' such that $w < z' < z$, both $([x, y'], [w, z']) \in Z$ and $[w, z] R_B [w, z']$ hold. If $x \in \mathbb{R}_1^i$ and $w \in \mathbb{R}_2^i$, for some $i \in \{3, 4\}$, and $y' \in \mathbb{R}_1^k$, for some $k \in \{1, 2, 3, 4\}$, then for any z' such that $w < z' < z$ and $z' \in \mathbb{R}_2^k$, it holds that $([x, y'], [w, z']) \in Z$ and $[w, z] R_B [w, z']$ (the existence of z' is guaranteed by density of $\mathbb{R}_2^k$ in $\mathbb{R}$). Finally, if $x \in \mathbb{R}_1^i$ and $w \in \mathbb{R}_2^{i'}$ for $i, i' \in \{3, 4\}$, with $i \neq i'$, and, in addition $y' \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$ (resp., $y' \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$), then for any z' such that $w < z' < z$ and $z' \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$ (resp., $z' \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$), it holds that $([x, y'], [w, z']) \in Z$ and $[w, z] R_B [w, z']$ (density of $\mathbb{R}_2^2$ and $\mathbb{R}_2^4$, resp., $\mathbb{R}_2^1$ and $\mathbb{R}_2^3$, in $\mathbb{R}$ is used).
- Let $X = E$. As $[x, y] R_E [x', y']$, we have that $y = y'$. We distinguish the following cases, where we tacitly use the density of the relevant sets in $\mathbb{R}$: (i) if $x' \in \mathbb{R}_1^1 \cup \mathbb{R}_1^2$, then we choose w' such that $w < w' < z$ and $w' \in \mathbb{R}_2^1$; (ii) if either $x' \in \mathbb{R}_1^3$ and $y \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$, or $x' \in \mathbb{R}_1^4$ and $y \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$, then we choose w' such that $w < w' < z$ and either $w' \in \mathbb{R}_2^3$ (if $z \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$), or $w' \in \mathbb{R}_2^4$ (if $z \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$); (iii) if either $x' \in \mathbb{R}_1^3$ and $y \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$, or $x' \in \mathbb{R}_1^4$ and $y \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$, then we choose w' such that $w < w' < z$ and either $w' \in \mathbb{R}_2^3$ (if $z \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$), or $w' \in \mathbb{R}_2^4$ (if $z \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$). In all cases, we have that $[w', z]$ is such that $([x', y], [w', z]) \in Z$ and $[w, z] R_E [w', z]$.
- Let $X = \overline{A}$. As $[x, y] R_{\overline{A}} [x', y']$, we have that $x = y'$. We distinguish the following cases: (i) if $x' \in \mathbb{R}_1^1 \cup \mathbb{R}_1^2$, then we choose w' such that $w' < w$ and $w' \in \mathbb{R}_2^1$; (ii) if either $x' \in \mathbb{R}_1^3$ and $x \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$, or $x' \in \mathbb{R}_1^4$ and $x \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$, then we choose w' such that $w' < w$ and either $w' \in \mathbb{R}_2^3$ (if $w \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$), or $w' \in \mathbb{R}_2^4$ (if $w \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$); (iii) if either $x' \in \mathbb{R}_1^3$ and $x \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$, or $x' \in \mathbb{R}_1^4$ and $x \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$, then we choose w' such that $w' < w$ and either $w' \in \mathbb{R}_2^3$ (if $w \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$), or $w' \in \mathbb{R}_2^4$ (if $w \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$). In all cases, we have that $[w', w]$ is such that $([x', x], [w', w]) \in Z$ and $[w, z] R_{\overline{A}} [w', w]$.
- Let $X = \overline{B}$. Since $[x, y] R_{\overline{B}} [x', y']$, we have that $x = x'$. If $x \in \mathbb{R}_1^1 \cup \mathbb{R}_1^2$ and $w \in \mathbb{R}_2^1 \cup \mathbb{R}_2^2$, then for any $z' > z$, both $([x, y'], [w, z']) \in Z$ and $[w, z] R_{\overline{B}} [w, z']$ hold. If $x \in \mathbb{R}_1^i$ and $w \in \mathbb{R}_2^i$, for some $i \in \{3, 4\}$, and $y' \in \mathbb{R}_1^k$, for some $k \in \{1, 2, 3, 4\}$, then for any $z' > z$ such that $z' \in \mathbb{R}_2^k$, it holds that $([x, y'], [w, z']) \in Z$ and $[w, z] R_{\overline{B}} [w, z']$ (the existence of z' is guaranteed by density of $\mathbb{R}_2^k$ in $\mathbb{R}$). Finally, if $x \in \mathbb{R}_1^i$ and $w \in \mathbb{R}_2^{i'}$ for $i, i' \in \{3, 4\}$, with $i \neq i'$, and, in addition

$y' \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$ (resp., $y' \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$), then for any $z' > z$ such that $z' \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$ (resp., $z' \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$), it holds that $([x, y'], [w, z']) \in Z$ and $[w, z] R_{\overline{B}} [w, z']$ (density of $\mathbb{R}_2^2$ and $\mathbb{R}_2^4$, resp., $\mathbb{R}_2^1$ and $\mathbb{R}_2^3$, in $\mathbb{R}$ is used).
- Let $X = \overline{E}$. Since $[x, y] R_{\overline{E}} [x', y']$, we have that $y = y'$. We distinguish the following cases: (i) if $x' \in \mathbb{R}_1^1 \cup \mathbb{R}_1^2$, then we choose w' such that $w' < w$ and $w' \in \mathbb{R}_2^1$; (ii) if either $x' \in \mathbb{R}_1^3$ and $y \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$, or $x' \in \mathbb{R}_1^4$ and $y \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$, then we choose w' such that $w' < w$ and either $w' \in \mathbb{R}_2^3$ (if $z \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$), or $w' \in \mathbb{R}_2^4$ (if $z \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$); (iii) if either $x' \in \mathbb{R}_1^3$ and $y \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$, or $x' \in \mathbb{R}_1^4$ and $y \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$, then we choose w' such that $w' < w$ and either $w' \in \mathbb{R}_2^3$ (if $z \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$), or $w' \in \mathbb{R}_2^4$ (if $z \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$). In all cases, we have that $[w', z]$ is such that $([x', y], [w', z]) \in Z$ and $[w, z] R_{\overline{E}} [w', z]$.

The backward condition follows from the forward one, by applying the usual argument based on the symmetry of Z. Therefore, Z is a $\mathsf{BE\overline{ABE}}$-bisimulation that violates $\langle A \rangle$. □

5.5 Completeness for $\langle D \rangle / \langle \overline{D} \rangle / \langle O \rangle / \langle \overline{O} \rangle$: Cases Lin and Den

In this section, we prove our completeness result for $\langle D \rangle$ and $\langle \overline{D} \rangle$ (Lemma 8), and for $\langle O \rangle$ and $\langle \overline{O} \rangle$ (Lemma 9).

Lemma 8 *Table 1 presents a complete set of optimal definabilities for $\langle D \rangle$ and $\langle \overline{D} \rangle$ relative to both classes* Lin *and* Den*.*

Proof Since $\langle D \rangle$ and $\langle \overline{D} \rangle$ are not symmetric, we have to solve both cases separately. According to Table 2, in order to deal with $\langle D \rangle$, we need to provide two bisimulations, namely an $\mathsf{ABO\overline{ABE}}$-bisimulation and an $\mathsf{AE\overline{ABEO}}$-bisimulation, that violate $\langle D \rangle$. In fact, it suffices to provide only the former bisimulation, thanks to the symmetry between $\mathsf{ABO\overline{ABE}}$ and $\mathsf{AE\overline{ABEO}}$. Similarly, in order to deal with $\langle \overline{D} \rangle$, we are supposed to provide two bisimulations, namely an $\mathsf{\overline{ABO}ABE}$-bisimulation and an $\mathsf{\overline{AE}ABEO}$-bisimulation, that violate $\langle \overline{D} \rangle$. Thanks to the symmetry between $\mathsf{\overline{ABO}ABE}$ and $\mathsf{\overline{AE}ABEO}$, we only give the former one.

Case $\mathsf{ABO\overline{ABE}}$. As a first step, we define a pair of functions that will be used in the definition of the models involved in the bisimulation relation Z. Let $\mathcal{P}(\mathbb{Q}) = \{\mathbb{Q}_q \mid q \in \mathbb{Q}\}$ and $\mathcal{P}(\overline{\mathbb{Q}}) = \{\overline{\mathbb{Q}}_q \mid q \in \mathbb{Q}\}$ be countably infinite partitions of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$, respectively, such that for every $q \in \mathbb{Q}$, both $\mathbb{Q}_q$ and $\overline{\mathbb{Q}}_q$ are dense in $\mathbb{R}$. For every $q \in \mathbb{Q}$, let $\mathbb{R}_q = \mathbb{Q}_q \cup \overline{\mathbb{Q}}_q$. We define a function $g : \mathbb{R} \to \mathbb{Q}$ that maps every real number x to the index q (a rational number) of the class $\mathbb{R}_q$ it belongs to. Formally, for every $x \in \mathbb{R}$, $g(x) = q$, where $q \in \mathbb{Q}$ is the unique rational number such that $x \in \mathbb{R}_q$. The two functions $f_1 : \mathbb{R} \to \mathbb{Q}$ and $f_2 : \mathbb{R} \to \mathbb{Q}$ are defined as follows:

$$f_1(x) = \begin{cases} g(x) & \text{if } x < g(x),\, x \neq 1, \text{ and } x \neq 0 \\ 2 & \text{if } x = 1 \\ \lceil x + 3 \rceil & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} g(x) & \text{if } x < g(x) \text{ and } x \notin [0, 3) \\ \lceil x + 3 \rceil & \text{otherwise} \end{cases}$$

It is not difficult to check that the above-defined functions f_i $(i \in \{1, 2\})$ satisfy the following properties:

(i) for every $x \in \mathbb{R}$, $f_i(x) > x$,
(ii) for every $x \in \mathbb{Q}$, both $f_i^{-1}(x) \cap \mathbb{Q}$ and $f_i^{-1}(x) \cap \overline{\mathbb{Q}}$ are left-unbounded (notice that surjectivity of f_i immediately follows), and
(iii) for every $x, y \in \mathbb{R}$, if $x < y$, then there exists $u_1 \in \mathbb{Q}$ (resp., $u_2 \in \overline{\mathbb{Q}}$) such that $x < u_1 < y$ (resp., $x < u_2 < y$) and $y < f_i(u_1)$ (resp., $y < f_i(u_2)$).

Now, we can define two models M_1 and M_2, built on the only proposition letter p, as follows: for each $i \in \{1,2\}$, $M_i = \langle \mathbb{I}(\mathbb{R}), V_i \rangle$, where $V_i : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ ($i \in \{1,2\}$) is defined as follows: $[x,y] \in V_i(p) \Leftrightarrow y \geq f_i(x)$. Finally, we define the relation Z as:

$$([x,y],[w,z]) \in Z \Leftrightarrow x \equiv w,\ y \equiv z, \text{ and } [x,y] \equiv_l [w,z],$$

where we define $u \equiv v \Leftrightarrow u \in \mathbb{Q}$ if and only if $v \in \mathbb{Q}$ and $[u,u'] \equiv_l [v,v'] \Leftrightarrow u' \sim f_1(u)$ and $v' \sim f_2(v)$, for $\sim \in \{<,=,>\}$.

Let us consider the interval $[0,3]$ in M_1 and the interval $[0,3]$ in M_2. It is immediate to see that these two intervals are Z-related. However, $M_1,[0,3] \Vdash \langle D \rangle p$ (as $M_1,[1,2] \Vdash p$), but $M_2,[0,3] \Vdash \neg\langle D \rangle p$.

We are left to show that Z is an $\mathsf{ABO\overline{ABE}}$-bisimulation between M_1 and M_2. Let $[x,y]$ and $[w,z]$ be two Z-related intervals. By definition, $y \sim f_1(x)$ and $z \sim f_2(w)$ for some $\sim \in \{<,=,>\}$. If $\sim \in \{=,>\}$, then both $[x,y]$ and $[w,z]$ satisfy p; otherwise, both of them satisfy $\neg p$. Thus, the local condition is satisfied. As for the forward condition, let $[x,y]$ and $[x',y']$ be two intervals in M_1 and $[w,z]$ an interval in M_2. We have to prove that if $[x,y]$ and $[w,z]$ are Z-related, then, for each modal operator $\langle X \rangle$ of $\mathsf{ABO\overline{ABE}}$ such that $[x,y]R_X[x',y']$, there exists an interval $[w',z']$ such that $[x',y']$ and $[w',z']$ are Z-related and $[w,z]R_X[w',z']$. Once again, we proceed by examining each case in turn.

- Let $X = A$. By definition of $\langle A \rangle$, $x' = y$ and we are forced to choose $w' = z$. By $y \equiv z$, it immediately follows that $x' \equiv w'$. We must find a point $z' > z$ such that $y' \equiv z'$ and both $y' \sim f_1(y)$ and $z' \sim f_2(z)$ for some $\sim \in \{<,=,>\}$. Let us suppose that $y' < f_1(y)$. In such a case, we choose a point z' such that $z < z' < f_2(z)$ and $y' \equiv z'$. The existence of such a point is guaranteed by property (i) of f_2 above and by the density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ in $\mathbb{R}$. Otherwise, if $y' = f_1(y)$, we choose $z' = f_2(z)$. By definition of f_1 and f_2 (the codomain of f_1 and f_2 is $\mathbb{Q}$), both y' and z' belong to $\mathbb{Q}$ and thus $y' \equiv z'$. Finally, if $y' > f_1(y)$, we choose $z' > f_2(z)$ such that $y' \equiv z'$. The existence of such a point is guaranteed by right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$, and the interval $[z,z']$ is such that $([x',y'],[z,z']) \in Z$ and $[w,z]R_A[z,z']$.
- Let $X = B$. In this case, $x = x'$ and $y' < y$. We distinguish the following cases.
  - If $y' > f_1(x)$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then $y > f_1(x)$ holds as well (as $y' < y$), which implies $z > f_2(w)$. Thus, we can choose any point $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$), with $f_2(w) < z' < z$ (the existence of such a point is guaranteed by density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$, respectively).
  - If $y' = f_1(x)$, then $y' \in \mathbb{Q}$ (by definition of f_1) and $y > f_1(x)$ holds (as $y' < y$). The latter implies $z > f_2(w)$, and thus we choose $z' = f_2(w)$. Note that $f_2(w) \in \mathbb{Q}$ by the definition of f_2.
  - If $y' < f_1(x)$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then we choose a point $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$) such that $w < z' < \min\{z, f_2(w)\}$ (the existence of such a point is guaranteed by density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$, respectively).

  In all cases, the interval $[w,z']$ is such that $([x,y'],[w,z']) \in Z$ and $[w,z]R_B[w,z']$.

- Let $X = O$. Firstly, we choose a point w' such that $w < w' < z$, $w' \in \mathbb{Q}$ if and only if $x' \in \mathbb{Q}$, and $f_2(w') > z$ (the existence of such a point is guaranteed by property (iii) of f_2 on page 24). Secondly, we choose a point z' such that $z' \in \mathbb{Q}$ if and only if $y' \in \mathbb{Q}$, and that
  (a) if $y' < f_1(x')$, then $z < z' < f_2(w')$ (density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ is used here),
  (b) if $y' > f_1(x')$, then $z' > f_2(w')$ (right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ is used here),
  (c) if $y' = f_1(x')$, then $z' = f_2(w')$.
  In all cases, the interval $[w', z']$ is such that $([x', y'], [w', z']) \in Z$ and $[w, z] R_O [w', z']$.
- Let $X = \overline{A}$. In this case, $y' = x$. We distinguish the following cases.
  - If $f_1(x') < y' (= x)$, then consider any point $\overline{w} \in \mathbb{Q}$, with $\overline{w} < w$. By property (ii) on page 24, there exist both a point $w'' \in \mathbb{Q}$ and a point $w''' \in \overline{\mathbb{Q}}$ such that $w'' < \overline{w}$, $w''' < \overline{w}$, and $f_2(w'') = f_2(w''') = \overline{w}$. We select $w' = w''$ if $x' \in \mathbb{Q}$, and $w' = w'''$ if $x' \in \overline{\mathbb{Q}}$.
  - If $f_1(x') > y' (= x)$, then consider any point $\overline{w} \in \mathbb{Q}$, with $w < \overline{w}$. By property (ii), there exist both a point $w'' \in \mathbb{Q}$ and a point $w''' \in \overline{\mathbb{Q}}$ such that $w'' < w$, $w''' < w$, and $f_2(w'') = f_2(w''') = \overline{w}$. We select $w' = w''$ if $x' \in \mathbb{Q}$, and $w' = w'''$ if $x' \in \overline{\mathbb{Q}}$.
  - If $f_1(x') = y' (= x)$, then $x, w \in \mathbb{Q}$. By property (ii), there exist both a point $w'' \in \mathbb{Q}$ and a point $w''' \in \overline{\mathbb{Q}}$ such that $w'' < w$, $w''' < w$, and $f_2(w'') = f_2(w''') = w$. We select $w' = w''$ if $x' \in \mathbb{Q}$, and $w' = w'''$ if $x' \in \overline{\mathbb{Q}}$.

  In all cases, the interval $[w', w]$ is such that $([x', x], [w', w]) \in Z$ and $[w, z] R_{\overline{A}} [w', w]$.
- Let $X = \overline{B}$. In this case $x' = x$. We distinguish the following cases.
  - If $y' < f_1(x)$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then $y < f_1(x)$ holds as well (as $y < y'$), which implies $z < f_2(w)$. Thus, we can choose any point $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$), with $z < z' < f_2(w)$ (the existence of such a point is guaranteed by density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$, respectively).
  - If $y' = f_1(x)$, then $y' \in \mathbb{Q}$ (by definition of f_1) and $y < f_1(x)$ holds (as $y < y'$). The latter implies $z < f_2(w)$, and thus we choose $z' = f_2(w)$. Note that $f_2(w) \in \mathbb{Q}$ by the definition of f_2.
  - If $y' > f_1(x)$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then we choose a point $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$) such that $z' > \max\{z, f_2(w)\}$ (the existence of such a point is guaranteed by right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$, respectively).

  In all cases, the interval $[w, z']$ is such that $([x', y'], [w, z']) \in Z$ and $[w, z] R_{\overline{B}} [w, z']$.
- Let $X = \overline{E}$. In this case $y = y'$ and $x' < x$. We distinguish the following cases.
  - If $f_1(x') < y' (= y)$, then consider any point $\overline{w} \in \mathbb{Q}$, with $\overline{w} < z$. By property (ii) on page 24, there exist both a point $w'' \in \mathbb{Q}$ and a point $w''' \in \overline{\mathbb{Q}}$ such that $w'' < w$, $w''' < w$, and $f_2(w'') = f_2(w''') = \overline{w}$. We select $w' = w''$ if $x' \in \mathbb{Q}$, and $w' = w'''$ if $x' \in \overline{\mathbb{Q}}$.
  - If $f_1(x') > y' (= y)$, then consider any point $\overline{w} \in \mathbb{Q}$, with $z < \overline{w}$. By property (ii) on page 24, there exist both a point $w'' \in \mathbb{Q}$ and a point $w''' \in \overline{\mathbb{Q}}$ such that $w'' < w$, $w''' < w$, and $f_2(w'') = f_2(w''') = \overline{w}$. We select $w' = w''$ if $x' \in \mathbb{Q}$, and $w' = w'''$ if $x' \in \overline{\mathbb{Q}}$.

- If $f_1(x') = y'(= y)$, then $y, z \in \mathbb{Q}$. By property (ii) on page 24, there exist both a point $w'' \in \mathbb{Q}$ and a point $w''' \in \overline{\mathbb{Q}}$ such that $w'' < w$, $w''' < w$, and $f_2(w'') = f_2(w''') = z$. We select $w' = w''$ if $x' \in \mathbb{Q}$, and $w' = w'''$ if $x' \in \overline{\mathbb{Q}}$.

In all cases, the interval $[w', z]$ is such that $([x', y], [w', z]) \in Z$ and $[w, z]R_{\overline{E}}[w', z]$.

The backward condition can be verified in a very similar way and the details are omitted. Thus, Z is an $\mathsf{ABO}\overline{\mathsf{ABE}}$-bisimulation that violates $\langle D \rangle$.

In order to deal with the operaor $\langle \overline{D} \rangle$, we provide an $\overline{\mathsf{ABO}}\mathsf{ABE}$-bisimulation that violates $\langle \overline{D} \rangle$. Such a bisimulation is very similar to the previous one, and it is defined as follows. The models M_1 and M_2 are defined as before, but they make use of a different pair of functions f_1, f_2 in the definition of the valuation functions (indeed, in this case, $f_1 = f_2$). Formally, for each $i \in \{1, 2\}$, $M_i = \langle \mathbb{I}(\mathbb{R}), V_i \rangle$, where $V_i : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ $(i \in \{1, 2\})$ is defined as follows: $[x, y] \in V_i(p) \Leftrightarrow y \geq f_i(x)$, where $f_1(= f_2) : \mathbb{R} \to \mathbb{Q}$ is such that:

$$f_1(x) = f_2(x) = \begin{cases} g(x) & \text{if } x < g(x) \leq 1 \\ \lceil x + 1 \rceil & \text{otherwise} \end{cases}$$

with g being the same function used before. It is not difficult to check that the newly-defined functions f_i $(i \in \{1, 2\})$ satisfy the following properties: (i) for every $x \in \mathbb{R}$, $x < f_i(x) < x + 2$, (ii) for every $y \in \mathbb{Q}$ and every $\epsilon > 0$, there exist $x_1, x_2 \in \mathbb{Q}$ and $\overline{x}_1, \overline{x}_2 \in \overline{\mathbb{Q}}$ such that $y - 1 < x_1 < \overline{x}_1 < y - 1 + \epsilon$, $y - \epsilon < x_2 < \overline{x}_2 < y$, and $f_i(x_1) = f_i(\overline{x}_1) = f_i(x_2) = f_i(\overline{x}_2) = y$, and (iii) for every $x, y \in \mathbb{R}$, if $x < y$, then there exists $u_1 \in \mathbb{Q}$ (resp., $u_2 \in \overline{\mathbb{Q}}$) such that $x < u_1 < y$ (resp., $x < u_2 < y$) and $y < f(u_1)$ (resp., $y < f(u_2)$). Finally, the relation Z is defined as before. By following an analogous technique to the one used above and making use of the properties of f_1 and f_2, it is not difficult to verify that Z is an $\overline{\mathsf{ABO}}\mathsf{ABE}$-bisimulation. Now, suppose that $0 \in \mathbb{Q}_q$, for some $q \in \mathbb{Q}$. By property (ii) of f_1, there exists $x \in \mathbb{Q}$ such that $-0.5 < x < 0$ and $f_1(x) = 0$. Thus, the interval $[x, 0.1]$ in M_1 is such that $f_1(x) < 0.1$. Consider the interval $[2, 4]$ in M_2. By property (i) of f_2, it must be $f_2(2) < 4$. Thus, $([x, 0.1], [2, 4]) \in Z$. However, on the one hand $M_1, [x, 0.1] \Vdash \langle \overline{D} \rangle \neg p$, because, for example, by property (ii) of f_1, there exists a point x' such that $0.5 < x' < x$ and $f_1(x') = 0.5$. Thus, $[x', 0.4]$ is such that $0.4 < f_1(x')$, which means that $M_1, [x', 0.4] \Vdash \neg p$. On the other hand, $M, [2, 4] \Vdash \neg \langle \overline{D} \rangle \neg p$, because every interval $[w, z]$, with $w < 2 < 4 < z$, is such that $f_2(w) < z$ (as $z > w + 2$), and thus $M, [w, z] \Vdash p$. This allows us to conclude that Z is an $\overline{\mathsf{ABO}}\mathsf{ABE}$-bisimulation that violates $\langle \overline{D} \rangle$. □

Lemma 9 *Table 1 presents a complete set of optimal definabilities for* $\langle O \rangle$ *and* $\langle \overline{O} \rangle$ *relative to both classes* Lin *and* Den.

Proof As usual, since $\langle O \rangle$ and $\langle \overline{O} \rangle$ are symmetric, we only solve the case of the operator $\langle O \rangle$, and, by symmetry, the claim holds also for $\langle \overline{O} \rangle$. According to Table 2, in order to deal with $\langle O \rangle$, we need to provide two bisimulations, namely an $\mathsf{ABE}\overline{\mathsf{AED}}$-bisimulation and an $\mathsf{ABD}\overline{\mathsf{ABE}}$-bisimulation, that violate $\langle O \rangle$.

Case $\mathsf{ABE}\overline{\mathsf{AED}}$. This bisimulation is very similar to those constructed for the operators $\langle E \rangle$ and $\langle \overline{E} \rangle$ in the proof of Lemma 6. Let $M_1 = \langle \mathbb{I}(\mathbb{R}), V_1 \rangle$ and $M_2 = \langle \mathbb{I}(\mathbb{R}), V_2 \rangle$ be two models over the set of proposition letters $\mathcal{AP} = \{p\}$, where the valuation functions $V_1 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ and $V_2 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ are, respectively, defined as follows:

- $[x, y] \in V_1(p) \overset{def}{\Leftrightarrow} x \in \mathbb{Q}$ iff $y \in \mathbb{Q}$ and

– $[w, z] \in V_2(p) \overset{def}{\Leftrightarrow} w \in \mathbb{Q}$ iff $z \in \mathbb{Q}$, and $([0, 3], [w, z]) \notin R_O$ (that is, it is not the case that $0 < w < 3 < z$).

Then, we define the relation Z between intervals of M_1 and intervals of M_2 as: $([x, y], [w, z]) \in Z \overset{def}{\Leftrightarrow} [x, y] \in V_1(p)$ iff $[w, z] \in V_2(p)$. It is immediate to see that $([0, 3], [0, 3]) \in Z$, $M_1, [0, 3] \Vdash \langle O\rangle p$, but $M_2, [0, 3] \Vdash \neg\langle O\rangle p$.

We show that Z is an $\mathsf{ABE}\overline{\mathsf{AED}}$-bisimulation between M_1 and M_2. The local condition immediately follows from the definition. As for the forward condition, it can be checked as follows. Let $[x, y]$ and $[w, z]$ be two Z-related intervals, and let us assume that $[x, y]R_X[x', y']$ holds for some $X \in \{A, B, E, \overline{A}, \overline{E}, \overline{D}\}$. We have to exhibit an interval $[w', z']$ such that $[x', y']$ and $[w', z']$ are Z-related, and $[w, z]$ and $[w', z']$ are R_X-related. We proceed by a case analysis on $X \in \{A, B, E, \overline{A}, \overline{E}, \overline{D}\}$.

– If $X = A$, then we distinguish the following cases: (a) if $0 < z < 3$, then we select a point z' such that $z < z' < 3$ and $z' \in \mathbb{Q}$ if and only if $y' \in \mathbb{Q}$ (the existence of such a point is guaranteed by density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$); (b) otherwise, we select a point z' such that $z' > z$ and $z' \in \mathbb{Q}$ if and only if $y' \in \mathbb{Q}$ (the existence of such a point is guaranteed by right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$). In both cases, the interval that $[z, z']$ is such that $[x', y']$ and $[z, z']$ are Z-related, and $[w, z]$ and $[z, z']$ are R_A-related.
– If $X = B$, the argument is similar to the previous one. We distinguish the following cases: (a) if $0 < w < 3$, then we choose a point z' such that $w < z' < \min\{3, z\}$ and $z' \in \mathbb{Q}$ if and only if $y' \in \mathbb{Q}$; (b) otherwise, we choose a point z' such that $w < z' < z$ and $z' \in \mathbb{Q}$ if and only if $y' \in \mathbb{Q}$. In both cases, the interval $[w, z']$ is such that $[x', y']$ and $[w, z']$ are Z-related, and $[w, z]$ and $[w, z']$ are R_B-related.
– If $X = E$, then we distinguish the following cases: (a) if $z > 3$, then we choose a point w' such that $\max\{3, w\} < w' < z$ and $w' \in \mathbb{Q}$ if and only if $x' \in \mathbb{Q}$; (b) otherwise, we choose a point w' such that $w < w' < z$ and $w' \in \mathbb{Q}$ if and only if $x' \in \mathbb{Q}$. In both cases, the interval $[w', z]$ is such that $[x', y']$ and $[w', z]$ are Z-related, and $[w, z]$ and $[w', z]$ are R_E-related.
– If $X = \overline{A}$, then we choose a point w' such that $w' < \min\{0, w\}$ and $w' \in \mathbb{Q}$ if and only if $x' \in \mathbb{Q}$. The interval $[w', w]$ is such that $[x', y']$ and $[w', w]$ are Z-related, and $[w, z]$ and $[w', w]$ are $R_{\overline{A}}$-related.
– If $X = \overline{E}$, then we choose a point w' such that $w' < \min\{0, w\}$ and $w' \in \mathbb{Q}$ if and only if $x' \in \mathbb{Q}$. The interval $[w', z]$ is such that $[x', y']$ and $[w', z]$ are Z-related, and $[w, z]$ and $[w', z]$ are $R_{\overline{E}}$-related.
– If $X = \overline{D}$, then we first choose a point w' such that $w' < \min\{0, w\}$ and $w' \in \mathbb{Q}$ if and only if $x' \in \mathbb{Q}$. Next, we choose a point z' such that $z' > z$ and $z' \in \mathbb{Q}$ if and only if $y' \in \mathbb{Q}$. The interval $[w', z']$ is such that $[x', y']$ and $[w', z']$ are Z-related, and $[w, z]$ and $[w', z']$ are $R_{\overline{D}}$-related.

The backward condition can be verified in a very similar way and thus we omit the details. Therefore, Z is an $\mathsf{ABE}\overline{\mathsf{AED}}$-bisimulation that violates $\langle O\rangle$.

<u>Case $\mathsf{ABD}\overline{\mathsf{ABE}}$</u>. The $\mathsf{ABD}\overline{\mathsf{ABE}}$-bisimulation that violates $\langle O\rangle$ has some similarities with the $\mathsf{ABO}\overline{\mathsf{ABE}}$-bisimulation that violates $\langle D\rangle$, presented in the proof of Lemma 8. However, we need to 'rearrange' the partitions of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ that we used in order to prove Lemma 8. More precisely, we still need two infinite and countable partitions $\mathcal{P}(\mathbb{Q})$ of $\mathbb{Q}$ and $\mathcal{P}(\overline{\mathbb{Q}})$ of $\overline{\mathbb{Q}}$, whose elements are dense in $\mathbb{R}$, but it is useful to provide a more suitable enumeration for both of them, as follows: $\mathcal{P}(\mathbb{Q}) = \{\mathbb{Q}^c_q \mid c \in \{a, b\}, q \in$

$\mathbb{Q}\}$ and $\mathcal{P}(\overline{\mathbb{Q}}) = \{\overline{\mathbb{Q}}_q^c \mid c \in \{a,b\}, q \in \mathbb{Q}\}$. Analogously to Lemma 8, we require these partitions to be such that, for each $c \in \{a,b\}, q \in \mathbb{Q}$, sets $\mathbb{Q}_q^c$ and $\overline{\mathbb{Q}}_q^c$ are dense in $\mathbb{R}$. Now, we define the partition $\mathcal{P}(\mathbb{R})$ of $\mathbb{R}$ as: $\mathcal{P}(\mathbb{R}) = \{\mathbb{R}_q^c \mid c \in \{a,b\}, q \in \mathbb{Q}\}$, where $\mathbb{R}_q^c = \mathbb{Q}_q^c \cup \overline{\mathbb{Q}}_q^c$, for each $c \in \{a,b\}$, $q \in \mathbb{Q}$. We use $\mathbb{Q}^c$ (resp., $\overline{\mathbb{Q}}^c$, $\mathbb{R}^c$) as an abbreviation for $\bigcup_{q\in\mathbb{Q}} \mathbb{Q}_q^c$ (resp., $\bigcup_{q\in\mathbb{Q}} \overline{\mathbb{Q}}_q^c$, $\bigcup_{q\in\mathbb{Q}} \mathbb{R}_q^c$), for each $c \in \{a,b\}$. In addition, we define $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{I}(\mathbb{R})$ as follows:

$$\begin{aligned} &\mathcal{S}_1 = \{[x,y] \mid x,y \in \mathbb{R}^c, c \in \{a,b\}\} \text{ and} \\ &\mathcal{S}_2 = \{[w,z] \mid w,z \in \mathbb{R}^c, c \in \{a,b\}\} \setminus \{[w,z] \mid 0 < w < 3 < z\}. \end{aligned}$$

Finally, for each $i \in \{1,2\}$, we use $\overline{\mathcal{S}}_i$ to denote the set $\mathbb{I}(\mathbb{R}) \setminus \mathcal{S}_i$. It is easy to verify that, for every pair of points $x, y \in \mathbb{I}(\mathbb{R})$, if $x < y$, then there exist $y_1, y_2, y_3, y_4 \in \mathbb{R}$ such that $x < y_i < y$ for each $i \in \{1,2,3,4\}$ and:

$$\begin{aligned} &y_1 \in \mathbb{Q} \text{ and } [x,y_1] \in \mathcal{S}_1 \text{ (resp., } \mathcal{S}_2), \\ &y_2 \in \mathbb{Q} \text{ and } [x,y_2] \in \overline{\mathcal{S}}_1 \text{ (resp., } \overline{\mathcal{S}}_2), \\ &y_3 \in \overline{\mathbb{Q}} \text{ and } [x,y_3] \in \mathcal{S}_1 \text{ (resp., } \mathcal{S}_2), \\ &y_4 \in \overline{\mathbb{Q}} \text{ and } [x,y_4] \in \overline{\mathcal{S}}_1 \text{ (resp., } \overline{\mathcal{S}}_2). \end{aligned} \tag{1}$$

We define now a pair of functions that will be used in the definition of the models involved in the bisimulation relation Z. Let $g : \mathbb{R} \to \mathbb{Q}$ be a function defined as follows (notice the strong similarity with the definition of g in Lemma 8): for each $x \in \mathbb{R}$, $g(x) = q$, where $q \in \mathbb{Q}$ is the unique rational number such that $x \in \mathbb{R}_q^a \cup R_q^b$. The functions $f_1 : \mathbb{R} \to \mathbb{Q}$ and $f_2 : \mathbb{R} \to \mathbb{Q}$ are defined as follows:

$$f_1(x) = \begin{cases} g(x) & \text{if } x < g(x) \\ \lceil x+3 \rceil & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} g(x) & \text{if } x < g(x) \text{ and } ([0,3],[x,g(x)]) \notin R_O \\ \lceil x+3 \rceil & \text{if } x \geq g(x) \text{ and } x \notin (0,3) \\ a_{n'} & \text{otherwise} \end{cases}$$

where $a_{n'}$ is the least element of the series $a_n = 3 - (\frac{1}{n})$ $(n \geq 1)$ such that $x < a_{n'}$. It is not hard to verify that the functions f_i $(i \in \{1,2\})$ fulfill the following conditions:

(*i*) $f_i(x) > x$ for every $x \in \mathbb{R}$;

(*ii*) for each $x \in \mathbb{Q}$, $f_i^{-1}(x) \cap \mathbb{Q}^a$, $f_i^{-1}(x) \cap \mathbb{Q}^b$, $f_i^{-1}(x) \cap \overline{\mathbb{Q}}^a$, and $f_i^{-1}(x) \cap \overline{\mathbb{Q}}^b$ are unbounded to left (notice that surjectivity of f_i immediately follows);

(*iii*) for each $x, y \in \mathbb{R}$, if $x < y$, then there exist:
- $u_1 \in \mathbb{Q}^a$ such that $x < u_1 < y$ and $y > f_i(u_1)$,
- $u_2 \in \mathbb{Q}^b$ such that $x < u_2 < y$ and $y > f_i(u_2)$,
- $u_3 \in \overline{\mathbb{Q}}^a$ such that $x < u_3 < y$ and $y > f_i(u_3)$, and
- $u_4 \in \overline{\mathbb{Q}}^b$ such that $x < u_4 < y$ and $y > f_i(u_4)$.

In addition, function f_2 satisfies the following property:

(*iv*) for each $w \in (0,3)$, $f_2(w) < 3$.

At this point, we are ready to define the models M_1 and M_2, and the bisimulation relation between their intervals. Let $i \in \{1,2\}$ and $M_i = \langle \mathbb{I}(\mathbb{R}), V_{f_i} \rangle$, where the valuation functions $V_i : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ is defined as follows:

$$[x,y] \in V_i(p) \overset{def}{\Leftrightarrow} \text{either } y = f_i(x) \text{ or both } y < f_i(x) \text{ and } [x,y] \in \mathcal{S}_i.$$

The relation Z is defined as follows:

$$([x,y],[w,z]) \in Z \overset{def}{\Leftrightarrow} x \equiv w,\ y \equiv z, \text{ and } [x,y] \equiv_l [w,z],$$

where the relations $\equiv$ and $\equiv_l$ are defined, respectively, thus:

$$x \equiv w \overset{def}{\Leftrightarrow} x \in \mathbb{Q} \text{ iff } w \in \mathbb{Q}$$

$$[x,y] \equiv_l [w,z] \overset{def}{\Leftrightarrow} \begin{cases} \text{either } y > f_1(x) \text{ and } z > f_2(w) \\ \text{or} \quad\ y = f_1(x) \text{ and } z = f_2(w) \\ \text{or} \quad\ y < f_1(x),\ z < f_2(w), \text{ and } ([x,y] \in \mathcal{S}_1 \text{ iff } [w,z] \in \mathcal{S}_2) \end{cases}$$

Now, by the definition of Z, we have that $([0,3],[0,3]) \in Z$ (notice that this is also consequence of the facts that $f_1(0) = f_2(0)$ and that $([0,3],[0,3]) \notin R_O$). Moreover, it is easy to see that $M_1,[0,3] \Vdash \langle O \rangle p$, while $M_2,[0,3] \Vdash \neg\langle O \rangle p$ (this is a direct consequence of property (iv) of f_2 and of the fact that $f_1(x) > 3$ for some $x \in (0,3)$).

We show that Z is an $\mathsf{ABD\overline{A}\overline{B}\overline{E}}$- bisimulation. For the local condition, consider the pair $([x,y],[w,z]) \in Z$. First, we assume that $[x,y] \in V_1(p)$ and we show that $[w,z] \in V_2(p)$ follows. Since $[x,y] \in V_1(p)$, either $y = f_1(x)$ holds or both $y < f_1(x)$ and $[x,y] \in \mathcal{S}_1$ hold. In the former case, by the definition of Z, it must be $z = f_2(w)$, which implies $[w,z] \in V_2(p)$. In the latter case, by the definition of Z, both $z < f_2(w)$ and $[w,z] \in \mathcal{S}_2$ hold, and thus $[w,z] \in V_2(p)$. Second, we assume that $[w,z] \in V_2(p)$ and we show that $[x,y] \in V_1(p)$ follows. Since $[w,z] \in V_2(p)$, either $z = f_2(w)$ holds or both $z < f_2(w)$ and $[w,z] \in \mathcal{S}_2$ hold. In the former case, by the definition of Z, it must be $y = f_1(x)$, which implies $[x,y] \in V_1(p)$. In the latter case, by the definition of Z, both $y < f_1(x)$ and $[x,y] \in \mathcal{S}_1$ hold, and thus $[x,y] \in V_1(p)$. In order to prove that the forward condition is satisfied, we assume that $([x,y],[w,z]) \in Z$ and $[x,y]R_X[x',y']$, for some $X \in \{A,B,D,\overline{A},\overline{B},\overline{E}\}$ and some $[x,y],[w,z],[x',y'] \in \mathbb{I}(\mathbb{R})$, and we show the existence of an interval $[w',z']$ such that $([x',y'],[w',z']) \in Z$ and $[w,z]R_X[w',z']$. As usual, we proceed by considering each case in turn.

- If $X = A$, then we distinguish three cases.
  - If $y' > f_1(x')$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then we select z' such that $z' > f_2(z)$ and $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$).
  - If $y' = f_1(x')$, then $y' \in \mathbb{Q}$, and we select $z' = f_2(z)$.
  - If $y' < f_1(x')$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then, by property (1) on page 28, there exists a point $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$), such that $z < z' < f_2(z)$ and that $[x',y'] \in \mathcal{S}_1$ if and only if $[z,z'] \in \mathcal{S}_2$ (notice that property (i) of f_2 plays a role here).

  In all cases, the interval $[z,z']$ is such that $([x',y'],[z,z']) \in Z$ and $[w,z]R_A[z,z']$.
- If $X = B$, then we distinguish three cases.
  - If $y' > f_1(x')$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then it must be $y > f_1(x)$ (as $y > y'$ and $x = x'$), which implies $z > f_2(w)$, and we select z' such that $f_2(w) < z' < z$ and $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$).
  - If $y' = f_1(x')$, then $y' \in \mathbb{Q}$ and $y > f_1(x)$ (as $y > y'$ and $x = x'$), which implies $z > f_2(w)$, and we select $z' = f_2(w)$.
  - If $y' < f_1(x')$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then, by property (1) on page 28, there exists a point $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$), such that $w < z' < f_2(w)$ and that $[x',y'] \in \mathcal{S}_1$ if and only if $[w,z'] \in \mathcal{S}_2$ (notice that property (i) of f_2 plays a role here).

In all cases, the interval $[w, z']$ is such that $([x', y'], [w, z']) \in Z$ and $[w, z] R_B [w, z']$.

- If $X = D$, then we first select a point w' such that $w < w' < z$, $w' \in \mathbb{Q}$ if and only if $x' \in \mathbb{Q}$, and such that $f_2(w') < z$ (the existence of such a point is guaranteed by property (iii) of f_2). Then, we select a point z' as follows.
  - If $y' > f_1(x')$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then we select z' such that $f_2(w) < z' < z$ and $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$).
  - If $y' = f_1(x')$, then $y' \in \mathbb{Q}$, and we select $z' = f_2(w)$. Notice that $z' \in \mathbb{Q}$ as well.
  - If $y' < f_1(x')$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then, by property (1) on page 28, there exists a point $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$), such that $w' < z' < f_2(w')$ and that $[x', y'] \in \mathcal{S}_1$ if and only if $[w', z'] \in \mathcal{S}_2$ (notice that property (i) of f_2 plays a role here).

  In all cases, the interval $[w', z']$ is such that $([x', y'], [w', z']) \in Z$ and $[w, z] R_D [w', z']$.
- If $X = \overline{A}$, then we distinguish three cases.
  - If $y' > f_1(x')$ and $x' \in \mathbb{Q}$ (resp., $x' \in \overline{\mathbb{Q}}$), then consider a point $\overline{z} \in \mathbb{Q}$ such that $\overline{z} < w$. We select a point $w' \in \mathbb{Q}$ (resp., $w' \in \overline{\mathbb{Q}}$) such that $w' < \overline{z} < w$ and $f_2(w') = \overline{z}$ (the existence of such a point is guaranteed by property (ii) of f_2).
  - If $y' = f_1(x')$ and $x' \in \mathbb{Q}$ (resp., $x' \in \overline{\mathbb{Q}}$), then $y' = x \in \mathbb{Q}$, which implies $w \in \mathbb{Q}$. We select a point $w' \in \mathbb{Q}$ (resp., $w' \in \overline{\mathbb{Q}}$) such that $w' < w$ and $f_2(w') = w$ (the existence of such a point is guaranteed by property (ii) of f_2).
  - If $y' < f_1(x')$ and $x' \in \mathbb{Q}$ (resp., $x' \in \overline{\mathbb{Q}}$), then consider a point $\overline{z} \in \mathbb{Q}$ such that $\overline{z} > w$. We select a point $w' \in \mathbb{Q}$ (resp., $w' \in \overline{\mathbb{Q}}$) such that $w' < \min\{0, w\}$, $f_2(w') = \overline{z}$, and that $[w', w] \in \mathcal{S}_2$ if and only if $[x', y'] \in \mathcal{S}_1$ (the existence of such a point is guaranteed by property (ii) of f_2). Notice that, since $w' < 0$, it is not the case that $[0, 3] R_O [w', w]$.

  In all cases, the interval $[w', w]$ is such that $([x', y'], [w', w]) \in Z$ and $[w, z] R_{\overline{A}} [w', w]$.
- If $X = \overline{B}$, then we distinguish three cases.
  - If $y' > f_1(x')$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then we select z' such that $z' > z$ and $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$),
  - If $y' = f_1(x')$, then $y' \in \mathbb{Q}$ and $y < f_1(x)$ (as $y < y'$ and $x = x'$), which implies $z < f_2(w)$, and we select $z' = f_2(w)$.
  - If $y' < f_1(x')$ and $y' \in \mathbb{Q}$ (resp., $y' \in \overline{\mathbb{Q}}$), then it must be the case that $y < f_1(x)$ (as $y < y'$ and $x = x'$). This yields $z < f_2(w)$, and we select $z' \in \mathbb{Q}$ (resp., $z' \in \overline{\mathbb{Q}}$) such that $z < z' < f_2(w)$ and that $[x', y'] \in \mathcal{S}_1$ if and only if $[w, z'] \in \mathcal{S}_2$ Notice that the existence of such a point strongly depends on the fact that it is not the case that $[0, 3] R_O [w, z']$. Indeed, suppose, towards a contradiction, that $[0, 3] R_O [w, z']$ holds, then we have $0 < w < 3 < z'$. From property (iv) of f_2, $0 < w < 3$ implies $f_2(w) < 3$, and thus $z' < 3$ (as $z' < f_2(w)$), contradicting the fact that $0 < w < 3 < z'$ holds.

  In all cases, the interval $[w, z']$ is such that $([x', y'], [w, z']) \in Z$ and $[w, z] R_{\overline{B}} [w, z']$.
- If $X = \overline{E}$, then we distinguish three cases.

- If $y' > f_1(x')$ and $x' \in \mathbb{Q}$ (resp., $x' \in \overline{\mathbb{Q}}$), then consider a point $\overline{z} \in \mathbb{Q}$ such that $\overline{z} < z$. We select a point $w' \in \mathbb{Q}$ (resp., $w' \in \overline{\mathbb{Q}}$) such that $w' < w$ and $f_2(w') = \overline{z}$ (the existence of such a point is guaranteed by property (ii) of f_2).
- If $y' = f_1(x')$ and $x' \in \mathbb{Q}$ (resp., $x' \in \overline{\mathbb{Q}}$), then $y' = y \in \mathbb{Q}$, which implies $z \in \mathbb{Q}$. We select a point $w' \in \mathbb{Q}$ (resp., $w' \in \overline{\mathbb{Q}}$) such that $w' < w$ and $f_2(w') = z$ (the existence of such a point is guaranteed by property (ii) of f_2).
- If $y' < f_1(x')$ and $x' \in \mathbb{Q}$ (resp., $x' \in \overline{\mathbb{Q}}$), then consider a point $\overline{z} \in \mathbb{Q}$ such that $\overline{z} > z$. We select a point $w' \in \mathbb{Q}$ (resp., $w' \in \overline{\mathbb{Q}}$) such that $w' < \min\{0, w\}$, $f_2(w') = \overline{z}$, and that $[w', w] \in \mathcal{S}_2$ if and only if $[x', y'] \in \mathcal{S}_1$ (the existence of such a point is guaranteed by property (ii) of f_2). Notice that, since $w' < 0$, it is not the case that $[0, 3]R_O[w', z]$.

In all cases, the interval $[w', z]$ is such that $([x', y'], [w', z]) \in Z$ and $[w, z]R_{\overline{E}}[w', z]$.

The backward condition can be verified in a very similar way and thus we omit the details. Therefore, Z is an $\mathsf{ABD}\overline{\mathsf{ABE}}$-bisimulation that violates $\langle O \rangle$. □

6 Harvest

In this paper, we compared and classified all fragments of HS with respect to their expressiveness, relative to the class of all linear orders and its subclass of all dense linear orders. For each of these classes, we identified a complete set of definabilities among HS modalities, valid in that class, thus obtaining a complete classification of the family of all $2^{12} = 4096$ fragments of HS with respect to their expressive power. The final outcome is that there are exactly 1347 expressively different fragments of HS, when we interpret them in the class of all linear orders, while such a number reduces to 966, when we restrict our attention to the subclass of all dense linear orders. Formally, the collection of results shown in the previous sections enables us to prove the following theorem.

Theorem 1 *Table 1 presents a complete set of optimal definabilities, relative to:*

- *the class* Lin *(definabilities in the groups on the top and in middle);*
- *the class* Den*, and, in general, every (left/right) symmetric class of dense linear orders containing at least one linear order isomorphic to* $\mathbb{R}$ *or to* $\mathbb{Q}$ *(all the definabilities).*

Proof For the class Lin, the class Den, and all symmetric classes of dense linear orders containing at least a linear order isomorphic to $\mathbb{R}$, the result is an immediate consequence of the results in Section 4 and Section 5. As for other symmetric classes of linear orders containing at least a linear order isomorphic to $\mathbb{Q}$, it is enough to observe that we have never made use of the Dedekind-completeness property (that distinguishes between $\mathbb{Q}$ and $\mathbb{R}$) and that, consequently, all the constructions given in Section 5 with respect to $\mathbb{R}$ can be easily adapted to $\mathbb{Q}$ instead. □

The proposed set of definability equations and the resulting classification of HS fragments are not appropriate any more if we change the semantics (from strict to non-strict) or if we interpret HS fragments in a different class of linear orders.

For instance, if the non-strict semantics is assumed, then $\langle A\rangle$ (resp., $\langle\overline{A}\rangle$) can be defined in $\overline{\mathsf{B}}\mathsf{E}$ (resp., $\mathsf{B}\overline{\mathsf{E}}$), as shown in [28]. Similarly, if we commit to the strict semantics, but we restrict our attention to the class of all discrete linear orders, $\langle A\rangle$ can be defined in $\overline{\mathsf{B}}\mathsf{E}$ as well: $\langle A\rangle p \equiv \varphi(p) \vee \langle E\rangle\varphi(p)$, where $\varphi(p)$ is a shorthand for $[E]\bot \wedge \langle\overline{B}\rangle([E][E]\bot \wedge \langle E\rangle(p \vee \langle\overline{B}\rangle p))$; likewise, $\langle\overline{A}\rangle$ is definable in $\mathsf{B}\overline{\mathsf{E}}$.

The classification of the expressive power of HS fragments with respect to other interesting classes of linear orderings, such as the class of all finite linear orders and the class of all discrete linear orders, is currently under investigation and will be reported in a forthcoming publication. The classification of HS fragments with respect to the various classes of linear orders when the non-strict semantics is assumed, as well as that of HS fragments enriched with point-based modalities borrowed from classical temporal logics [19], are still open problems. As a further research direction, it would also be natural to study extensions of classic logical formalisms with Allen's relations between intervals. As a contribution to that line of research, Conradie and Sciavicco identify in [16] the set of expressively different extensions of first-order logic with Allen's relations between intervals.

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