A Cancellation Theorem for BCCSP*

Luca Aceto[1], Wan Fokkink[2,3], and Anna Ingolfsdottir[1]

[1] Reykjavík University, School of Computer Science
Kringlan 1, 103 Reykjavík, Iceland
[2] Vrije Universiteit, Section Theoretical Computer Science
Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
[3] CWI, Embedded Systems Group
Kruislaan 413, 1098 SJ Amsterdam, The Netherlands
luca@ru.is,wanf@cs.vu.nl,annai@ru.is

Abstract. This paper presents a cancellation theorem for the preorders in van Glabbeek's linear time-branching time spectrum over BCCSP. Apart from having some intrinsic interest, the proven cancellation result plays a crucial role in the study of the cover equations, in the sense of Fokkink and Nain, that characterize the studied semantics. The techniques used in the proof of the cancellation theorem may also have some independent interest.

1 Introduction

The lack of consensus on what constitutes an appropriate notion of observable behaviour for reactive systems has led to a large number of proposals for behavioural equivalences and preorders for concurrent processes. In his by now classic paper [8], van Glabbeek presented the linear time-branching time spectrum of behavioural preorders and equivalences for finitely branching, concrete, sequential processes. The semantics in this spectrum are based on simulation notions and on decorated traces. (Figure 1 on page 6 depicts the linear time-branching time spectrum.)

Van Glabbeek [8] studied the semantics in his spectrum in the setting of the process algebra BCCSP, which contains only the basic process algebraic operators from CCS [12] and CSP [11], but is sufficiently powerful to express all finite synchronization trees. In the aforementioned reference, van Glabbeek gave, amongst a wealth of other results, (in)equational axiomatizations for the preorders and equivalences in the spectrum, such that two closed BCCSP terms can be equated by the axioms if, and only if, they are related by the preorder or equivalence in question. Groote [9] obtained ω-completeness results for most of the axiomatizations, in case the alphabet of actions is infinite. (An axiomatization E is *ω-complete* when an equation can be derived from E if, and only if, all of its closed instantiations can be derived from E.) The papers [1, 4, 5] offer

* The first and third author were partly supported by the project "The Equational Logic of Parallel Processes" (nr. 060013021) of The Icelandic Research Fund.

positive and negative results on the existence of finite (in)equational axiomatizations for several behavioural equivalences and preorders in the spectrum over the language BCCSP, both in the setting of finite and infinite sets of actions.

Fokkink and Nain developed in [6] a technique for studying the equational theory of BCCSP modulo the semantics in the spectrum and applied it to the setting of failures semantics. The aim of their approach is to obtain an explicit description of the equational theory for a particular semantics. The central idea is that if an equation $t \approx u$ is sound for BCCSP modulo some semantics in the linear time-branching time spectrum, then $u + t \approx t$ and $t + u \approx u$ are sound as well; and from the last two equations one can derive $t \approx u$. This implies that it is sufficient to consider only sound equations of the form $x + u \approx u$ and $at + u \approx u$ (where a denotes an action and t, u are BCCSP terms). These are called the *cover equations*. When the cover equations have been classified, one can proceed in two ways. Either one can determine an infinite family of cover equations that obstructs a finite basis, or one can isolate a finite basis among the cover equations.

In order to limit further the form of the cover equations that need to be considered, one usually tries to establish the following properties for the equivalence $\simeq$ at hand:

1. If $at + u + bv \simeq u + bv$ with $a \neq b$, then $at + u \simeq u$.
2. If $t \simeq u$, then t and u contain the same variables, at the same depth in their syntax trees.
3. If $t + x \simeq u + x$, and x is not a summand of $t + u$, then $t \simeq u$.

If the properties above hold, then it suffices to consider only cover equations of the form $at + au_1 + \ldots + au_n \approx au_1 + \ldots + au_n$.

It is easy to show that the second property holds for all equivalences finer than, or as fine as, partial trace equivalence, in case $|A| > 1$ (see [5], and cf. Lemma 3). The first and third properties have to be proved for each equivalence separately. Proving the first property is generally easy, but proving the third property can be a challenge.

Fokkink and Nain [7] proved this third property for failures semantics, with the aim to obtain an ω-completeness result for this semantics; their proof is rather delicate. To the best of our knowledge, failures semantics has so far been the only semantics in the spectrum for which the above result has been published. In this paper, we provide a proof of the above-mentioned property for all of the other semantics in the linear time-branching time spectrum. We actually prove the property for the preorder versions of these semantics, denoted by $\precsim$, since this constitutes a stronger property than for the corresponding equivalences $\simeq$. Despite the naturalness of the statement, which appears obvious, these proofs are far from trivial, and quite technical.

The proof technique that we employ to prove the above result also has some independent interest. Suppose that $t \not\precsim u$, meaning that $\sigma(t) \not\precsim \sigma(u)$ for some closed substitution σ. The challenge is to adapt σ into a distinguishing substitution ρ such that $\rho(t+x) \not\precsim \rho(u+x)$. This substitution ρ is obtained by adapting the value of $\sigma(x)$, where the new value $\rho(x)$ is based on the characteristics of the preorder under consideration; in some cases it is simply the constant $\mathbf{0}$ that does not exhibit any behaviour, while in others it requires an intricate recursive definition. We use this technique to prove the third property for ready trace, failure trace, readies, possible futures and possible worlds semantics. For the other semantics, we employ a similar approach, based on suitable transformations of closed substitutions, to show, conversely, that if $t + x \precsim u + x$, then $\sigma(t) \precsim \sigma(u)$ holds for each closed substitution σ.

The paper is organized as follows. Section 2 presents preliminaries on the language BCCSP and its transition-system semantics. The linear time-branching time spectrum is introduced in Section 3. We then present the main theorem in the paper (Section 4), whose proof takes up the whole of Section 5. We end the paper with some conclusions (Section 6), including a remark to the effect that the cancellation property also holds for bisimilarity.

2 Preliminaries

Syntax of BCCSP BCCSP(A) is a basic process algebra for expressing finite process behaviour. Its syntax consists of closed (process) terms p, q, r, s that are constructed from a constant $\mathbf{0}$, a binary operator $\_ + \_$ called *alternative composition*, and unary *prefix* operators $a\_$, where a ranges over some nonempty set A of *actions* (with typical elements a, b, c). We write $|A|$ for the cardinality of the set A.

Open terms t, u, v can moreover contain occurrences of variables from a countably infinite set V (with typical elements x, y, z).

A (closed) substitution maps variables in V to (closed) terms. For every term t and substitution σ, the term $\sigma(t)$ is obtained by replacing every occurrence of a variable x in t by $\sigma(x)$. Note that $\sigma(t)$ is closed if σ is a closed substitution.

Transition rules Intuitively, closed BCCSP(A) terms represent finite process behaviours, where $\mathbf{0}$ does not exhibit any behaviour, $p+q$ is the nondeterministic choice between the behaviours of p and q, and ap executes action a to transform into p. This intuition is captured, in the style of Plotkin, by the transition rules below, which give rise to A-labelled transitions between closed terms.

$$\frac{}{ax \xrightarrow{a} x} \qquad \frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'} \qquad \frac{y \xrightarrow{a} y'}{x + y \xrightarrow{a} y'}$$

The operational semantics is extended to open terms by assuming that variables do not exhibit any behaviour.

3 The Linear Time-Branching Time Spectrum

Van Glabbeek presented in [8] the linear time-branching time spectrum of behavioural semantics for finitely branching, concrete processes. In this section, for the sake of completeness, we define the semantics in this spectrum. We refer the interested reader to [8] for motivation, examples and a wealth of results on these semantics.

A *labelled transition system* contains a set of *states*, with typical element s, and a set of transitions $s \xrightarrow{a} s'$, where a ranges over some set of labels A. The set $\mathcal{I}(s)$ of *initial actions* of s consists of those labels a for which there exists a transition $s \xrightarrow{a} s'$.

First we define five variations on the notion of simulation.

Definition 1 (Simulations). *Assume a labelled transition system.*

- *A binary relation* $\mathcal{R}$ *on states is a* simulation *if* $s_0 \mathcal{R} s_1$ *and* $s_0 \xrightarrow{a} s_0'$ *imply* $s_1 \xrightarrow{a} s_1'$ *for some* s_1' *with* $s_0' \mathcal{R} s_1'$.
- *A simulation* $\mathcal{R}$ *is a* completed simulation *if* $s_0 \mathcal{R} s_1$ *and* $\mathcal{I}(s_0) = \emptyset$ *imply* $\mathcal{I}(s_1) = \emptyset$.
- *A simulation* $\mathcal{R}$ *is a* ready simulation *if* $s_0 \mathcal{R} s_1$ *and* $a \notin \mathcal{I}(s_0)$ *imply* $a \notin \mathcal{I}(s_1)$.
- *A simulation* $\mathcal{R}$ *is a* 2-nested simulation *if* $s_1 \mathcal{S} s_0$ *holds for some simulation* $\mathcal{S}$ *whenever* $s_0 \mathcal{R} s_1$.
- *A* bisimulation *is a symmetric simulation.*

Next we define six types of decorated versions of traces.

Definition 2 (Decorated Traces). *Assume a labelled transition system.*

- *A sequence of actions* $a_1 \ldots a_n$, *with* $n \geq 0$, *is a* (partial) trace *of a state* s_0 *if there is a sequence of transitions* $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \ldots s_{n-1} \xrightarrow{a_n} s_n$. *It is a* completed trace *of* s_0 *if moreover* $\mathcal{I}(s_n) = \emptyset$.
- *A pair* $(a_1 \ldots a_n, X)$, *with* $n \geq 0$ *and* $X \subseteq A$, *is a* ready pair *of a state* s_0 *if there is a sequence of transitions* $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \ldots s_{n-1} \xrightarrow{a_n} s_n$ *with* $\mathcal{I}(s_n) = X$. *It is a* failure pair *of* s_0 *if* $\mathcal{I}(s_n) \cap X = \emptyset$.
- *A sequence* $X_0 a_1 X_1 \ldots a_n X_n$, *with* $n \geq 0$ *and* $X_i \subseteq A$, *is a* ready trace *of a state* s_0 *if there is a sequence of transitions* $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \ldots s_{n-1} \xrightarrow{a_n} s_n$ *with* $\mathcal{I}(s_i) = X_i$ *for* $i = 0, \ldots, n$. *It is a* failure trace *of* s_0 *if* $\mathcal{I}(s_i) \cap X_i = \emptyset$ *for* $i = 0, \ldots, n$.

In what follows, we shall often write

- $s_0 \xrightarrow{a_1 \ldots a_n} s_n$ if there is a sequence of transitions $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \ldots s_{n-1} \xrightarrow{a_n} s_n$,
- $s_0 \xrightarrow{a_1 \ldots a_n}$ if there is some s_n such that $s_0 \xrightarrow{a_1 \ldots a_n} s_n$ and
- $s_0 \not\xrightarrow{a_1 \ldots a_n}$ if $s_0 \xrightarrow{a_1 \ldots a_n}$ does not hold.

Definition 3 (Depth). *The* depth *of a term* t, *denoted by* $depth(t)$, *is the length of a longest trace of* t.

Finally, we define two semantics based on possible futures and on possible worlds.

Definition 4 (Possible Futures/Worlds). *Assume a labelled transition system.*

– *A pair* $(a_1 \dots a_n, X)$*, with* $n \geq 0$ *and* $X \subseteq A^*$*, is a* possible future *of a state* s_0 *if there is a sequence of transitions* $s_0 \stackrel{a_1}{\to} s_1 \stackrel{a_2}{\to} \dots s_{n-1} \stackrel{a_n}{\to} s_n$ *where* X *is the set of traces of* s_n*.*
– *A state* s *is* deterministic *if for each* $a \in \mathcal{I}(s)$ *there is exactly one state* s' *such that* $s \stackrel{a}{\to} s'$*, and moreover* s' *is deterministic.*
A state s *is a* possible world *of a state* s_0 *if* s *is deterministic and* $s \mathrel{\mathcal{R}} s_0$ *for some ready simulation* $\mathcal{R}$*.*

Two states s and s' are related by the *simulation*, *ready simulation*, *2-nested simulation* or *completed simulation preorder* if there exists a simulation, ready simulation, 2-nested simulation or completed simulation $\mathcal{R}$, respectively, with $s \mathrel{\mathcal{R}} s'$. They are *bisimilar* if there is a bisimulation that relates them. They are related by the *possible futures*, *possible worlds*, *ready traces*, *failure traces*, *readies*, *failures*, *completed traces*, or *partial traces preorder* if the set of possible futures, possible worlds, ready traces, failure traces, ready pairs, failure pairs, completed traces, or traces of the former is included in that of the latter, respectively.

Figure 1 depicts the linear time-branching time spectrum, where a directed edge from one semantics to another means that the source of the edge is finer than the target. We use $\precsim$ to denote a preorder in this spectrum. When we want to refer to a specific preorder in the spectrum, we shall subscribe the symbol $\precsim$ with the initials of the intended semantics. For instance, we shall use $\precsim_{\mathrm{RS}}$ to denote the ready simulation preorder, $\precsim_{\mathrm{S}}$ for the simulation preorder, $\precsim_{\mathrm{F}}$ for the failures preorder, $\precsim_{\mathrm{CT}}$ for the completed traces preorder, and $\precsim_{\mathrm{PT}}$ for the partial traces preorder.

Remark 1. We note that for each of the preorders in the spectrum, if $p \precsim q$, then $depth(p) \leq depth(q)$. In addition, for the 2-nested simulation and the possible futures preorder, if $p \precsim q$, then the closed terms p and q have the same depth and the same set of traces.

Each preorder in the linear time-branching time spectrum is a *precongruence* over the algebra of closed BCCSP(A) terms. That is, $p_1 \precsim q_1$ and $p_2 \precsim q_2$ imply $ap_1 \precsim aq_1$, for each $a \in A$, and $p_1 + p_2 \precsim q_1 + q_2$.

Given a preorder $\precsim$ over closed terms, for open terms t and u, we define $t \precsim u$ if $\rho(t) \precsim \rho(u)$ for each closed substitution ρ.

The core axioms A1–4 for BCCSP(A) given below are ω-complete [13], and sound and ground-complete [10, 12] modulo bisimulation equivalence, which is the finest semantics in the linear time-branching time spectrum.

$$\begin{array}{ll} \mathrm{A1} & x + y \approx y + x \\ \mathrm{A2} & (x + y) + z \approx x + (y + z) \\ \mathrm{A3} & x + x \approx x \\ \mathrm{A4} & x + \mathbf{0} \approx x \end{array}$$

In the remainder of this paper, process terms are considered modulo A1–4. A term x or at is a *summand* of each term $x + u$ or $at + u$, respectively. We use *summation* $\sum_{i=1}^{n} t_i$ (with $n \geq 0$) to denote $t_1 + \ldots + t_n$, where the empty sum

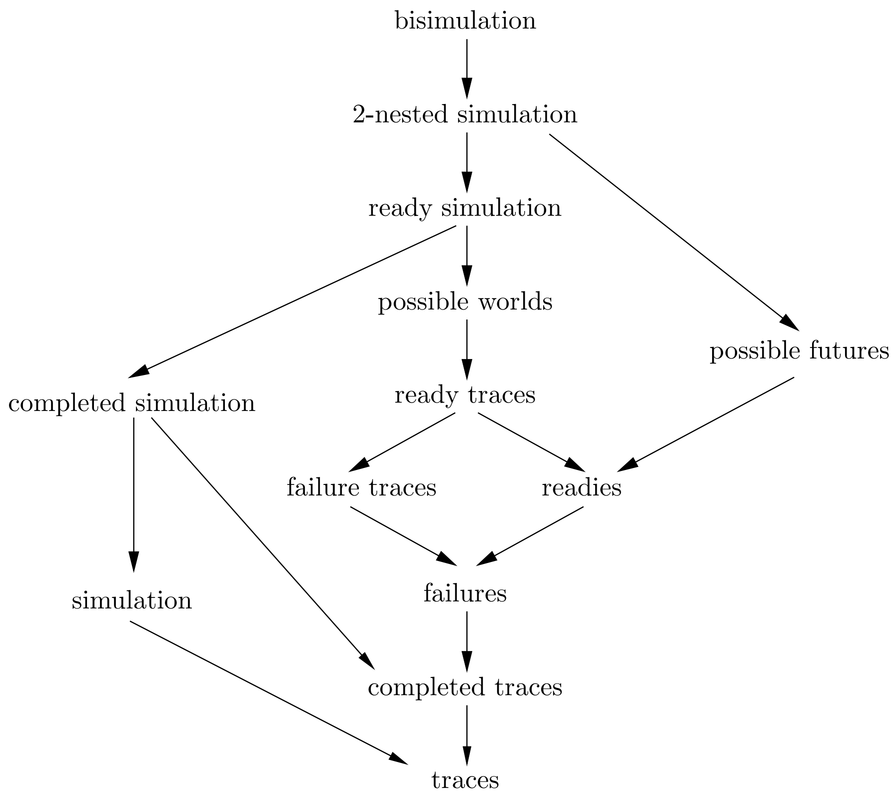

Fig. 1. The linear time-branching time spectrum

denotes **0**. As binding convention, alternative composition and summation bind more weakly than prefixing. Modulo the equations A1–4 each BCCSP(A) term t can be written in the form $\sum_{i=1}^{n} t_i$, where each t_i is either a variable or is of the form at' for some action a and term t'.

4 The Cancellation Result

The following theorem, which states a kind of cancellation result for the preorders in the spectrum, is the main result of the paper.

Theorem 1. *Let $\precsim$ be a preorder in the linear time-branching time spectrum. If $t + x \precsim u + x$, and x is not a summand of $t + u$, then $t \precsim u$.*

The remainder of this paper will be devoted to a detailed proof of the above result. Before embarking on its proof, let us remark in passing that Theorem 1 needs to be shown separately for each preorder in the linear time-branching time spectrum. Despite the naturalness of its statement, which appears obvious, these proofs are not trivial, and quite technical. Fokkink and Nain [7] proved the instance of the above result for failures semantics, with the aim to obtain an ω-completeness result for this semantics, and their proof is rather delicate. The proofs of the statement for the other semantics in the spectrum, which we present in the following section, are also challenging.

Remark 2. The condition in Theorem 1 that x is not a summand of $t + u$ is essential. For instance, $x + x \precsim_{\mathrm{PT}} \mathbf{0} + x$, but $x \not\precsim_{\mathrm{PT}} \mathbf{0}$. And $\mathbf{0} + x \precsim_{\mathrm{CT}} x + x$, but $\mathbf{0} \not\precsim_{\mathrm{CT}} x$.

5 Proof of Theorem 1

In this section, we collect the proof of Theorem 1 for each of the behavioural preorders in the linear time-branching time spectrum ranging between the 2-nested simulation and partial traces preorders. (As we remark in Section 6, Theorem 1 also holds for bisimulation equivalence.)

Throughout this section, we use $\sigma_{\mathbf{0}}$ to stand for the closed substitution mapping each variable to $\mathbf{0}$. For each closed substitution σ, variable x, and closed term p, we use the notation $\sigma[x \mapsto p]$ to stand for the closed substitution mapping x to p, and acting like σ on all other variables.

The following lemmas will find repeated application in the proof of Theorem 1.

Lemma 1. *Let σ be a closed substitution. Then $\sigma(t) \stackrel{a_1 \ldots a_n}{\rightarrow} p$, for some closed term p, sequence of actions $a_1 \ldots a_n$ and $n \geq 0$, iff there are a $j \leq n$ and a term t' such that $t \stackrel{a_1 \ldots a_j}{\rightarrow} t'$ and*

1. *either $j = n$ and $\sigma(t') = p$*
2. *or $j < n$ and $\sigma(x) \stackrel{a_{j+1} \ldots a_n}{\rightarrow} p$, for some summand x of t'.*

Proof. Both implications can be shown by induction on the structure of t. The details are tedious, but not hard, and are therefore omitted. □

The following two lemmas collect some well-known facts regarding the partial and completed traces preorders (see e.g. [5]).

Lemma 2. *If $t \precsim_{\mathrm{PT}} u$, then all variables in t also occur in u.*

Proof. Suppose, towards a contradiction, that there exists some x occurring in t that does not occur in u. Let $\sigma = \sigma_{\mathbf{0}}[x \mapsto a^{depth(u)+1}\mathbf{0}]$. Then

$$depth(\sigma(t)) \geq depth(\sigma(x)) = depth(u) + 1 > depth(\sigma(u)) \ ,$$

which by Remark 1 contradicts $t \precsim_{\mathrm{PT}} u$. □

Lemma 3. *If $t + x \precsim u$, and either $\precsim \subseteq \precsim_{\mathrm{CT}}$, or $\precsim \subseteq \precsim_{\mathrm{PT}}$ and $|A| > 1$, then x is a summand of u.*

Proof. We distinguish the two cases in the statement of the lemma.
CASE 1: $\precsim \subseteq \precsim_{\mathrm{CT}}$.

Pick some $a \in A$, and consider the closed substitution

$$\sigma = \sigma_{\mathbf{0}}[x \mapsto a^{depth(u)+1}\mathbf{0}] \ .$$

Clearly, $a^{depth(u)+1}$ is a completed trace of $\sigma(t+x)$, so it must be a completed trace of $\sigma(u)$. By Lemma 1, this implies that x is a summand of u.
CASE 2: $\precsim \subseteq \precsim_{\mathrm{PT}}$ and $|A| > 1$.

Pick some distinct $a, b \in A$, and consider the closed substitution

$$\sigma = \sigma_{\mathbf{0}}[x \mapsto a^{depth(u)}b\mathbf{0}] \ .$$

Then $a^{depth(u)}b$ is a partial trace of $\sigma(t+x)$, so it must be a partial trace of $\sigma(u)$. By Lemma 1, this implies that x is a summand of u. □

Remark 3. If $|A| = 1$, then $\precsim_{\mathrm{PT}}$ and $\precsim_{\mathrm{S}}$ coincide—see, e.g., [2]. For this special case, Lemma 3 fails. Namely, let $A = \{a\}$. Then $x \precsim ax$ is sound for $\precsim_{\mathrm{PT}}$ (and $\precsim_{\mathrm{S}}$).

5.1 Proof of Theorem 1 for $\precsim_{\mathrm{CT}}$

We begin our proof of Theorem 1 for $\precsim_{\mathrm{CT}}$ by stating a useful lemma.

Lemma 4. *Let $t = \sum_{i \in I} x_i$ and $u = \sum_{k \in K} b_k.u_k + \sum_{j \in J} y_j$, where I and J are finite index sets. Then $t \precsim_{\mathrm{CT}} u$ iff $K = \emptyset$ and $\{x_i \mid i \in I\} = \{y_j \mid j \in J\}$; that is, $t = u$.*

Proof. The "if" implication is trivial, since then t and u are bisimilar. We therefore focus on establishing the implication from left to right. First note that K must be empty, because otherwise $\sigma_{\mathbf{0}}(u)$ would not have the empty string ε as one of its completed traces, contradicting $t \precsim_{\mathrm{CT}} u$. We now prove that $\{x_i \mid i \in I\} = \{y_j \mid j \in J\}$.

To this end, we begin by observing that each x_i must occur as a summand of u by Lemma 3. We are therefore left to prove that each y_j is also a summand of t. To see that this does hold, pick an action $a \in A$, and consider the closed substitution $\sigma = \sigma_{\mathbf{0}}[y_j \mapsto a\mathbf{0}]$. The only completed trace of $\sigma(u)$ is a. It follows that y_j must be a summand of t. Indeed, if y_j is not a summand of t, then $\sigma(t) = \sigma_{\mathbf{0}}(t) = \mathbf{0}$ has only the empty string ε as completed trace, contradicting $t \precsim_{\mathrm{CT}} u$. □

We are now ready to prove that Theorem 1 holds for $\precsim_{\mathrm{CT}}$.

Proof of Theorem 1 for $\precsim_{\mathrm{CT}}$ Assume that $t + x \precsim_{\mathrm{CT}} u + x$, and x is not a summand of $t+u$. Let σ be a closed substitution. We prove that each completed trace of $\sigma(t)$ is also a completed trace of $\sigma(u)$. This is immediate from the proviso of the theorem if $\sigma(x) = \mathbf{0}$. Assume therefore that $\sigma(x) \neq \mathbf{0}$.

Let $a_1 \ldots a_n$ be a completed trace of $\sigma(t)$—that is, $\sigma(t) \stackrel{a_1 \ldots a_n}{\to} \mathbf{0}$. If $n = 0$, then $\sigma(t) = \mathbf{0}$. This means that $t = \sum_{i \in I} x_i$ for some set of variables $\{x_i \mid i \in I\}$

such that $\sigma(x_i) = \mathbf{0}$ for each $i \in I$. Note that, by the proviso of the theorem, $x \neq x_i$ for each $i \in I$. Since $t + x \precsim_{\mathrm{CT}} u + x$, Lemma 4 yields that $u = t$, and therefore $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$.

Assume now that $n \geq 1$. Since $\sigma(t) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$, Lemma 1 yields that there are a $j \leq n$ and a term t' such that $t \stackrel{a_1 \ldots a_j}{\longrightarrow} t'$ and

1. either $j = n$ and $\sigma(t') = \mathbf{0}$,
2. or $j < n$ and $\sigma(y) \stackrel{a_{j+1} \ldots a_n}{\longrightarrow} \mathbf{0}$, for some summand y of t'.

In the former case, $t' = \sum_{m \in M} z_m$ for some collection $\{z_m \mid m \in M\}$ of variables such that $\sigma(z_m) = \mathbf{0}$ for each $m \in M$. By assumption, $\sigma(x) \neq \mathbf{0}$, so $z_m \neq x$ for each $m \in M$. We wish to argue that $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$. Pick an $\ell > n$. By Lemma 1,

$$\sigma[x \mapsto a^\ell \mathbf{0}](t) \stackrel{a_1 \ldots a_n}{\longrightarrow} \sigma[x \mapsto a^\ell \mathbf{0}](t') = \sigma(t') = \mathbf{0} \ .$$

Since $t+x \precsim_{\mathrm{CT}} u+x$ and $n \geq 1$, the term $\sigma[x \mapsto a^\ell \mathbf{0}](u+x)$ also affords $a_1 \ldots a_n$ as one of its completed traces. As $\ell > n$, it follows that $\sigma[x \mapsto a^\ell \mathbf{0}](u) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$. Using Lemma 1 and the assumption that $\ell > n$, we may conclude that $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$, which was to be shown.

In the latter case, it suffices to show that $u \stackrel{a_1 \ldots a_j}{\longrightarrow} u'$ for some u' that has y as a summand. Pick an $N > \mathit{depth}(u)$. By Lemma 1, $\sigma_{\mathbf{0}}[y \mapsto a^N \mathbf{0}](t)$ affords the completed trace $a_1 \ldots a_j a^N$. Since $j + N \geq 1$, $a_1 \ldots a_j a^N$ is also a completed trace of $\sigma_{\mathbf{0}}[y \mapsto a^N \mathbf{0}](t + x)$, and therefore of $\sigma_{\mathbf{0}}[y \mapsto a^N \mathbf{0}](u + x)$. Note that if $j = 0$, then $y \neq x$, because x is not a summand of t by the proviso of the theorem. Hence it follows that $a_1 \ldots a_j a^N$ is also a completed trace of $\sigma_{\mathbf{0}}[y \mapsto a^N \mathbf{0}](u)$. Let $b_1 \ldots b_{N+j} = a_1 \ldots a_j a^N$. Since $N > \mathit{depth}(u)$, by Lemma 1, $u \stackrel{b_1 \ldots b_k}{\longrightarrow} u'$ and $\sigma_{\mathbf{0}}[y \mapsto a^N \mathbf{0}](z) \stackrel{b_{k+1} \ldots b_{N+j}}{\longrightarrow} \mathbf{0}$ for some term u', variable z and $k < N$, where u' has z as a summand. Since $N + j > k$, it follows that $z = y$, $k = j$ and $b_{k+1} \ldots b_{N+j} = a^N$. Concluding, $u \stackrel{a_1 \ldots a_j}{\longrightarrow} u'$ where u' has y as a summand. Since $\sigma(y) \stackrel{a_{j+1} \ldots a_n}{\longrightarrow} \mathbf{0}$ and $j < n$, by Lemma 1, $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$, which was to be shown.

This concludes the proof for $\precsim_{\mathrm{CT}}$. □

5.2 Proof of Theorem 1 for the Simulation Preorders

In this section, we collect the proof of Theorem 1 for the ready simulation, completed simulation, simulation and 2-nested simulation preorders.

Proof of Theorem 1 for $\precsim_{\mathbf{RS}}$ Assume that $t + x \precsim_{\mathrm{RS}} u + x$, and x is not a summand of $t + u$. Let σ be a closed substitution. We prove that $\sigma(t) \precsim_{\mathrm{RS}} \sigma(u)$.

In order to prove that $\sigma(t) \precsim_{\mathrm{RS}} \sigma(u)$, we need to show the following two claims:

1. if $\sigma(t) \stackrel{a}{\rightarrow} p$, then $\sigma(u) \stackrel{a}{\rightarrow} q$ for some q such that $p \precsim_{\mathrm{RS}} q$, and
2. $\mathcal{I}(\sigma(u)) \subseteq \mathcal{I}(\sigma(t))$.

We prove these two claims separately.

PROOF OF CLAIM 1. Suppose that $\sigma(t) \xrightarrow{a} p$. Either this transition is due to a variable summand y of t such that $\sigma(y) \xrightarrow{a} p$, or there is a summand at' of t such that $p = \sigma(t')$. In the former case, $y \neq x$ by the proviso of the theorem. Since $t + x \precsim_{\mathrm{RS}} u + x$, by Lemma 3, y is also a summand of u. It follows that $\sigma(u) \xrightarrow{a} p$, and we are done.

Suppose now that there is a summand at' of t such that $p = \sigma(t')$. If $\sigma(y) \xrightarrow{a} q$ for some variable summand y of u and closed term q such that $p \precsim_{\mathrm{RS}} q$, we are done. Assume therefore that, for each closed term r and variable summand y of u,

$$\sigma(y) \xrightarrow{a} r \text{ implies } p \not\precsim_{\mathrm{RS}} r \ . \tag{1}$$

We claim that $p \precsim_{\mathrm{RS}} \sigma(u')$ for some summand au' of u. We proceed with the proof of this claim by distinguishing two cases, depending on whether x occurs in t' or not.

– CASE x DOES NOT OCCUR IN t'. Let $N \geq \mathit{depth}(\sigma(t))$. Define the closed term s as follows:

$$s = \sum_{b \in \mathcal{I}(\sigma(x))} b a^N \mathbf{0} \ .$$

Since at' is a summand of t and x does not occur in t',

$$\sigma[x \mapsto s](t + x) \xrightarrow{a} \sigma[x \mapsto s](t') = \sigma(t') = p \ .$$

So, since $t + x \precsim_{\mathrm{RS}} u + x$,

$$\sigma[x \mapsto s](u + x) \xrightarrow{a} q \ ,$$

for some q such that $p \precsim_{\mathrm{RS}} q$. Note that $p \not\precsim_{\mathrm{RS}} a^N \mathbf{0}$, because $\mathit{depth}(p) < N$. And by assumption (1), for variable summands $y \neq x$ of u, $\sigma[x \mapsto s](y) \xrightarrow{a} r$ implies $p \not\precsim_{\mathrm{RS}} r$. Hence, u must have a summand of the form au' such that

$$p \precsim_{\mathrm{RS}} \sigma[x \mapsto s](u') \ .$$

We now prove, by induction on the depth of p, that, as $\mathit{depth}(p) < N$,

$$p \precsim_{\mathrm{RS}} \sigma(u') \ .$$

First of all, note that

$$\mathcal{I}(\sigma(u')) = \mathcal{I}(\sigma[x \mapsto s](u')) = \mathcal{I}(p)$$

because $\mathcal{I}(\sigma(x)) = \mathcal{I}(s)$.

Suppose that $p \xrightarrow{b} p'$. We prove that $\sigma(u') \xrightarrow{b} q'$ for some q' such that $p' \precsim_{\mathrm{RS}} q'$. Since $p \precsim_{\mathrm{RS}} \sigma[x \mapsto s](u')$, there is a q'' such that $\sigma[x \mapsto s](u') \xrightarrow{b} q''$ and

$p' \precsim_{\mathrm{RS}} q''$. Note that this transition cannot be due to a summand x of u', because $depth(p') \leq N-2$ and therefore $p' \not\precsim_{\mathrm{RS}} a^N\mathbf{0}$. If the transition

$$\sigma[x \mapsto s](u') \xrightarrow{b} q''$$

is due to a variable summand $y \neq x$ of u', then $\sigma(u') \xrightarrow{b} q''$ also holds, and we are done. Otherwise, there is a summand bu'' of u' such that $q'' = \sigma[x \mapsto s](u'')$. As $p' \precsim_{\mathrm{RS}} \sigma[x \mapsto s](u'')$ and $depth(p') < depth(p) < N$, the induction hypothesis yields that $p' \precsim_{\mathrm{RS}} \sigma(u'')$. Since $\sigma(u') \xrightarrow{b} \sigma(u'')$, we are done.

Therefore $p \precsim_{\mathrm{RS}} \sigma(u')$, as claimed above. Since $\sigma(u) \xrightarrow{a} \sigma(u')$, we are done.

– Case x occurs in t'. In this case,

$$depth(p) = depth(\sigma(t')) \geq depth(\sigma(x)) \ .$$

Since each a-derivative of $\sigma(x)$ has depth smaller than that of p, it cannot simulate p. Since $\sigma(t+x) \precsim_{\mathrm{RS}} \sigma(u+x)$ and $\sigma(t+x) \xrightarrow{a} p$, it follows that $\sigma(u) \xrightarrow{a} \sigma(u')$ and $p \precsim_{\mathrm{RS}} \sigma(u')$ for some summand au' of u.

Proof of Claim 2. Assume that $a \in \mathcal{I}(\sigma(u))$. Since x is not a summand of u by the proviso of the theorem, $a \in \mathcal{I}(\sigma[x \mapsto \mathbf{0}](u+x))$. As $t+x \precsim_{\mathrm{RS}} u+x$, it follows that $a \in \mathcal{I}(\sigma[x \mapsto \mathbf{0}](t+x))$. This implies $a \in \mathcal{I}(\sigma(t))$, which was to be shown.

This concludes the proof for $\precsim_{\mathrm{RS}}$. □

Proof of Theorem 1 for $\precsim_{\mathrm{CS}}$ Assume that $t+x \precsim_{\mathrm{CS}} u+x$, and x is not a summand of $t+u$. Let σ be a closed substitution. We prove that $\sigma(t) \precsim_{\mathrm{CS}} \sigma(u)$.

In order to prove that $\sigma(t) \precsim_{\mathrm{CS}} \sigma(u)$, we need to show the following two claims:

1. if $\sigma(t) \xrightarrow{a} p$, then $\sigma(u) \xrightarrow{a} q$ for some q such that $p \precsim_{\mathrm{CS}} q$, and
2. if $\sigma(t) = \mathbf{0}$, then $\sigma(u) = \mathbf{0}$.

We prove these two claims separately.

Proof of Claim 1. Suppose that $\sigma(t) \xrightarrow{a} p$. We show that $\sigma(u) \xrightarrow{a} q$ for some q such that $p \precsim_{\mathrm{CS}} q$. This is immediate if there is a variable summand y of u such that $\sigma(y) \xrightarrow{a} q$ and $p \precsim_{\mathrm{CS}} q$.

Assume therefore that, for each closed term r and variable summand y of u,

$$\sigma(y) \xrightarrow{a} r \text{ implies } p \not\precsim_{\mathrm{CS}} r \ . \tag{2}$$

By Lemma 3, it follows that $\sigma(t) \xrightarrow{a} p$, because t has a summand at' such that $\sigma(t') = p$.

We claim that $p \precsim_{\mathrm{CS}} \sigma(u')$ for some summand au' of u. We proceed with the proof of this claim by distinguishing two cases, depending on whether x occurs in t' or not.

– CASE x DOES NOT OCCUR IN t'. Let $N > \mathit{depth}(\sigma(t))$. Since at' is a summand of t and x does not occur in t,

$$\sigma[x \mapsto a^N\mathbf{0}](t+x) \xrightarrow{a} \sigma[x \mapsto a^N\mathbf{0}](t') = \sigma(t') = p \ .$$

Therefore

$$\sigma[x \mapsto a^N\mathbf{0}](u+x) \xrightarrow{a} q \ ,$$

for some q such that $p \precsim_{\mathrm{CS}} q$. Note that $p \not\precsim_{\mathrm{CS}} a^{N-1}\mathbf{0}$, because $\mathit{depth}(p) < N-1$ and $a^{N-1}\mathbf{0}$ affords only a completed trace of length $N-1$. Hence, by assumption (2), u must have a summand of the form au' such that

$$p \precsim_{\mathrm{CS}} \sigma[x \mapsto a^N\mathbf{0}](u') \ .$$

We now prove, by induction on the depth of p, that, as $\mathit{depth}(p) < N-1$,

$$p \precsim_{\mathrm{CS}} \sigma(u') \ .$$

First of all, if $p = \mathbf{0}$, then $\sigma[x \mapsto a^N\mathbf{0}](u') = \mathbf{0}$, which yields that x does not occur in u'. Therefore, $\sigma[x \mapsto a^N\mathbf{0}](u') = \sigma(u')$, and we are done.
Suppose that $p \xrightarrow{b} p'$. We prove that $\sigma(u') \xrightarrow{b} q'$ for some q' such that $p' \precsim_{\mathrm{CS}} q'$. Since $p \precsim_{\mathrm{CS}} \sigma[x \mapsto a^N\mathbf{0}](u')$, there is a q'' such that $\sigma[x \mapsto a^N\mathbf{0}](u') \xrightarrow{b} q''$ and $p' \precsim_{\mathrm{CS}} q''$. Note that this transition cannot be due to a summand x of u', because $\mathit{depth}(p') < N-2$ and therefore $p' \not\precsim_{\mathrm{CS}} a^{N-1}\mathbf{0}$. If the transition $\sigma[x \mapsto a^N\mathbf{0}](u') \xrightarrow{b} q''$ is due to a variable summand $y \neq x$ of u', then $\sigma(u') \xrightarrow{b} q''$ also holds, and we are done. Otherwise, there is a summand bu'' of u' such that

$$q'' = \sigma[x \mapsto a^N\mathbf{0}](u'') \ .$$

As $p' \precsim_{\mathrm{CS}} \sigma[x \mapsto a^N\mathbf{0}](u'')$ and $\mathit{depth}(p') < \mathit{depth}(p) < N-1$, the induction hypothesis yields that $p' \precsim_{\mathrm{CS}} \sigma(u'')$. Since $\sigma(u') \xrightarrow{b} \sigma(u'')$, we are done.
Therefore $p \precsim_{\mathrm{CS}} \sigma(u')$, as claimed above. Since $\sigma(u) \xrightarrow{a} \sigma(u')$, we are done.

– CASE x OCCURS IN t'. In this case,

$$\mathit{depth}(p) = \mathit{depth}(\sigma(t')) \geq \mathit{depth}(\sigma(x)) \ .$$

Since each a-derivative of $\sigma(x)$ has depth smaller than that of p, it cannot simulate p. As $\sigma(t+x) \precsim_{\mathrm{CS}} \sigma(u+x)$ by the proviso of the theorem, and $\sigma(t+x) \xrightarrow{a} p$, it follows that $\sigma(u) \xrightarrow{a} \sigma(u')$ and $p \precsim_{\mathrm{CS}} \sigma(u')$ for some summand au' of u.

PROOF OF CLAIM 2. Assume that $\sigma(t) = \mathbf{0}$. This means that t is a sum of variables, and $t = u$ by Lemma 4. (Recall that $\precsim_{\mathrm{CS}} \subseteq \precsim_{\mathrm{CT}}$; see Figure 1 on page 6.) Hence $\sigma(u) = \mathbf{0}$.

This concludes the proof for $\precsim_{\mathrm{CS}}$. □

Proof of Theorem 1 for $\precsim_{\mathrm{S}}$ Assume that $t + x \precsim_{\mathrm{S}} u + x$, and x is not a summand of $t + u$. Let σ be a closed substitution. We prove that $\sigma(t) \precsim_{\mathrm{S}} \sigma(u)$. Suppose that $\sigma(t) \xrightarrow{a} p$. We show that $\sigma(u) \xrightarrow{a} q$ for some q such that $p \precsim_{\mathrm{S}} q$.

Assume, first of all, that the transition $\sigma(t) \xrightarrow{a} p$ is due to a variable summand y of t, that is $\sigma(y) \xrightarrow{a} p$. In this case, $y \neq x$ by the proviso of the theorem. If $|A| > 1$, then $t + x \precsim_{\mathrm{S}} u + x$ yields that y is also a variable summand of u (Lemma 3), and we are done, because $\sigma(u) \xrightarrow{a} p$. If $A = \{a\}$, then Lemma 2 yields that y occurs in u. Therefore, $u \xrightarrow{a^n} u'$ for some $n \geq 0$ and term u' having y as a summand. If $n = 0$, then $\sigma(u) \xrightarrow{a} p$ as above, and we are done. If instead $u \xrightarrow{a} u_1 \xrightarrow{a^{n-1}} u'$ for some u_1, then $\sigma(u) \xrightarrow{a} \sigma(u_1)$. Moreover, we claim that $p \precsim_{\mathrm{S}} \sigma(u_1)$. Indeed, since $u_1 \xrightarrow{a^{n-1}} u'$, the variable y occurs in u_1. This yields that

$$depth(p) < depth(\sigma(y)) \leq depth(\sigma(u_1)) \enspace .$$

The claim now follows because, in the presence of a single action a, $\precsim_{\mathrm{PT}}$ and $\precsim_{\mathrm{S}}$ coincide—see, e.g., [2]—, and therefore

$$depth(p) < depth(\sigma(u_1)) \text{ implies } p \precsim_{\mathrm{S}} \sigma(u_1) \enspace .$$

We are now left to examine the case in which $\sigma(t) \xrightarrow{a} p$, because t has a summand at' such that $p = \sigma(t')$. If $\sigma(y) \xrightarrow{a} q$ for some variable summand y of u and closed term q such that $p \precsim_{\mathrm{S}} q$, we are done. Assume therefore that, for each closed term r and variable summand y of u,

$$\sigma(y) \xrightarrow{a} r \text{ implies } p \not\precsim_{\mathrm{S}} r \enspace . \tag{3}$$

We claim that $p \precsim_{\mathrm{S}} \sigma(u')$ for some summand au' of u.

We proceed with the proof of this claim by distinguishing two cases, depending on whether x occurs in t' or not.

- Case x does not occur in t'. In this case,

  $$\sigma[x \mapsto \mathbf{0}](t + x) \xrightarrow{a} \sigma[x \mapsto \mathbf{0}](t') = \sigma(t') = p \enspace .$$

  Since $t + x \precsim_{\mathrm{S}} u + x$, there is a closed term q such that $\sigma[x \mapsto \mathbf{0}](u) \xrightarrow{a} q$ and $p \precsim_{\mathrm{S}} q$. By (3), it follows that $q = \sigma[x \mapsto \mathbf{0}](u')$ for some summand au' of u. Since $\mathbf{0} \precsim_{\mathrm{S}} \sigma(x)$ and $\precsim_{\mathrm{S}}$ is a precongruence, we may conclude that

  $$p \precsim_{\mathrm{S}} q = \sigma[x \mapsto \mathbf{0}](u') \precsim_{\mathrm{S}} \sigma(u') \enspace .$$

  It follows that $\sigma(u) \xrightarrow{a} \sigma(u')$ and $p \precsim_{\mathrm{S}} \sigma(u')$, and we are done.
- Case x occurs in t'. In this case,

  $$depth(p) = depth(\sigma(t')) \geq depth(\sigma(x)) \enspace .$$

  Since each a-derivative of $\sigma(x)$ has depth smaller than that of p, it cannot simulate p. As $\sigma(t + x) \precsim_{\mathrm{S}} \sigma(u + x)$ by the proviso of the theorem, and $\sigma(t + x) \xrightarrow{a} p$, it follows by (3) that $\sigma(u) \xrightarrow{a} \sigma(u')$ and $p \precsim_{\mathrm{S}} \sigma(u')$ for some summand au' of u.

This concludes the proof for $\precsim_{\mathrm{S}}$. ☐

Proof of Theorem 1 for $\precsim_{\mathbf{2NS}}$ Assume that $t + x \precsim_{2NS} u + x$, and x is not a summand of $t+u$. Let σ be a closed substitution. We prove that $\sigma(t) \precsim_{2NS} \sigma(u)$.

In order to prove that $\sigma(t) \precsim_{2NS} \sigma(u)$, we need to show the following two claims:

1. if $\sigma(t) \xrightarrow{a} p$, then $\sigma(u) \xrightarrow{a} q$ for some q such that $p \precsim_{2NS} q$, and
2. $\sigma(u) \precsim_{S} \sigma(t)$.

We prove these two claims separately.

Proof of Claim 1. Suppose that $\sigma(t) \xrightarrow{a} p$. Either this transition is due to a variable summand y of t such that $\sigma(y) \xrightarrow{a} p$, or there is a summand at' of t such that $p = \sigma(t')$. In the former case, $y \neq x$ by the proviso of the theorem. Therefore, by Lemma 3, y is also a summand of u. It follows that $\sigma(u) \xrightarrow{a} p$, and we are done.

Suppose now that there is a summand at' of t such that $p = \sigma(t')$. If $\sigma(y) \xrightarrow{a} q$ for some variable summand y of u and closed term q such that $p \precsim_{2NS} q$, we are done. Assume therefore that, for each closed term r and variable summand y of u,

$$\sigma(y) \xrightarrow{a} r \text{ implies } p \not\precsim_{2NS} r \ . \tag{4}$$

We claim that $p \precsim_{2NS} \sigma(u')$ for some summand au' of u. We proceed with the proof of this claim by distinguishing two cases, depending on whether x occurs in t' or not.

– Case x does not occur in t'. Let $N \geq \mathit{depth}(\sigma(t))$. Since at' is a summand of t and x does not occur in t',

$$\sigma[x \mapsto a^{N+1}\mathbf{0}](t + x) \xrightarrow{a} \sigma[x \mapsto a^{N+1}\mathbf{0}](t') = \sigma(t') = p \ .$$

Therefore

$$\sigma[x \mapsto a^{N+1}\mathbf{0}](u + x) \xrightarrow{a} q \ ,$$

for some closed term q such that $p \precsim_{2NS} q$. Note that $p \not\precsim_{2NS} a^N\mathbf{0}$, because

$$\mathit{depth}(p) < \mathit{depth}(\sigma(t)) \leq N = \mathit{depth}(a^N\mathbf{0})$$

and, as observed in Remark 1, 2-nested simulation only relates terms with equal depth. Hence, by assumption (4), u must have a summand of the form au' such that

$$p \precsim_{2NS} \sigma[x \mapsto a^{N+1}\mathbf{0}](u') \ .$$

As $\mathit{depth}(p) < N$, this is only possible if x does not occur in u', or else $\sigma[x \mapsto a^{N+1}\mathbf{0}](u')$ would have depth at least $N+1$ and could not be simulated by p. Therefore,

$$\sigma[x \mapsto a^{N+1}\mathbf{0}](u') = \sigma(u') \ .$$

Since $\sigma(u) \xrightarrow{a} \sigma(u')$, we are done.

– CASE x OCCURS IN t'. In this case,

$$depth(p) = depth(\sigma(t')) \geq depth(\sigma(x)) \ .$$

Since each a-derivative of $\sigma(x)$ has depth smaller than that of p, it cannot simulate p. Since $\sigma(t+x) \precsim_{2\mathrm{NS}} \sigma(u+x)$ and $\sigma(t+x) \xrightarrow{a} p$, it follows that $\sigma(u) \xrightarrow{a} \sigma(u')$ and $p \precsim_{2\mathrm{NS}} \sigma(u')$ for some summand au' of u.

PROOF OF CLAIM 2. Since $t + x \precsim_{2\mathrm{NS}} u + x$ by assumption, we have that $u + x \precsim_{\mathrm{S}} t + x$. We showed earlier that Theorem 1 holds for $\precsim_{\mathrm{S}}$, and therefore $u \precsim_{\mathrm{S}} t$. This yields, in particular, that $\sigma(u) \precsim_{\mathrm{S}} \sigma(t)$.

This concludes the proof for $\precsim_{2\mathrm{NS}}$. □

5.3 Proof of Theorem 1 for $\precsim_{\mathrm{PT}}$

Assume that $t + x \precsim_{\mathrm{PT}} u + x$, and x is not a summand of $t + u$. We shall show that $t \precsim_{\mathrm{PT}} u$. This follows from the result for $\precsim_{\mathrm{S}}$ if $|A| = 1$. Indeed, in that case, $\precsim_{\mathrm{PT}}$ and $\precsim_{\mathrm{S}}$ coincide—see, e.g., [2].

Assume therefore that $|A| > 1$, so that there are two distinct actions a, b in A. Let σ be a closed substitution. We prove that each trace $a_1 \ldots a_n$ of $\sigma(t)$ is also a trace of $\sigma(u)$.

Since $\sigma(t) \xrightarrow{a_1 \ldots a_n}$, by Lemma 1 we have that

1. either $t \xrightarrow{a_1 \ldots a_n}$
2. or there are a $j < n$, a variable y and a term t' such that $t \xrightarrow{a_1 \ldots a_j} t'$, y is a summand of t', and $\sigma(y) \xrightarrow{a_{j+1} \ldots a_n}$.

In the former case, by Lemma 1 $\sigma_{\mathbf{0}}(t+x) \xrightarrow{a_1 \ldots a_n}$. Thus also $\sigma_{\mathbf{0}}(u+x) \xrightarrow{a_1 \ldots a_n}$. It follows that $u \xrightarrow{a_1 \ldots a_n}$, and thus $\sigma(u) \xrightarrow{a_1 \ldots a_n}$, which was to be shown.

Consider the latter case. If $j = 0$, then $y \neq x$ is a summand of t. By Lemma 3, it is also a summand of u, and therefore $\sigma(u) \xrightarrow{a_1 \ldots a_n}$, which was to be shown. Assume therefore that $j \geq 1$. Let $N \geq depth(u)$. By Lemma 1

$$\sigma_{\mathbf{0}}[y \mapsto a^N b\mathbf{0}](t+x) \xrightarrow{a_1 \ldots a_j a^N b} \ ,$$

so also $\sigma_{\mathbf{0}}[y \mapsto a^N b\mathbf{0}](u+x) \xrightarrow{a_1 \ldots a_j a^N b}$. Since $j \geq 1$, this implies $\sigma_{\mathbf{0}}[y \mapsto a^N b\mathbf{0}](u) \xrightarrow{a_1 \ldots a_j a^N b}$. Let $b_1 \ldots b_{N+j+1} = a_1 \ldots a_j a^N b$. Since $N \geq depth(u)$, by Lemma 1 $u \xrightarrow{b_1 \ldots b_k} u'$ and $\sigma_{\mathbf{0}}[y \mapsto a^N b\mathbf{0}](z) \xrightarrow{b_{k+1} \ldots b_{N+j+1}}$ for some term u', variable z and $k \leq N$, where u' has z as a summand. Since $N + j + 1 > k$, it follows that $z = y$, $k = j$ and $b_{k+1} \ldots b_{N+j+1} = a^N b$. Concluding, $u \xrightarrow{a_1 \ldots a_j} u'$ where u' has y as a summand. Since $\sigma(y) \xrightarrow{a_{j+1} \ldots a_n}$ and $j < n$, using Lemma 1 we infer that $\sigma(u) \xrightarrow{a_1 \ldots a_n}$, which was to be shown.

This concludes the proof for $\precsim_{\mathrm{PT}}$. □

5.4 Proof of Theorem 1 for $\precsim_{\mathbf{RT}}$

We begin by stating a useful lemma relating the ready traces of a term $\sigma(t)$, where σ is a closed substitution, to the action transitions and ready traces of the term t and of the terms $\sigma(x)$ for each variable x occurring in t.

Lemma 5. *Let σ be a closed substitution, and t a term.*

1. *Assume that $X_0 b_1 X_1 \ldots b_k X_k$ is a ready trace of $\sigma(t)$. Then*
   (a) *either there are terms $t_1, \ldots, t_k$ such that*

   $$t = t_0 \stackrel{b_1}{\to} t_1 \ldots t_{k-1} \stackrel{b_k}{\to} t_k$$

   and $\mathcal{I}(\sigma(t_i)) = X_i$, for each $0 \leq i \leq k$,
   (b) *or $t = t_0 \stackrel{b_1}{\to} t_1 \ldots t_{i-1} \stackrel{b_i}{\to} t_i$ for some $0 \leq i < k$, and terms $t_1, \ldots, t_i$ such that*
      i. *$\mathcal{I}(\sigma(t_j)) = X_j$, for each $0 \leq j \leq i$, and*
      ii. *$t_i = y + t'$ for some variable y and term t' such that*

      $$\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \ldots b_k X_k$$

      is a ready trace of $\sigma(y)$.
2. *Assume that $t = t_0 \stackrel{b_1}{\to} t_1 \ldots t_{k-1} \stackrel{b_k}{\to} t_k$ and $\mathcal{I}(\sigma(t_i)) = X_i$, for each $0 \leq i \leq k$. Then $X_0 b_1 X_1 \ldots b_k X_k$ is a ready trace of $\sigma(t)$.*
3. *Assume that $t = t_0 \stackrel{b_1}{\to} t_1 \ldots t_{i-1} \stackrel{b_i}{\to} t_i$ for some $0 \leq i < k$, and terms $t_1, \ldots, t_i$ such that*
   (a) *$\mathcal{I}(\sigma(t_j)) = X_j$, for each $0 \leq j \leq i$, and*
   (b) *$t_i = y + t'$ for some variable y and term t' such that*

   $$\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \ldots b_k X_k$$

   is a ready trace of $\sigma(y)$.
   Then $X_0 b_1 X_1 \ldots b_k X_k$ is a ready trace of $\sigma(t)$.

We are now ready to prove Theorem 1 for $\precsim_{\mathrm{RT}}$. Assume that $t \not\precsim_{\mathrm{RT}} u$, and x is not a summand of $t + u$. We shall show that $t + x \not\precsim_{\mathrm{RT}} u + x$.

Since $t \not\precsim_{\mathrm{RT}} u$, there is a closed substitution σ such that $\sigma(t) \not\precsim_{\mathrm{RT}} \sigma(u)$. This means that there is a ready trace $X_0 b_1 X_1 \ldots b_k X_k$ of $\sigma(t)$ that is not a ready trace of $\sigma(u)$. In the remainder of the proof, we use this information to construct a closed substitution ρ such that $\rho(t + x) \not\precsim_{\mathrm{RT}} \rho(u + x)$, thus establishing our claim that $t + x \not\precsim_{\mathrm{RT}} u + x$.

Suppose that $\mathcal{I}(\sigma(t)) \neq \mathcal{I}(\sigma(u))$. As x is not a summand of $t+u$, then clearly $\sigma[x \mapsto \mathbf{0}](t + x) \not\precsim_{\mathrm{RT}} \sigma[x \mapsto \mathbf{0}](u + x)$. Hence, $t + x \not\precsim_{\mathrm{RT}} u + x$, which was to be shown.

So we may assume that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u)) = X_0$. In particular this implies that $k > 0$.

Our order of business now will be to construct a closed substitution ρ with the following properties:

1. $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x))$, and $\rho(y) = \sigma(y)$ for each variable $y \neq x$,
2. $\rho(x)$ and $\sigma(x)$ have the same ready traces of length smaller than k, and
3. $\rho(x)$ does not have any ready pairs of the form $(c_1 \dots c_k, X_k)$.

Before giving the construction of ρ, we shall argue that from these three properties it follows that

$$\rho(t + x) \not\lesssim_{\mathrm{RT}} \rho(u + x) \ .$$

Observe, first of all, that $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t + x)$. To see this, recall that, as $X_0 b_1 X_1 \dots b_k X_k$ is a ready trace of $\sigma(t)$, by Lemma 5(1) we have that

1. either there are terms $t_1, \dots, t_k$ such that

$$t = t_0 \xrightarrow{b_1} t_1 \dots t_{k-1} \xrightarrow{b_k} t_k$$

   and $\mathcal{I}(\sigma(t_i)) = X_i$, for each $0 \leq i \leq k$,
2. or $t = t_0 \xrightarrow{b_1} t_1 \dots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 \leq i < k$, and terms $t_1, \dots, t_i$ such that
   (a) $\mathcal{I}(\sigma(t_j)) = X_j$, for each $0 \leq j \leq i$, and
   (b) $t_i = y + t'$ for some variable y and term t' such that

$$\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \dots b_k X_k$$

   is a ready trace of $\sigma(y)$.

(Note that, in light of Lemma 3 and our assumptions that $X_0 b_1 X_1 \dots b_k X_k$ is not a ready trace of $\sigma(u)$ and $k > 0$, in the latter case $i > 0$. Indeed, if $i = 0$, then y would also be a variable summand of u, and $X_0 b_1 X_1 \dots b_k X_k$ would be a ready trace of $\sigma(u)$.) We proceed to prove that $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t + x)$ by considering the two possibilities above separately.

– Suppose that $t = t_0 \xrightarrow{b_1} t_1 \dots t_{k-1} \xrightarrow{b_k} t_k$ and $\mathcal{I}(\sigma(t_i)) = X_i$, for each $0 \leq i \leq k$. By property 1 of ρ, $\mathcal{I}(\rho(t_i)) = \mathcal{I}(\sigma(t_i)) = X_i$ for each $0 \leq i \leq k$. So $X_0 b_1 X_1 \dots b_k X_k$ is also a ready trace of $\rho(t)$. By property 1 of ρ, $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t + x)$, as claimed.
– Suppose that $t = t_0 \xrightarrow{b_1} t_1 \dots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 < i < k$, and terms $t_1, \dots, t_i$ such that
  1. $\mathcal{I}(\sigma(t_j)) = X_j$, for each $0 \leq j \leq i$, and
  2. $t_i = y + t'$ for some variable y and term t' such that

$$\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \dots b_k X_k$$

  is a ready trace of $\sigma(y)$.

  If $y \neq x$, then $\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \dots b_k X_k$ is a ready trace of $\rho(y)$, by property 1 of ρ. By Lemma 5(3) and property 1 of ρ, $X_0 b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t)$. Since $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x))$, we may conclude that $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t + x)$, as claimed.

If $y = x$, then $\mathcal{I}(\sigma(y))\, b_{i+1}X_{i+1}\dots b_kX_k$ is a ready trace of $\rho(x)$, by property 2 of ρ, because $i > 0$. By Lemma 5(3) and property 1 of ρ,

$$X_0b_1X_1\dots b_kX_k$$

is a ready trace of $\rho(t)$. Since $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x))$, we may again conclude that $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1X_1\dots b_kX_k$ is a ready trace of $\rho(t+x)$, as claimed.

We now prove that $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1X_1\dots b_kX_k$ is *not* a ready trace of $\rho(u+x)$. Since $k > 0$, this follows if we can argue that $\mathcal{I}(\sigma(x))\, b_1X_1\dots b_kX_k$ is not a ready trace of $\rho(x)$ and $X_0b_1X_1\dots b_kX_k$ is not a ready trace of $\rho(u)$. To this end, note, first of all, that $\mathcal{I}(\sigma(x))\, b_1X_1\dots b_kX_k$ is not a ready trace of $\rho(x)$ by property 3 of ρ. Therefore, we are left to show that $X_0b_1X_1\dots b_kX_k$ is not a ready trace of $\rho(u)$.

By Lemma 5(1), $X_0b_1X_1\dots b_kX_k$ is a ready trace of $\rho(u)$ only if

1. either there are terms $u_1,\dots,u_k$ such that

$$u = u_0 \xrightarrow{b_1} u_1 \dots u_{k-1} \xrightarrow{b_k} u_k$$

   and $\mathcal{I}(\rho(u_i)) = X_i$, for each $0 \le i \le k$,
2. or $u = u_0 \xrightarrow{b_1} u_1 \dots u_{i-1} \xrightarrow{b_i} u_i$ for some $0 \le i < k$, and terms $u_1,\dots,u_i$ such that
   (a) $\mathcal{I}(\rho(u_j)) = X_j$, for each $0 \le j \le i$, and
   (b) $u_i = z + u'$ for some variable z and term u' such that

$$\mathcal{I}(\rho(z))\, b_{i+1}X_{i+1}\dots b_kX_k$$

   is a ready trace of $\rho(z)$.

We now proceed to argue that both of these possibilities contradict our assumption that $X_0b_1X_1\dots b_kX_k$ is not a ready trace of $\sigma(u)$. Indeed, in the former case, we could conclude that $X_0b_1X_1\dots b_kX_k$ is a ready trace of $\sigma(u)$ using property 1 of ρ and Lemma 5(2). In the latter case, we could reach the same conclusion using properties 1 and 2 of ρ and Lemma 5(3).

All that we are left to do to complete the proof for this case is to construct a closed substitution ρ having properties 1–3. We begin by defining, for each closed term p, $n \ge 0$ and set of actions X, the closed term $\pi_n^X(p)$ as follows:

$$\begin{aligned}\pi_0^X(p) &= \sum\{a\mathbf{0} \mid a \in \mathcal{I}(p) \cap X\} + \sum\{aa\mathbf{0} \mid a \in \mathcal{I}(p) - X\}\\ \pi_{n+1}^X(p) &= \sum\{a\pi_n^X(p') \mid p \xrightarrow{a} p'\}\ .\end{aligned}$$

Take $\rho = \sigma[x \mapsto \pi_{k-1}^{X_k}(\sigma(x))]$. By definition, $\mathcal{I}(\pi_n^X(p)) = \mathcal{I}(p)$, for each closed term p, $n \ge 0$ and $X \subseteq A$. Therefore ρ meets property 1.

We claim that $\rho(x)$ and $\sigma(x)$ have the same ready traces of length smaller than k. This follows immediately from the following two observations:

- $\mathcal{I}(\pi_n^{X_k}(p)) = \mathcal{I}(p)$, for each p and $n \ge 0$, and

– for all closed terms p, q, action c and $n > 0$,

$$p \xrightarrow{c} q \text{ iff } \pi_n^{X_k}(p) \xrightarrow{c} \pi_{n-1}^{X_k}(q) \ .$$

So ρ enjoys property 2.

Finally, to see that ρ meets property 3, assume that $\pi_{k-1}^{X_k}(\sigma(x)) \xrightarrow{c_1 \ldots c_k} q$ for some sequence $c_1 \ldots c_k$ of actions and closed term q. It is not hard to see that either $X_k \neq \emptyset$ and $q = \mathbf{0}$, or $q = a\mathbf{0}$ for some $a \notin X_k$. In both cases, $\mathcal{I}(q) \neq X_k$.

This concludes the proof for $\precsim_{\mathrm{RT}}$. □

5.5 Proof of Theorem 1 for $\precsim_{\mathbf{FT}}$

We begin by stating a useful lemma relating the failure traces of a term $\sigma(t)$, where σ is a closed substitution, to the action transitions and failure traces of the term t and of the terms $\sigma(x)$ for each variable x occurring in t.

Lemma 6. *Let σ be a closed substitution, and t be a term.*

1. *Assume that $X_0 b_1 X_1 \ldots b_k X_k$ is a failure trace of $\sigma(t)$. Then*
   (a) *either there are terms $t_1, \ldots, t_k$ such that*

   $$t = t_0 \xrightarrow{b_1} t_1 \ldots t_{k-1} \xrightarrow{b_k} t_k$$

   and $\mathcal{I}(\sigma(t_i)) \cap X_i = \emptyset$, for each $0 \leq i \leq k$,

   (b) *or $t = t_0 \xrightarrow{b_1} t_1 \ldots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 \leq i < k$, and terms $t_1, \ldots, t_i$ such that*
      i. *$\mathcal{I}(\sigma(t_j)) \cap X_j = \emptyset$, for each $0 \leq j \leq i$, and*
      ii. *$t_i = y + t'$ for some variable y and term t' such that*

      $$\emptyset \, b_{i+1} X_{i+1} \ldots b_k X_k$$

      is a failure trace of $\sigma(y)$.
2. *Assume that $t = t_0 \xrightarrow{b_1} t_1 \ldots t_{k-1} \xrightarrow{b_k} t_k$ and $\mathcal{I}(\sigma(t_i)) \cap X_i = \emptyset$, for each $0 \leq i \leq k$. Then $X_0 b_1 X_1 \ldots b_k X_k$ is a failure trace of $\sigma(t)$.*
3. *Assume that $t = t_0 \xrightarrow{b_1} t_1 \ldots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 \leq i < k$, and terms $t_1, \ldots, t_i$ such that*
   (a) *$\mathcal{I}(\sigma(t_j)) \cap X_j = \emptyset$, for each $0 \leq j \leq i$, and*
   (b) *$t_i = y + t'$ for some variable y and term t' such that*

   $$\emptyset \, b_{i+1} X_{i+1} \ldots b_k X_k$$

   is a failure trace of $\sigma(y)$.

   Then $X_0 b_1 X_1 \ldots b_k X_k$ is a failure trace of $\sigma(t)$.

We are now ready to prove Theorem 1 for $\precsim_{\mathrm{FT}}$. Assume that $t \not\precsim_{\mathrm{FT}} u$, and x is not a summand of $t + u$. We shall show that $t + x \not\precsim_{\mathrm{FT}} u + x$.

Since $t \not\precsim_{\mathrm{FT}} u$, there is a closed substitution σ such that $\sigma(t) \not\precsim_{\mathrm{FT}} \sigma(u)$. This means that there is a failure trace $X_0 b_1 X_1 \ldots b_k X_k$ of $\sigma(t)$ that is not a failure trace of $\sigma(u)$. In the remainder of the proof, we use this information to construct a closed substitution ρ such that $\rho(t + x) \not\precsim_{\mathrm{FT}} \rho(u + x)$, thus establishing our claim that $t + x \not\precsim_{\mathrm{FT}} u + x$.

Suppose that $\mathcal{I}(\sigma(t)) \neq \mathcal{I}(\sigma(u))$. As x is not a summand of $t+u$, then clearly $\sigma[x \mapsto \mathbf{0}](t + x) \not\precsim_{\mathrm{FT}} \sigma[x \mapsto \mathbf{0}](u + x)$. Hence, $t + x \not\precsim_{\mathrm{FT}} u + x$, which was to be shown.

So we may assume that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$. In particular this implies that $k > 0$. We distinguish two cases, depending on whether $k = 1$ or $k > 1$.

- Case $k = 1$ Our order of business now will be to construct a closed substitution ρ with the following properties:
  1. $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x)) \cap X_1$, and $\rho(y) = \sigma(y)$ for each variable $y \neq x$, and
  2. $\rho(x)$ does not have any failure pairs of the form (c_1, X_1).

  Before giving the construction of ρ, we shall argue that from these two properties it follows that
  $$\rho(t + x) \not\precsim_{\mathrm{FT}} \rho(u + x) \ .$$
  Observe, first of all, that (b_1, X_1) is a failure pair of $\rho(t + x)$. To see this, recall that, as $X_0 b_1 X_1$ is a failure trace of $\sigma(t)$, by Lemma 6(1) we have two possibilities. Either there are is a term t' such that $t \xrightarrow{b_1} t'$ and $\mathcal{I}(\sigma(t')) \cap X_1 = \emptyset$. Or $t = y + t'$ for some variable y and term t' such that (b_1, X_1) is a failure pair of $\sigma(y)$.

  In the second case, in light of Lemma 3, y would also be a variable summand of u, and $X_0 b_1 X_1$ would be a failure trace of $\sigma(u)$, because $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$. This contradicts one of our assumptions.

  So we can assume that that $t \xrightarrow{b_1} t'$ with $\mathcal{I}(\sigma(t')) \cap X_1 = \emptyset$. By property 1 of ρ, $\mathcal{I}(\sigma(t')) \cap X_1 = \emptyset$ implies that $\mathcal{I}(\rho(t')) \cap X_1 = \emptyset$. So (b_1, X_1) is a failure pair of $\rho(t)$. We conclude that (b_1, X_1) is a failure pair of $\rho(t+x)$, as claimed.

  We now prove that (b_1, X_1) is *not* a failure pair of $\rho(u + x)$. This follows if we can argue that (b_1, X_1) is neither a failure pair of $\rho(x)$ nor a failure pair of $\rho(u)$. To this end, note, first of all, that (b_1, X_1) is not a failure pair of $\rho(x)$ by property 2 of ρ. Therefore, we are left to show that (b_1, X_1) is not a failure pair of $\rho(u)$.

  By Lemma 6(1), (b_1, X_1) is a failure pair of $\rho(u)$ only if
  - either there are is a term u' such that $u \xrightarrow{b_1} u'$ and $\mathcal{I}(\rho(u')) \cap X_1 = \emptyset$;
  - or $u = z + u'$ for some variable z and term u' such that (b_1, X_1) is a failure pair of $\rho(z)$.

  We now proceed to argue that both of these possibilities contradict our assumption that $X_0 b_1 X_1$ is not a failure trace of $\sigma(u)$. Indeed, in the former case, we could conclude that $X_0 b_1 X_1$ is a failure trace of $\sigma(u)$ using our assumption that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$, property 1 of ρ and Lemma 6(2). In the latter case, by assumption $z \neq x$, so $\rho(z) = \sigma(z)$ by property 1 of ρ. Hence again we could conclude that $X_0 b_1 X_1$ is a failure trace of $\sigma(u)$ using our assumption that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$ and Lemma 6(3).

All that we are left to do to complete the proof for this case is to construct a closed substitution ρ having properties 1–2. We begin by defining, for each closed term p, the closed term $\mathrm{chop}^{X_1}(p)$ as follows:

$$\mathrm{chop}^{X_1}(p) = \sum \{ aa\mathbf{0} \mid a \in \mathcal{I}(p) \cap X_1 \} \ .$$

Take $\rho = \sigma[x \mapsto \mathrm{chop}^{X_1}(\sigma(x))]$. By definition, $\mathcal{I}(\mathrm{chop}^{X_1}(p)) = \mathcal{I}(p) \cap X_1$, for each closed term p. Therefore ρ meets property 1.

To see that ρ meets property 2, assume that $\mathrm{chop}^{X_1}(\sigma(x)) \xrightarrow{c_1} q$ for some action c_1 and closed term q. Then clearly $c_1 \in X_1$ and $q = c_1\mathbf{0}$. Therefore $\rho(x)$ does not have any failure pairs of the form (c_1, X_1).

– Case $k > 1$ Our order of business now will be to construct a closed substitution ρ with the following properties:
  1. $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x))$, and $\rho(y) = \sigma(y)$ for each variable $y \neq x$,
  2. $\rho(x)$ and $\sigma(x)$ have the same failure traces of the form

$$Y_0 c_1 Y_1 \ldots Y_{\ell-1} c_\ell X_k$$

  for $\ell < k$, and
  3. $\rho(x)$ does not have any failure pairs of the form $(c_1 \ldots c_k, X_k)$.

Before giving the construction of ρ, we shall argue that from these three properties it follows that

$$\rho(t + x) \not\lesssim_{\mathrm{FT}} \rho(u + x) \ .$$

Observe, first of all, that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t + x)$. To see this, recall that, as $X_0 b_1 X_1 \ldots b_k X_k$ is a failure trace of $\sigma(t)$, by Lemma 6(1) we have that
  1. either there are terms $t_1, \ldots, t_k$ such that

$$t = t_0 \xrightarrow{b_1} t_1 \ldots t_{k-1} \xrightarrow{b_k} t_k$$

  and $\mathcal{I}(\sigma(t_i)) \cap X_i = \emptyset$, for each $0 \leq i \leq k$,
  2. or $t = t_0 \xrightarrow{b_1} t_1 \ldots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 \leq i < k$, and terms $t_1, \ldots, t_i$ such that
     (a) $\mathcal{I}(\sigma(t_j)) \cap X_j = \emptyset$, for each $0 \leq j \leq i$, and
     (b) $t_i = y + t'$ for some variable y and term t' such that

$$\emptyset\, b_{i+1} X_{i+1} \ldots b_k X_k$$

     is a failure trace of $\sigma(y)$.

(Note that, in light of Lemma 3 and our assumptions that $X_0 b_1 X_1 \ldots b_k X_k$ is not a failure trace of $\sigma(u)$ and $k > 0$, in the latter case $i > 0$. Indeed, if $i = 0$, then y would also be a variable summand of u, and $X_0 b_1 X_1 \ldots b_k X_k$ would be a failure trace of $\sigma(u)$, because $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$.) We proceed to prove that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t + x)$ by considering the two possibilities above separately.

- Suppose that $t = t_0 \xrightarrow{b_1} t_1 \ldots t_{k-1} \xrightarrow{b_k} t_k$ and $\mathcal{I}(\sigma(t_i)) \cap X_i = \emptyset$, for each $0 \leq i \leq k$. By property 1 of ρ, $\mathcal{I}(\rho(t_i)) = \mathcal{I}(\sigma(t_i))$ for each $0 \leq i \leq k$. So $\emptyset b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t)$. We conclude that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t + x)$, as claimed.
- Suppose that $t = t_0 \xrightarrow{b_1} t_1 \ldots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 < i < k$, and terms $t_1, \ldots, t_i$ such that
  1. $\mathcal{I}(\sigma(t_j)) \cap X_j = \emptyset$, for each $0 \leq j \leq i$, and
  2. $t_i = y + t'$ for some variable y and term t' such that
  $$\emptyset\, b_{i+1} X_{i+1} \ldots b_k X_k$$
  is a failure trace of $\sigma(y)$.

  If $y \neq x$, then $\emptyset\, b_{i+1} X_{i+1} \ldots b_k X_k$ is a failure trace of $\rho(y)$, by property 1 of ρ. By Lemma 6(3) and property 1 of ρ, it follows that $\emptyset b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t)$. We conclude that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t + x)$, as claimed.

  If $y = x$, then $\emptyset\, b_{i+1} X_{i+1} \ldots b_k X_k$ is a failure trace of $\rho(x)$, by property 2 of ρ, because $i > 0$. By Lemma 6(3) and property 1 of ρ, we have that $\emptyset b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t)$. We may again conclude that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t + x)$, as claimed.

We now prove that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is *not* a failure trace of $\rho(u + x)$. This follows if we can argue that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is neither a failure trace of $\rho(x)$ nor a failure trace of $\rho(u)$. To this end, note, first of all, that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is not a failure trace of $\rho(x)$ by property 3 of ρ. Therefore, we are left to show that $\emptyset b_1 X_1 \ldots b_k X_k$ is not a failure trace of $\rho(u)$.

By Lemma 6(1), $\emptyset b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(u)$ only if

1. either there are terms $u_1, \ldots, u_k$ such that
$$u = u_0 \xrightarrow{b_1} u_1 \ldots u_{k-1} \xrightarrow{b_k} u_k$$
and $\mathcal{I}(\rho(u_i)) \cap X_i = \emptyset$, for each $1 \leq i \leq k$,
2. or $u = u_0 \xrightarrow{b_1} u_1 \ldots u_{i-1} \xrightarrow{b_i} u_i$ for some $0 \leq i < k$, and terms $u_1, \ldots, u_i$ such that
   (a) $\mathcal{I}(\rho(u_j)) \cap X_j = \emptyset$, for each $1 \leq j \leq i$, and
   (b) $u_i = z + u'$ for some variable z and term u' such that
   $$\emptyset\, b_{i+1} X_{i+1} \ldots b_k X_k$$
   is a failure trace of $\rho(z)$.

We now proceed to argue that both of these possibilities contradict our assumption that $X_0 b_1 X_1 \ldots b_k X_k$ is not a failure trace of $\sigma(u)$. Indeed, in the former case, we could conclude that $X_0 b_1 X_1 \ldots b_k X_k$ is a failure trace of $\sigma(u)$ using our assumption that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$, property 1 of ρ and Lemma 6(2). In the latter case, we could reach the same conclusion using our assumption that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$, properties 1 and 2 of ρ and Lemma 6(3).

All that we are left to do to complete the proof for this case is to construct a closed substitution ρ having properties 1–3. We begin by defining, for each closed term p, and $n \geq 0$, the closed term $\mathrm{chop}_n^{X_k}(p)$ as follows:

$$\begin{aligned} \mathrm{chop}_0^{X_k}(p) &= \sum\{a a\mathbf{0} \mid a \in \mathcal{I}(p) \cap X_k\} \\ \mathrm{chop}_{n+1}^{X_k}(p) &= \sum\{a\,\mathrm{chop}_n^{X_k}(p') \mid p \xrightarrow{a} p'\} \ . \end{aligned}$$

Take $\rho = \sigma[x \mapsto \mathrm{chop}_{k-1}^{X_k}(\sigma(x))]$. By definition, $\mathcal{I}(\mathrm{chop}_n^{X_k}(p)) = \mathcal{I}(p)$, for each closed term p, and $n > 0$. Since $k-1 > 0$, ρ meets property 1.
We claim that $\rho(x)$ and $\sigma(x)$ have the same failure traces of length smaller than k. This follows immediately from the following three observations:

- for each closed term p and $n > 0$,

$$\mathcal{I}(\mathrm{chop}_n^{X_k}(p)) = \mathcal{I}(p) \ ,$$

- for all closed terms p, q, action c and $n > 0$,

$$p \xrightarrow{c} q \text{ iff } \mathrm{chop}_n^{X_k}(p) \xrightarrow{c} \mathrm{chop}_{n-1}^{X_k}(q) \ , \text{ and}$$

- for each closed term p,

$$\mathcal{I}(p) \cap X_k = \emptyset \text{ iff } \mathcal{I}(\mathrm{chop}_0^{X_k}(p)) \cap X_k = \emptyset \ .$$

So ρ enjoys property 2.
Finally, to see that ρ meets property 3, assume that $\mathrm{chop}_{k-1}^{X_k}(\sigma(x)) \xrightarrow{c_1 \ldots c_k} q$ for some sequence $c_1 \ldots c_k$ of actions and closed term q. It is not hard to see that then $c_k \in X_k$ and $q = c_k\mathbf{0}$. Therefore $\rho(x)$ does not have any failure pairs of the form $(c_1 \ldots c_k, X_k)$.

This concludes the proof for $\precsim_{\mathrm{FT}}$. □

5.6 Proof of Theorem 1 for $\precsim_{\mathbf{R}}$

We begin by stating a useful lemma relating the ready pairs of a closed term $\sigma(t)$, where σ is a closed substitution, to the action transitions and ready pairs of t and of the closed terms $\sigma(x)$ for each variable x occurring in t.

Lemma 7. *Let σ be a closed substitution, and let t be a term.*

1. *Assume that $(b_1 \ldots b_k, X)$ is a ready pair of $\sigma(t)$. Then*
   (a) *either $t \xrightarrow{b_1 \ldots b_k} t'$ and $\mathcal{I}(\sigma(t')) = X$, for some t',*
   (b) *or $t \xrightarrow{b_1 \ldots b_i} y + t'$ for some $i < k$, variable y and term t' such that $(b_{i+1} \ldots b_k, X)$ is a ready pair of $\sigma(y)$.*
2. *Assume that $t \xrightarrow{b_1 \ldots b_k} t'$ for some t'. Then $(b_1 \ldots b_k, \mathcal{I}(\sigma(t')))$ is a ready pair of $\sigma(t)$.*

3. *Assume that* $t \stackrel{b_1 \ldots b_i}{\longrightarrow} y + t'$ *for some* $i < k$*, variable* y *and term* t' *such that* $(b_{i+1} \ldots b_k, X)$ *is a ready pair of* $\sigma(y)$*. Then* $(b_1 \ldots b_k, X)$ *is a ready pair of* $\sigma(t)$*.*

We are now ready to prove Theorem 1 for $\precsim_{\mathrm{R}}$. This proof is very similar to the proof for failure semantics from [7]. Assume that $t \not\precsim_{\mathrm{R}} u$, and x is not a summand of $t + u$. We shall show that $t + x \not\precsim_{\mathrm{R}} u + x$.

Since $t \not\precsim_{\mathrm{R}} u$, there is a closed substitution σ such that $\sigma(t) \not\precsim_{\mathrm{R}} \sigma(u)$. This means that there is a ready pair $(b_1 \ldots b_k, X)$ of $\sigma(t)$ that is not a ready pair of $\sigma(u)$. In the remainder of the proof, we use this information to construct a closed substitution ρ such that $\rho(t+x) \not\precsim_{\mathrm{R}} \rho(u+x)$, thus establishing our claim that $t + x \not\precsim_{\mathrm{R}} u + x$.

Suppose that $\mathcal{I}(\sigma(t)) \neq \mathcal{I}(\sigma(u))$. As x is not a summand of $t+u$, then clearly $\sigma[x \mapsto \mathbf{0}](t + x) \not\precsim_{\mathrm{R}} \sigma[x \mapsto \mathbf{0}](u + x)$. Hence, $t + x \not\precsim_{\mathrm{R}} u + x$, which was to be shown.

So we may assume that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u)) = X$. In particular this implies that $k > 0$.

As in the proof for $\precsim_{\mathrm{RT}}$, we define the closed substitution ρ by $\rho = \sigma[x \mapsto \pi^X_{k-1}(\sigma(x))]$, where the closed term $\pi^X_{k-1}(\sigma(x))$ is defined as on page 18. We observed in the proof for $\precsim_{\mathrm{RT}}$ that (stronger versions of) the following properties hold for ρ:

1. $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x))$, and $\rho(y) = \sigma(y)$ for each variable $y \neq x$,
2. $\rho(x)$ and $\sigma(x)$ have the same ready pairs of length smaller than k, and
3. $\rho(x)$ does not have any ready pairs of the form $(c_1 \ldots c_k, X)$.

We shall argue that

$$\rho(t + x) \not\precsim_{\mathrm{R}} \rho(u + x) \ ,$$

showing that $t + x \not\precsim_{\mathrm{R}} u + x$, as claimed.

Observe, first of all, that $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t+x)$. To see this, recall that, as $(b_1 \ldots b_k, X)$ is a ready pair of $\sigma(t)$, by Lemma 7(1) we have that

– either $t \stackrel{b_1 \ldots b_k}{\longrightarrow} t'$ and $\mathcal{I}(\sigma(t')) = X$, for some t',
– or $t \stackrel{b_1 \ldots b_i}{\longrightarrow} y+t'$ for some $i < k$, variable y and term t' such that $(b_{i+1} \ldots b_k, X)$ is a ready pair of $\sigma(y)$.

(Note that, in light of Lemma 3 and our assumptions that $(b_1 \ldots b_k, X)$ is not a ready pair of $\sigma(u)$ and $k > 0$, in the latter case $i > 0$. Indeed, if $i = 0$, then y would also be a variable summand of u, and $(b_1 \ldots b_k, X)$ would be a ready pair of $\sigma(u)$.) We proceed to prove that $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t + x)$ by considering the two possibilities above separately.

– Suppose that $t \stackrel{b_1 \ldots b_k}{\longrightarrow} t'$ and $\mathcal{I}(\sigma(t')) = X$, for some t'. Lemma 7(2) yields that $\rho(t) \stackrel{b_1 \ldots b_k}{\longrightarrow} \rho(t')$. Moreover, by property 1 of ρ, $\mathcal{I}(\rho(t')) = \mathcal{I}(\sigma(t')) = X$. Since $k > 0$, $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t + x)$, as claimed.

– Suppose that $t \stackrel{b_1 \ldots b_i}{\rightarrow} y + t'$ for some $0 < i < k$, variable y and term t' such that $(b_{i+1} \ldots b_k, X)$ is a ready pair of $\sigma(y)$. In this case,

$$\rho(t) \stackrel{b_1 \ldots b_i}{\rightarrow} \rho(y + t') \ .$$

If $y \neq x$, then $(b_{i+1} \ldots b_k, X)$ is a ready pair of $\rho(y)$, by property 1 of ρ. Since $i < k$, by Lemma 7(3), $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t)$. Since $k > 0$, $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t + x)$, as claimed.
If $y = x$, then $(b_{i+1} \ldots b_k, X)$ is a ready pair of $\rho(x)$, by property 2 of ρ, because $i > 0$. Since $i < k$, by Lemma 7(3), $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t)$. Since $k > 0$, $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t + x)$, as claimed.

We now prove that $(b_1 \ldots b_k, X)$ is *not* a ready pair of $\rho(u + x)$. To this end, note, first of all, that $(b_1 \ldots b_k, X)$ is not a ready pair of $\rho(x)$ by property 3 of ρ. Since $k > 0$, it suffices to show that $(b_1 \ldots b_k, X)$ is not a ready pair of $\rho(u)$. By Lemma 7(1), $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(u)$ only if

– either $u \stackrel{b_1 \ldots b_k}{\rightarrow} u'$ and $\mathcal{I}(\rho(u')) = X$, for some u',
– or $u \stackrel{b_1 \ldots b_i}{\rightarrow} y + u'$ for some $i < k$, variable y and term u' such that

$$(b_{i+1} \ldots b_k, X)$$

is a ready pair of $\rho(y)$.

We now proceed to argue that both of these possibilities contradict our assumption that $(b_1 \ldots b_k, X)$ is not a ready pair of $\sigma(u)$. Indeed, in the former case, we could conclude that $(b_1 \ldots b_k, X)$ is a ready pair of $\sigma(u)$ using property 1 of ρ and Lemma 7(2). In the latter case, we could reach the same conclusion using properties 1 and 2 of ρ and Lemma 7(3).

This concludes the proof for $\precsim_{\mathrm{R}}$. □

5.7 Proof of Theorem 1 for $\precsim_{\mathrm{PF}}$

Assume that $t \not\precsim_{\mathrm{PF}} u$, and x is not a summand of $t + u$. We shall show that $t + x \not\precsim_{\mathrm{PF}} u + x$.

Since $t \not\precsim_{\mathrm{PF}} u$, there is a closed substitution σ such that $\sigma(t) \not\precsim_{\mathrm{PF}} \sigma(u)$. This means that there is a possible future $(a_1 \ldots a_k, X)$ of $\sigma(t)$ that is not a possible future of $\sigma(u)$. In the remainder of the proof, we use this information to construct a closed substitution ρ such that $\rho(t + x) \not\precsim_{\mathrm{PF}} \rho(u + x)$, thus establishing our claim that $t + x \not\precsim_{\mathrm{PF}} u + x$.

If $k = 0$, then $\sigma(t)$ and $\sigma(u)$ do not have the same set of traces. So t and u are not partial trace equivalent. From the cancellation result for $\precsim_{\mathrm{PT}}$ that we proved earlier it then follows that $t+x$ and $u+x$ are not partial trace equivalent. By Remark 1, this implies that $t + x \not\precsim_{\mathrm{PF}} t + u$, and we are done.

So we can assume that $k > 0$. Then $\sigma(t) \stackrel{a_1 \ldots a_k}{\rightarrow} p$ for some closed term p whose set of traces is X. Since $\sigma(t) \stackrel{a_1 \ldots a_k}{\rightarrow} p$, we have that

1. either $t \stackrel{a_1 \ldots a_k}{\rightarrow} t'$ for some t' such that $\sigma(t') = p$
2. or there are an $\ell < k$, a variable y and a term t' such that $t \stackrel{a_1 \ldots a_\ell}{\rightarrow} t'$, the variable y is a summand of t', and $\sigma(y) \stackrel{a_{\ell+1} \ldots a_k}{\rightarrow} p$.

In the former case, we observe, first of all, that if x occurs in t', then the depth of p is at least that of $\sigma(x)$. Since $k > 0$, it follows that $(a_1 \ldots a_k, X)$ is not a possible future of $\sigma(x)$. Recall that it is not a possible future of $\sigma(u)$ either. Therefore, as $k > 0$, $(a_1 \ldots a_k, X)$ is not a possible future of $\sigma(u + x)$. On the other hand, $\sigma(t + x)$ does have possible future $(a_1 \ldots a_k, X)$, because $k > 0$. Hence, $t + x \not\precsim_{\mathrm{PF}} u + x$, as claimed.

Consider now the former case where x does not occur in t', and the latter case with $y \neq x$. Let $N = \mathit{depth}(p)$. It is not hard to see, using Lemma 1, that in either case, $\sigma[x \mapsto a^{k+N+1}\mathbf{0}](t + x)$ has the possible future $(a_1 \ldots a_k, X)$. But $\sigma[x \mapsto a^{k+N+1}\mathbf{0}](u+x)$ does not. For else, using the depth of $a^{k+N+1}\mathbf{0}$ is greater than p, it would not be hard to see that $(a_1 \ldots a_k, X)$ would be a possible future of $\sigma(u)$, contradicting our assumption. Therefore, $t + x \not\precsim_{\mathrm{PF}} u + x$, as claimed.

Consider now the latter case with $y = x$. Then, by the assumption of the theorem, $\ell > 0$. For the sake of clarity, we recall that $t \stackrel{a_1 \ldots a_\ell}{\rightarrow} t'$, $1 \leq \ell < k$, x is a summand of t', and $\sigma(x) \stackrel{a_{\ell+1} \ldots a_k}{\rightarrow} p$. Let $M > \mathit{depth}(u)$. By Lemma 1, $\sigma_{\mathbf{0}}[x \mapsto a^M\mathbf{0}](t)$ affords the completed trace $a_1 \ldots a_\ell a^M$, and therefore so does $\sigma_{\mathbf{0}}[x \mapsto a^M\mathbf{0}](t+x)$. We claim that $\sigma_{\mathbf{0}}[x \mapsto a^M\mathbf{0}](u+x)$ does not have $a_1 \ldots a_\ell a^M$ as a completed trace. Clearly, $a^M\mathbf{0}$ does not have $a_1 \ldots a_\ell a^M$ as a completed trace, because $\ell > 0$. Suppose now, towards a contradiction, that $a_1 \ldots a_\ell a^M$ is a completed trace of $\sigma_{\mathbf{0}}[x \mapsto a^M\mathbf{0}](u)$. We can now copy the reasoning in the final paragraph of the proof of Theorem 1 for $\precsim_{\mathrm{CT}}$ on page 9. Let $c_1 \ldots c_{M+\ell} = a_1 \ldots a_\ell a^M$. Since $M > \mathit{depth}(u)$, by Lemma 1, $u \stackrel{c_1 \ldots c_m}{\rightarrow} u'$ and $\sigma_{\mathbf{0}}[x \mapsto a^M\mathbf{0}](z) \stackrel{c_{m+1} \ldots c_{M+\ell}}{\rightarrow} \mathbf{0}$ for some term u', variable z and $m < M$, where u' has z as a summand. Since $M + \ell > m$, it follows that $z = x$, $m = \ell$ and $c_{m+1} \ldots c_{M+\ell} = a^M$. Concluding, $u \stackrel{a_1 \ldots a_\ell}{\rightarrow} u'$ where u' has x as a summand. Then, since $\ell < k$, by Lemma 1, $\sigma(u)$ affords the possible future $(a_1 \ldots a_k, X)$, contradicting our assumption. Hence, $a_1 \ldots a_\ell a^M$ is not a completed trace of $\sigma_{\mathbf{0}}[x \mapsto a^M\mathbf{0}](u)$. So $\sigma_{\mathbf{0}}[x \mapsto a^M\mathbf{0}](t + x) \not\precsim_{\mathrm{CT}} \sigma_{\mathbf{0}}[x \mapsto a^M\mathbf{0}](u + x)$. Since $\precsim_{\mathrm{PF}}$ is included $\precsim_{\mathrm{CT}}$, we may infer that $t + x \not\precsim_{\mathrm{PF}} u + x$, and we are done.

This concludes the proof for $\precsim_{\mathrm{PF}}$. ☐

5.8 Proof of Theorem 1 for $\precsim_{\mathrm{PW}}$

We begin by offering a reformulation of the definition of $\precsim_{\mathrm{PW}}$ that will be useful in the proof to follow.

Definition 5. *A closed term ap is a* prefixed possible world *of a closed term q if:*

1. *p is deterministic, and*
2. *$q \stackrel{a}{\rightarrow} q'$ for some closed term q' such that $p \precsim_{RS} q'$.*

For closed terms r and s, we define $r \sqsubseteq_{PW} s$ if:

1. *the prefixed possible worlds of r are also prefixed possible worlds of s, and*
2. $\mathcal{I}(r) = \mathcal{I}(s)$.

The relation $\sqsubseteq_{\mathrm{PW}}$ is lifted to open terms in the standard fashion; see page 3.

Lemma 8. $\precsim_{\mathrm{PW}}$ *and* $\sqsubseteq_{\mathrm{PW}}$ *coincide over* $\mathrm{BCCSP}(A)$.

Proof. It suffices to show the statement for closed terms. Assume that $r \precsim_{\mathrm{PW}} s$. We prove that $r \sqsubseteq_{\mathrm{PW}} s$ also holds. To this end, observe, first of all, that $\mathcal{I}(r) = \mathcal{I}(s)$, since $\precsim_{\mathrm{PW}}$ is included in $\precsim_{\mathrm{R}}$. We are therefore left to show that the prefixed possible worlds of r are also prefixed possible worlds of s.

Suppose that ap is a prefixed possible world of r. It is not hard to see that $ap + p'$ is a possible world of r, for some p'. As $r \precsim_{\mathrm{PW}} s$, it follows that $ap + p'$ is also a possible world of s. We may therefore conclude that ap is a prefixed possible world of s, which was to be shown.

Assume now that $r \sqsubseteq_{\mathrm{PW}} s$. We prove that $r \precsim_{\mathrm{PW}} s$ also holds. Observe, first of all, that $\mathcal{I}(r) = \mathcal{I}(s)$ by our assumption that $r \sqsubseteq_{\mathrm{PW}} s$. Let p be a possible world of r. Then p is deterministic and $p \precsim_{\mathrm{RS}} r$. Since p is deterministic, for each $a \in \mathcal{I}(p)$ there is a unique closed term p_a such that $p \xrightarrow{a} p_a$. Moreover,

$$p = \sum_{a \in \mathcal{I}(p)} ap_a$$

and $\mathcal{I}(p) = \mathcal{I}(r) = \mathcal{I}(s)$. As $p \precsim_{\mathrm{RS}} r$, for each $a \in \mathcal{I}(p)$ there is a closed term r_a such that $p_a \precsim_{\mathrm{RS}} r_a$. Since p_a is itself deterministic, ap_a is a prefixed possible world of r, for each $a \in \mathcal{I}(p)$. As $r \sqsubseteq_{\mathrm{PW}} s$ by assumption, it follows that ap_a is also a prefixed possible world of s for each $a \in \mathcal{I}(p)$. We conclude that p is a possible world of s, which was to be shown. $\square$

We are now ready to prove Theorem 1 for $\precsim_{\mathrm{PW}}$. In light of the above lemma, it suffices to prove this statement for $\sqsubseteq_{\mathrm{PW}}$. Assume that $t \not\sqsubseteq_{\mathrm{PW}} u$, and x is not a summand of $t + u$. We shall show that $t + x \not\sqsubseteq_{\mathrm{PW}} u + x$.

Since $t \not\sqsubseteq_{\mathrm{PW}} u$, there is a closed substitution σ such that $\sigma(t) \not\sqsubseteq_{\mathrm{PW}} \sigma(u)$.

If $\mathcal{I}(\sigma(t)) \neq \mathcal{I}(\sigma(u))$, then, reasoning as in the proof for $\precsim_{\mathrm{R}}$, it is easy to prove that $t + x \not\sqsubseteq_{\mathrm{PW}} u + x$. So we can assume that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$.

Because of this assumption, there is a prefixed possible world ap of $\sigma(t)$ that is not a prefixed possible world of $\sigma(u)$. Our order of business will now be to construct a closed substitution ρ with the following properties:

1. $\rho(y) = \sigma(y)$ for each variable $y \neq x$,
2. $\rho(x)$ and $\sigma(x)$ have the same prefixed possible worlds of depth at most $depth(p)$, and
3. $\rho(x)$ does not have any completed traces of length $depth(p) + 1$.

Before giving the construction of ρ, we shall argue that from these three properties it follows that

$$\rho(t + x) \not\sqsubseteq_{\mathrm{PW}} \rho(u + x) \ .$$

In view of properties 1 and 2, it is not hard to see that for any term v,

(i) $\sigma(v)$ and $\rho(v)$ have the same prefixed possible worlds of depth at most $depth(p)$, and
(ii) if v does not have a summand x, then $\sigma(v)$ and $\rho(v)$ have the same prefixed possible worlds of depth at most $depth(p)+1$.

We prove these two claims by induction on $depth(v)$. Suppose that

$$v = \sum_{i\in I} a_i v_i + \sum_{j\in J} y_j \ .$$

By induction, for $i \in I$, claim (i) yields that $\rho(v_i)$ and $\sigma(v_i)$ have the same prefixed possible worlds of depth at most $depth(p)$. This implies (cf. Lemma 8) that $\rho(a_i v_i)$ and $\sigma(a_i v_i)$ have the same prefixed possible worlds of depth at most $depth(p)+1$. And for $j \in J$, if $y_j \neq x$, then by property 1, $\rho(y_j) = \sigma(y_j)$, so they have the same prefixed possible worlds. This completes the proof of claim (ii). Finally, if $y_j = x$ for some $j \in J$, then by property 2, $\rho(x)$ and $\sigma(x)$ have the same prefixed possible worlds of depth at most $depth(p)$. Hence we can conclude that claim (i) also holds.

By assumption, x is not a summand of t, and ap is a prefixed possible world of $\sigma(t)$. So by claim (ii), ap is a prefixed possible world of $\rho(t)$, and so also of $\rho(t+x)$.

By assumption, x is not a summand of u, and ap is not a prefixed possible world of $\sigma(u)$. So by claim (ii), ap is not a prefixed possible world of $\rho(u)$. Moreover, by property 3, ap is not a prefixed possible world of $\rho(x)$. Hence, ap is not a prefixed possible world of $\rho(u+x)$.

Since ap is a prefixed possible world of $\rho(t+x)$ and not of $\rho(u+x)$, we conclude that $t+x \not\sqsubseteq_{\mathrm{PW}} u+x$, which was to be proved.

All that we are left to do to complete the proof for this case is to construct a closed substitution ρ having properties 1–3. We define

$$\rho = \sigma[x \mapsto \pi^{\emptyset}_{depth(p)}(\sigma(x))] \ ,$$

where the closed term $\pi^{\emptyset}_{depth(p)}(\sigma(x))$ is defined as on page 18. Property 1 trivially holds. And property 2 follows immediately from the following two observations:

– $\mathcal{I}(\pi^{\emptyset}_n(q)) = \mathcal{I}(q)$, for each q and $n \geq 0$, and
– for all closed terms q, r, action c and $n > 0$,

$$q \xrightarrow{c} r \text{ iff } \pi^{\emptyset}_n(q) \xrightarrow{c} \pi^{\emptyset}_{n-1}(r) \ .$$

Finally, to see that ρ meets property 3, assume that $\pi^{\emptyset}_{depth(p)}(\sigma(x)) \stackrel{c_1\ldots c_{depth(p)+1}}{\longrightarrow} r$ for some sequence $c_1 \ldots c_{depth(p)+1}$ of actions and closed term r. It is not hard to see that then $r = a\mathbf{0}$ for some $a \in A$.

This concludes the proof for $\precsim_{\mathrm{PW}}$. □

6 Concluding Remarks

In this paper, we have proved that all of the preorders in the linear time-branching time spectrum enjoy the, to our mind very natural and elegant, cancellation property stated in Theorem 1. It is remarkable that the proof of such a simply-stated and natural property turns out to be non-trivial for most of the semantics in the spectrum. We trust that the cancellation property stated in Theorem 1 will make it easier to apply the cover-equations approach due to Fokkink and Nain in the study of the equational theory of BCCSP modulo the behavioural preorders in the spectrum. We also hope that the proof techniques that we have employed in the proof of Theorem 1 may have some independent interest, and that they may be applicable in other contexts.

We conclude this paper by remarking that Theorem 1 also holds modulo bisimulation equivalence [12]. This follows because an equation $t \approx u$ is valid modulo bisimulation equivalence over the language BCCSP iff

- t and u have the same variable summands;
- for each summand at' of t, there is a summand au' of u such that t' and u' are bisimulation equivalent; and
- for each summand au' of u, there is a summand at' of t such that t' and u' are bisimulation equivalent.

(See, e.g., [13, page 41].) Therefore t and u are bisimulation equivalent if so are $t + x$ and $u + x$, provided that x is *not* a summand of $t + u$.

References

1. L. Aceto, W. Fokkink, R.J. van Glabbeek, and A. Ingolfsdottir. Nested semantics over finite trees are equationally hard. *Information and Computation*, 191(2):203–232, 2004.
2. L. Aceto, W. Fokkink, and A. Ingolfsdottir. A menagerie of non-finitely based process semantics over BPA*—from ready simulation to completed traces. *Mathematical Structures in Computer Science*, 8(3):193–230, 1998.
3. J.A. Bergstra and J.W. Klop. Fixed point semantics in process algebras. Report IW 206, Mathematisch Centrum, Amsterdam, 1982.
4. S. Blom, W. Fokkink, and S. Nain. On the axiomatizability of ready traces, ready simulation and failure traces. In *Proc. ICALP'03*, LNCS 2719, pp. 109–118. Springer, 2003.
5. T. Chen, W. Fokkink, B. Luttik, and S. Nain. On finite alphabets and infinite bases. *Information and Computation*, To appear.
6. W. Fokkink and S. Nain. On finite alphabets and infinite bases: from ready pairs to possible worlds. In *Proc. FoSSaCS'04*, LNCS 2987, pp. 182–194. Springer, 2004.
7. W. Fokkink and S. Nain. A finite basis for failure semantics. In *Proc. ICALP'05*, LNCS 3580, pp. 755–765. Springer, 2005.
8. R.J. van Glabbeek. The linear time – branching time spectrum I. The semantics of concrete, sequential processes. In J.A. Bergstra, A. Ponse, and S.A. Smolka, eds, *Handbook of Process Algebra*, pp. 3–99. Elsevier, 2001.

9. J.F. Groote. A new strategy for proving ω-completeness with applications in process algebra. In *Proc. CONCUR'90*, LNCS 458, pp. 314–331. Springer, 1990.
10. M.C.B. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32(1):137–161, 1985.
11. C.A.R. Hoare. *Communicating Sequential Processes*. Prentice-Hall, 1985.
12. R. Milner. *Communication and Concurrency*. Prentice-Hall, 1989.
13. F. Moller. *Axioms for Concurrency*. PhD thesis, Department of Computer Science, University of Edinburgh, July 1989. Report CST-59-89. Also published as ECS-LFCS-89-84.