On the specification of modal systems: a comparison of three frameworks☆

Luca Aceto[a], Ignacio Fábregas[b], David de Frutos-Escrig[b], Anna Ingólfsdóttir[a], Miguel Palomino[b]

[a] *ICE-TCS, School of Computer Science, Reykjavik University, Iceland*
[b] *Departamento de Sistemas Informáticos y Computación, Universidad Complutense de Madrid, Spain*

Abstract

This paper studies the relationships between three notions of behavioural preorder that have been proposed in the literature: refinement over modal transition systems, and the covariant-contravariant simulation and the partial bisimulation preorders over labelled transition systems. It is shown that there are mutual translations between modal transition systems and labelled transition systems that preserve, and reflect, refinement and the covariant-contravariant simulation preorder. The translations are also shown to preserve the modal properties that can be expressed in the logics that characterize those preorders. A translation from labelled transition systems modulo the partial bisimulation preorder into the same model modulo the covariant-contravariant simulation preorder is also offered, together with some evidence that the former model is less expressive than the latter. In order to gain more insight into the relationships between modal transition systems modulo refinement and labelled transition systems modulo the covariant-contravariant simulation preorder, their connections are also phrased and studied in the context of institutions.

1. Introduction

Modal transition systems (MTSs) have been proposed in, e.g., [16, 17] as a model of reactive computation based on states and transitions that naturally supports a notion of *refinement* that is akin to the notion of implication in logical specification languages. (See the paper [6] for a thorough analysis of the connections between specifications given in terms of MTSs and logical specifications in the setting of a modal logic that characterizes refinement.) In an MTS,

☆Research supported by Spanish projects DESAFIOS10 TIN2009-14599-C03-01, TESIS TIN2009-14321-C02-01 and PROMETIDOS S2009/TIC-1465, and the NILS Mobility Project (Abel Extraordinary Chair programme). Luca Aceto and Anna Ingólfsdóttir have been partially supported by the project 'Processes and Modal Logics' (project nr. 100048021) of the Icelandic Fund for Research.

transitions come in two flavours: the *may* transitions and the *must* transitions, with the requirement that each must transition is also a may transition. The idea behind the notion of refinement over MTSs is that, in order to implement correctly a specification, an implementation should exhibit all the transitions that are required by the specification (these are the must transitions in the MTS that describes the specification) and may provide the transitions that are allowed by the specification (these are the may transitions in the MTS that describes the specification).

The formalism of modal transition systems is intuitive, has several variants with varying degrees of expressive power and complexity—see, e.g., the survey paper [3]—and has recently been used as a suitable model for the specification of service-oriented applications. In particular, results on the supervisory control (in the sense of Ramadge and Wonham [20]) of systems whose specification is given in that formalism have been presented in, e.g., [7, 11].

The very recent development of the notion of *partial bisimulation* in the setting of labelled transition systems (LTSs) presented in [4, 5] has been explicitly motivated by the desire to develop a process-algebraic model within which one can study topics in the field of supervisory control. A partial bisimulation is a variation on the classic notion of bisimulation [18, 19] in which two LTSs are only required to fulfil the bisimulation conditions on a subset B of the collection of actions; transitions labelled by actions not in B are treated as in the standard simulation preorder. Intuitively, one may think of the actions in B as corresponding to the uncontrollable events—see [4, page 4]. The aforementioned paper offers a thorough development of the basic theory of partial bisimulation.

Another recent proposal for a simulation-based behavioural relation over LTSs, called the *covariant-contravariant simulation preorder*, has been put forward in [8], and its theory has been investigated further in [9]. This notion of simulation between LTSs is based on considering a partition of their set of actions into three sets: the collection of covariant actions, that of contravariant actions and the set of bivariant actions. Intuitively, one may think of the covariant actions as being under the control of the specification LTS, and transitions with such actions as their label should be simulated by any correct implementation of the specification. On the other hand, the contravariant actions may be considered as being under the control of the implementation (or of the environment) and transitions with such actions as their label should be simulated by the specification. The bivariant actions are treated as in the classic notion of bisimulation.

It is natural to wonder whether there are any relations among these three formalisms. In particular, one may ask oneself whether it is possible to offer mutual translations between specifications given in those state-transition-based models that preserve, and reflect, the appropriate notions of behavioural preorder as well as properties expressed in the modal logics that accompany them—see, e.g., [4, 6, 9]. The aim of this study is to offer an answer to this question.

In this paper, we study the relationships between refinement over modal transition systems, and the covariant-contravariant simulation and the partial bisimulation preorders over labelled transition systems. We offer mutual translations between modal transition systems and labelled transition systems that preserve, and reflect, refinement and the covariant-contravariant simulation preorder, as well as the modal properties that can be expressed in the logics that characterize those preorders. We also give a translation from labelled transition systems modulo the partial bisimulation preorder into the same model modulo the covariant-contravariant simulation preorder, together with some evidence that the former model is less expressive than the latter. Finally, in order to gain more insight into the relationships between modal transition systems modulo refinement and labelled transition systems modulo the covariant-contravariant simulation preorder, we phrase and study their connections in the context of institutions [12].

The developments in this paper indicate that the formalism of MTSs may be seen as a common ground within which one can embed LTSs modulo the covariant-contravariant simulation preorder or partial bisimilarity. Moreover, there are some interesting, and non-obvious, corollaries that one may infer from the translations we provide. See Section 5, where we use our translations to show, e.g., that checking whether two states in an LTS are related by the covariant-contravariant simulation preorder can always be reduced to an equivalent check in a setting without bivariant actions, and provide a more detailed analysis of the translations. The study of the relative expressive power of different formalisms is, however, an art as well as a science, and may yield different answers depending on the conceptual framework that one adopts for the comparison. For instance, at the level of institutions [12], we provide an institution morphism from the institution corresponding to the theory of MTSs modulo refinement into the institution corresponding to the theory of LTSs modulo the covariant-contravariant simulation preorder. However, we conjecture that there is no institution morphism in the other direction. The work presented in the study opens several interesting avenues for future research, and settling the above conjecture is one of a wealth of research questions we survey in Section 9.

The remainder of the paper is organized as follows. Section 2 is devoted to preliminaries. In particular, in that section, we provide all the necessary background on modal and labelled transition systems, modal refinement and the covariant-contravariant simulation preorder, and the modal logics that characterize those preorders. In Section 3, we show how one can translate LTSs modulo the covariant-contravariant simulation preorder into MTSs modulo refinement. Section 4 presents the converse translation. We discuss the mutual translations between LTSs and MTSs in Section 5. As described in Section 6, the translation from MTSs and their modal logic to the realm of LTSs modulo the covariant-contravariant simulation preorder can be used to transfer the characteristic-formula result from [6] to one for LTSs modulo the covariant-contravariant simulation preorder. Section 7 offers a translation from LTSs modulo partial bisimilarity into LTSs modulo the covariant-contravariant simulation preorder. In Section 8, we study the relationships between modal transition systems modulo refinement and labelled transition systems modulo the covariant-contravariant simulation preorder in the context of institutions. Section 9 concludes the paper and offers a number of directions for future research that we plan to pursue.

This article is a substantially expanded version of the conference paper [1]. Apart from including the proofs of all the technical results, which were announced without proof in the conference publication with the exception of three propositions, as well as further remarks and explanations, the following contributions are new in this version of the paper:

- the discussion of the translation $\mathcal{MC}$ from Boudol-Larsen modal formulae to covariant-contravariant formulae presented on pages 9–10;
- the discussion of the translation $\mathcal{C}^{-1}$ from covariant-contravariant formulae to Boudol-Larsen modal formulae presented on page 13;
- the material in Section 6; and
- the material on page 28 regarding a conjecture from [1].

2. Preliminaries

We begin by introducing modal transition systems, with their associated notion of (modal) refinement, and labelled transition systems modulo the covariant-contravariant simulation preorder. We refer the reader to, e.g., [6, 16, 17] and [8, 9] for more information, motivation and examples.

2.1. Modal transition systems and refinement

Definition 1. For a set of actions A, a *modal transition* system (MTS) is a triple $(P, \rightarrow_\diamond, \rightarrow_\Box)$, where P is a set of states and $\rightarrow_\diamond, \rightarrow_\Box \subseteq P \times A \times P$ are transition relations such that $\rightarrow_\Box \subseteq \rightarrow_\diamond$.

An MTS is *image finite* iff the set $\{p' \mid p \xrightarrow{a}_\diamond p'\}$ is finite for each $p \in P$ and $a \in A$.

The transitions in $\rightarrow_\Box$ are called the *must transitions* and those in $\rightarrow_\diamond$ are the *may transitions*. In an MTS, each must transition is also a may transition, which intuitively means that any required transition is also allowed.

In what follows, we often identify an MTS, or a transition system of any of the types that we consider in this paper, with its set of states. In case we wish to make clear the 'ambient' transition system in which a state p lives, we write (P, p) to indicate that p is to be viewed as a state in P.

The notion of (modal) refinement $\sqsubseteq$ over MTSs that we now proceed to introduce is based on the idea that if $p \sqsubseteq q$ then q is a 'refinement' of the specification p. In that case, intuitively, q may be obtained from p by possibly

- removing some of its may transitions and/or
- turning some of its may transitions into must transitions.

Definition 2. A relation $R \subseteq P \times Q$ is a *refinement relation* between two modal transition systems if, whenever $p \; R \; q$:

- $p \xrightarrow{a}_{\Box} p'$ implies that there exists some q' such that $q \xrightarrow{a}_{\Box} q'$ and $p' \; R \; q'$;
- $q \xrightarrow{a}_{\Diamond} q'$ implies that there exists some p' such that $p \xrightarrow{a}_{\Diamond} p'$ and $p' \; R \; q'$.

We write $\sqsubseteq$ for the largest refinement relation.

Example 1. Consider the MTS U over the set of actions A with u as its only state, and transitions $u \xrightarrow{a}_{\Diamond} u$ for each $a \in A$. It is well known, and not hard to see, that $u \sqsubseteq p$ holds for each state p in any MTS over action set A. The state u is often referred to as the *loosest (or universal) specification*.

Definition 3. Given a set of actions A, the collection of *Boudol-Larsen's modal formulae [6]* is given by the following grammar:

$$\varphi ::= \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid [a]\varphi \mid \langle a \rangle \varphi \qquad (a \in A).$$

The semantics of these formulae with respect to an MTS P and a state $p \in P$ is defined by means of the satisfaction relation $\models$, which is the least relation satisfying the following clauses:

$(P, p) \models \top$.

$(P, p) \models \varphi_1 \wedge \varphi_2$ if $(P, p) \models \varphi_1$ and $(P, p) \models \varphi_2$.

$(P, p) \models \varphi_1 \vee \varphi_2$ if $(P, p) \models \varphi_1$ or $(P, p) \models \varphi_2$.

$(P, p) \models [a]\varphi$ if $(P, p') \models \varphi$ for all $p \xrightarrow{a}_{\Diamond} p'$.

$(P, p) \models \langle a \rangle \varphi$ if $(P, p') \models \varphi$ for some $p \xrightarrow{a}_{\Box} p'$.

We say that a formula is *existential* if it does not contain occurrences of $[a]$-operators, $a \in A$.

For example, the state u in the MTS U from Example 1 satisfies neither the formula $\langle a \rangle \top$ nor the formula $[a]\bot$. Indeed, it is not hard to see that (U, u) satisfies a formula φ if, and only if, φ is a tautology.

The following result stems from [6].

Proposition 1. *Let p, q be states in image-finite MTSs over the set of actions A. Then $p \sqsubseteq q$ iff the collection of Boudol-Larsen's modal formulae satisfied by p is included in the collection of formulae satisfied by q.*

2.2. Labelled transition systems and covariant-contravariant simulation

A labelled transition system (LTS) is just an MTS with $\rightarrow_\diamond = \rightarrow_\Box$. In what follows, we write $\rightarrow$ for the transition relation in an LTS.

Definition 4. Let P and Q be two LTSs over the set of actions A, and let $\{A^r, A^l, A^{bi}\}$ be a partition of A[1]. An (A^r, A^l)*-simulation* (or just a *covariant-contravariant simulation* when the partition of the set of actions A is understood from the context) between P and Q is a relation $R \subseteq P \times Q$ such that, whenever $p\ R\ q$, we have:

- For all $a \in A^r \cup A^{bi}$ and all $p \xrightarrow{a} p'$, there exists some $q \xrightarrow{a} q'$ with $p'\ R\ q'$.
- For all $a \in A^l \cup A^{bi}$ and all $q \xrightarrow{a} q'$, there exists some $p \xrightarrow{a} p'$ with $p'\ R\ q'$.

We will write $p \lesssim_{cc} q$ if there exists a covariant-contravariant simulation R such that $p\ R\ q$.

The actions in the set A^r are sometimes called *covariant*, those in A^l are *contravariant* and the ones in A^{bi} are *bivariant*. When working with covariant-contravariant simulations, we shall sometimes refer to the triple (A^r, A^l, A^{bi}) as the *signature* of the corresponding LTS, and we will say that such a system is a covariant-contravariant LTS.

Example 2. Assume that $a \in A^r$ and $b \in A^l$. Consider the LTSs described by the CCS [18] terms $p = a + b$, $q = a$ and $r = b$. Then $r \lesssim_{cc} p \lesssim_{cc} q$, but none of the converse relations holds.

Definition 5. *Covariant-contravariant modal logic* has almost the same syntax as the one for modal refinement:

$$\varphi ::= \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid [b]\varphi \mid \langle a \rangle \varphi \qquad (a \in A^r \cup A^{bi}, b \in A^l \cup A^{bi}).$$

However, the semantics differs for the modal operators, since we interpret formulae over ordinary LTSs:

$(P, p) \models [b]\varphi$ if $(P, p') \models \varphi$ for all $p \xrightarrow{b} p'$.

$(P, p) \models \langle a \rangle \varphi$ if $(P, p') \models \varphi$ for some $p \xrightarrow{a} p'$.

For example, both p and q from Example 2 satisfy the formula $\langle a \rangle \top$, while r does not. On the other hand, q satisfies the formula $[b]\bot$, but neither p nor r do.

The following result stems from [9].

Proposition 2. *Let p, q be states in image-finite LTSs with the same signature. Then $p \lesssim_{cc} q$ iff the collection of covariant-contravariant modal formulae satisfied by p is included in the collection of covariant-contravariant modal formulae satisfied by q.*

[1]Note that any of the sets A^r, A^l and A^{bi} may be empty. Our use of the word 'partition' is therefore non-standard.

3. From covariant-contravariant simulations to modal refinements

We start our study of the connections between MTSs modulo refinement and LTSs modulo the covariant-contravariant simulation preorder, by showing that LTSs modulo $\lesssim_{cc}$ may be translated into MTSs modulo $\sqsubseteq$. Such a translation preserves, and reflects, those preorders and the satisfaction of modal formulae. This result is, at first, a bit surprising, since covariant-contravariant systems look more expressive than modal systems because they contain three different kinds of actions, which moreover are totally independent from each other, while modal systems only contain two kinds of transitions, which besides are strongly related, since any must transition is also a may one.

Definition 6. Let P be a covariant-contravariant LTS with signature $\{A^r, A^l, A^{bi}\}$. Then the associated MTS $\mathcal{M}(P)$ is constructed as follows:

- The set of actions of $\mathcal{M}(P)$ is $A = A^r \cup A^l \cup A^{bi}$.
- The set of states of $\mathcal{M}(P)$ is that of P plus a new state u.
- For each transition $p \xrightarrow{a} p'$ in P, add a may transition $p \xrightarrow{a}_\diamond p'$ in $\mathcal{M}(P)$.
- For each transition $p \xrightarrow{a} p'$ in P with $a \in A^r \cup A^{bi}$, add a must transition $p \xrightarrow{a}_\Box p'$ in $\mathcal{M}(P)$.
- For each a in A^r and state p, add the transition $p \xrightarrow{a}_\diamond u$ to $\mathcal{M}(P)$, as well as transitions $u \xrightarrow{a}_\diamond u$ for each action $a \in A$.
- There are no other transitions in $\mathcal{M}(P)$.

The following proposition essentially states that the translation $\mathcal{M}$ is correct.

Proposition 3. *A relation R is a covariant-contravariant simulation between LTSs P and Q iff $\mathcal{M}(R)$ is a refinement between $\mathcal{M}(P)$ and $\mathcal{M}(Q)$, where $\mathcal{M}(R) = R \cup \{(u, q) \mid q$ a state of $\mathcal{M}(Q)\}$.*

Proof. We prove the two implications separately.

($\Rightarrow$) Assume that R is a covariant-contravariant simulation. We shall prove that $\mathcal{M}(R)$ is a refinement.

Suppose that $p\ R\ q$ and $q \xrightarrow{a}_\diamond q'$ in $\mathcal{M}(Q)$. By the definition of $\mathcal{M}(Q)$, the transition $q \xrightarrow{a} q'$ is in Q. If $a \in A^l \cup A^{bi}$, since $p\ R\ q$ and R is a covariant-contravariant simulation, we have that $p \xrightarrow{a} p'$ in P for some p' such that $p'\ R\ q'$. By the construction of $\mathcal{M}(P)$, it holds that $p \xrightarrow{a}_\diamond p'$ and we are done. If $a \in A^r$, then $p \xrightarrow{a}_\diamond u$ and $u\ \mathcal{M}(R)\ q'$, as required.

Assume now that $p\ R\ q$ and $p \xrightarrow{a}_\Box p'$ in $\mathcal{M}(P)$. Then $p \xrightarrow{a} p'$ in P with $a \in A^r \cup A^{bi}$. As R is a covariant-contravariant simulation, it follows that $q \xrightarrow{a} q'$ in Q for some q' such that $p'\ R\ q'$. Since $a \in A^r \cup A^{bi}$, there is a must transition $q \xrightarrow{a}_\Box q'$ in $\mathcal{M}(Q)$, and we are done. To finish the proof of this implication, recall that, as shown in Example 1, each state q is a refinement of u.

($\Leftarrow$) Assume that $\mathcal{M}(R)$ is a refinement. We shall prove that R is a covariant-contravariant simulation.

Suppose that $p\ R\ q$ and $q \xrightarrow{a} q'$ in Q with $a \in A^l \cup A^{bi}$. Then $q \xrightarrow{a}_\diamond q'$ in $\mathcal{M}(Q)$. Since $\mathcal{M}(R)$ is a refinement, in $\mathcal{M}(P)$ we have that $p \xrightarrow{a}_\diamond p'$ for some p' (different from u, because $a \notin A^r$) such that $p'\ R\ q'$. By the construction of $\mathcal{M}(P)$, it follows that $p \xrightarrow{a} p'$ in P and we are done.

Suppose now that $p\ R\ q$ and $p \xrightarrow{a} p'$ in P with $a \in A^r \cup A^{bi}$. Then $p \xrightarrow{a}_\Box p'$ in $\mathcal{M}(P)$. Since $\mathcal{M}(R)$ is a refinement, there is some q' (again, different from u) such that $q \xrightarrow{a}_\Box q'$ in $\mathcal{M}(Q)$ and $p'\ R\ q'$. By the construction of $\mathcal{M}(Q)$, it follows that $q \xrightarrow{a} q'$ in Q and we are done. □

Remark 1. As witnessed by the proof of the above proposition, the role of the transitions $p \xrightarrow{a}_\diamond u$ in $\mathcal{M}(P)$ with $a \in A^r$, where u is the loosest specification from Example 1, is to satisfy 'for free' the proof obligations that are generated, in the setting of modal refinement, by representing A^r-labelled transitions in an LTS P by means of must transitions in $\mathcal{M}(P)$. This is in the spirit of the developments in [15], where the standard simulation preorder is cast in a coalgebraic framework by phrasing it in the setting of bisimilarity. The coalgebraic recasting of simulation as a bisimulation is done in such a way that the added proof obligations that are present in the definition of bisimilarity are automatically satisfied.

Corollary 4. *Let P and Q be two LTSs with the same signature, and let $p \in P$ and $q \in Q$. Then $(P,p) \lesssim_{cc} (Q,q)$ iff $(\mathcal{M}(P),p) \sqsubseteq (\mathcal{M}(Q),q)$.*

Definition 7. Let us extend $\mathcal{M}$ to translate formulae over the modal logic that characterizes the covariant-contravariant simulation preorder to the modal logic for modal transition systems by simply defining $\mathcal{M}(\varphi) = \varphi$.

Proposition 5. *If P is an LTS and φ is a formula of the logic that characterizes covariant-contravariant simulation, then for each $p \in P$:*

$$(P,p) \models \varphi \iff (\mathcal{M}(P),p) \models \mathcal{M}(\varphi).$$

Proof. By structural induction on φ. The only non-trivial cases are the ones corresponding to the modal operators, which we detail below. (In all the following proofs, the steps labelled 'IH' are those that use the induction hypothesis.)

- $\langle a \rangle \varphi$, with $a \in A^r \cup A^{bi}$.

$$\begin{aligned}
(P,p) \models \langle a \rangle \varphi &\iff \text{there is } p \xrightarrow{a} p' \text{ in } P \text{ with } (P,p') \models \varphi \\
&\overset{\text{IH}}{\iff} \text{there is } p \xrightarrow{a}_\Box p' \text{ in } \mathcal{M}(P) \text{ with } (\mathcal{M}(P),p') \models \mathcal{M}(\varphi) \\
&\iff (\mathcal{M}(P),p) \models \langle a \rangle \mathcal{M}(\varphi) \\
&\iff (\mathcal{M}(P),p) \models \mathcal{M}(\langle a \rangle \varphi)
\end{aligned}$$

- $[a]\varphi$, with $a \in A^l \cup A^{bi}$.

$$\begin{array}{rcl}(P,p) \models [a]\varphi & \Longleftrightarrow & (P,p') \models \varphi \text{ for all } p \xrightarrow{a} p' \text{ in } P \\ & \overset{\text{IH}}{\Longleftrightarrow} & (\mathcal{M}(P),p') \models \mathcal{M}(\varphi) \text{ for all } p \xrightarrow{a}_{\diamond} p' \text{ in } \mathcal{M}(P) \\ & & (\text{note that } p \xrightarrow{a}_{\diamond} u \text{ only for } a \in A^r) \\ & \Longleftrightarrow & (\mathcal{M}(P),p) \models [a]\mathcal{M}(\varphi) \\ & \Longleftrightarrow & (\mathcal{M}(P),p) \models \mathcal{M}([a]\varphi)\end{array}$$

□

It is natural to wonder whether it is possible to provide a version of Proposition 5 for formulae in Boudol-Larsen modal logic. In particular, it would be interesting to characterize the collections of formulae in Boudol-Larsen modal logic whose satisfaction is preserved by $\mathcal{M}$, in a suitable technical sense. In order to address this question, let $\{A^r, A^l, A^{bi}\}$ be the signature of some LTS P and let $A = A^r \cup A^l \cup A^{bi}$.

Define the transformation $\mathcal{MC}$ from Boudol-Larsen formulae over A to covariant-contravariant formulae over the signature $\{A^r, A^l, A^{bi}\}$ as follows:

Definition 8. $\mathcal{MC}$ is the unique homomorphism satisfying:

- $\mathcal{MC}(\langle a\rangle\varphi) = \begin{cases} \langle a\rangle\mathcal{MC}(\varphi) & \text{if } a \in A^r \cup A^{bi} \\ \bot & \text{otherwise} \end{cases}$
- $\mathcal{MC}([a]\varphi) = \begin{cases} [a]\mathcal{MC}(\varphi) & \text{if } a \in A^l \cup A^{bi} \\ \top & \text{otherwise} \end{cases}$

The interplay between the transformation function $\mathcal{M}$ between LTSs and MTSs, and the function $\mathcal{MC}$ operating on Boudol-Larsen formulae is fully described by the following results.

Proposition 6. *Let P an LTS over signature $\{A^r, A^l, A^{bi}\}$ and let $p \in P$. Suppose that φ is a formula in Boudol-Larsen modal logic over $A = A^r \cup A^l \cup A^{bi}$. Then the following statements hold:*

1. *If $(\mathcal{M}(P),p) \models \varphi$ then $(P,p) \models \mathcal{MC}(\varphi)$.*
2. *If $(P,p) \models \mathcal{MC}(\varphi)$ and (either φ is existential or $A^r = \emptyset$) then $(\mathcal{M}(P),p) \models \varphi$.*

Proof. We prove the two statements separately.

1. We proceed by induction on the structure of φ and focus on the cases involving the modal operators.

- Case $\varphi = \langle a\rangle\varphi'$. Assume that $(\mathcal{M}(P), p) \models \langle a\rangle\varphi'$. This means that there is some p' such that $p \xrightarrow{a}_{\Box} p'$ in $\mathcal{M}(P)$ and $(\mathcal{M}(P), p') \models \varphi'$. By the definition of $\mathcal{M}$, $a \in A^r \cup A^{bi}$ and $p \xrightarrow{a} p'$. Moreover, by the inductive hypothesis, $(P, p) \models \mathcal{MC}(\varphi')$. Therefore, $(P, p) \models \langle a\rangle\mathcal{MC}(\varphi')$, and since $a \in A^r \cup A^{bi}$, $(P, p) \models \mathcal{MC}(\langle a\rangle\varphi')$.
- Case $\varphi = [a]\varphi'$. Assume that $(\mathcal{M}(P), p) \models [a]\varphi'$. If $a \in A^r$ then there is nothing to prove, since $\mathcal{MC}(\varphi) = \top$. Assume therefore that $a \in A^l \cup A^{bi}$. We will prove that $(p, p) \models [a]\mathcal{MC}(\varphi')$. To this end, suppose that $p \xrightarrow{a} p'$ in P with $a \in A^l \cup A^{bi}$. By the definition of $\mathcal{M}$ we have that $p \xrightarrow{a}_{\Diamond} p'$ in $\mathcal{M}(P)$. Since $(\mathcal{M}(P), p) \models [a]\varphi'$, it follows that $(\mathcal{M}(P), p') \models \varphi'$. The Inductive hypothesis yields $(P, p') \models \mathcal{MC}(\varphi')$, which was to be shown.

2. Assume that $(P, p) \models \mathcal{MC}(\varphi)$ and (either φ is existential or $A^r = \emptyset$). We show that $(\mathcal{M}(P), p) \models \varphi$ by induction on the structure of φ. Again, the only interesting cases are those dealing with the modal operators.

   - Case $\varphi = \langle a\rangle\varphi'$. Since $(P, p) \models \mathcal{MC}(\varphi)$, we have that $a \in A^r \cup A^{bi}$ and that $p \xrightarrow{a} p'$ for some p' such that $(P, p') \models \mathcal{MC}(\varphi')$. Since φ' is either existential or $A^r = \emptyset$, we may apply the inductive hypothesis to infer that $(\mathcal{M}(P), p') \models \varphi'$. By the definition of $\mathcal{M}$, we have that $p \xrightarrow{a}_m ustp'$. Therefore $(\mathcal{M}(P), p) \models \langle a\rangle\varphi'$, which was to be shown.
   - Case $\varphi = [a]\varphi'$. Since φ is not existential, we have that $A^r = \emptyset$. So $\mathcal{MC}([a]\varphi') = [a]\mathcal{MC}(\varphi')$ and $(P, p) \models [a]\mathcal{MC}(\varphi')$ by assumption.

     Let $p \xrightarrow{a}_{\Diamond} p'$ in $\mathcal{M}(P)$. As $a \in A^r \cup A^{bi}$, it follows that $p \xrightarrow{a} p'$ in P. Therefore $(P, p') \models \mathcal{MC}(\varphi')$. By induction, $(\mathcal{M}(P), p') \models \varphi'$. Since $p \xrightarrow{a}_{\Diamond} p'$ was chosen arbitrary, it follows that $(\mathcal{M}(P), p) \models [a]\varphi'$, and we are done. □

Remark 2. The proviso that φ is existential or $A^r = \emptyset$ is necessary in statement 2 of the above proposition. To see this, assume that $a \in A^r$ and consider the Boudol-Larsen formula $[a]\bot$. Then the LTS described by the covariant-contravariant system term 0 satisfies $\top = \mathcal{MC}([a]\bot)$. On the other hand, $0 \xrightarrow{a}_{\Diamond} u$ holds in $\mathcal{M}(0)$, and therefore $(\mathcal{M}(0), 0) \not\models [a]\bot$. This point is related to some observations we shall present in Section 8.

Remark 3 (Open question). Is there a (compositional) translation from Boudol-Larsen logic to covariant-contravariant modal logic such that

$$(P, p) \models \varphi \text{ implies } (\mathcal{M}(P), p) \models \varphi$$

for all LTSs P, states $p \in P$ and formulae φ?

4. From modal refinements to covariant-contravariant simulations

We next show that MTSs modulo $\sqsubseteq$ may be translated into LTSs modulo $\lesssim_{cc}$. As the one studied in the previous section, our translation preserves, and reflects, those preorders and the satisfaction of modal formulae. This is, to our mind, a less surprising result than the one presented in the previous section, even if in order to obtain it we have to introduce two "copies" of each action $a \in A$: one covariant $cv(a) \in A^r$ to represent must transitions, and another contravariant $ct(a) \in A^l$ to represent may transitions. As a matter of fact, we do not need the additional generality that is offered by the possibility of also having bivariant actions in the signature to adequately represent any MTS.

Definition 9. Let M be an MTS with set of actions A. The LTS $\mathcal{C}(M)$, with signature $A^r = \{\mathrm{cv}(a) \mid a \in A\}$, $A^l = \{\mathrm{ct}(a) \mid a \in A\}$ and $A^{bi} = \emptyset$, is constructed as follows:

- The set of states of $\mathcal{C}(M)$ is the same as that of M.
- For each transition $p \xrightarrow{a}_{\Diamond} p'$ in M, add $p \xrightarrow{\mathrm{ct}(a)} p'$ to $\mathcal{C}(M)$.
- For each transition $p \xrightarrow{a}_{\Box} p'$ in M, add $p \xrightarrow{\mathrm{cv}(a)} p'$ to $\mathcal{C}(M)$.
- There are no other transitions in $\mathcal{C}(M)$.

Observe that the LTSs obtained as a translation of an MTS have the following properties:

1. $A^{bi} = \emptyset$ and
2. there is a bijection $h : A^r \to A^l$ such that if $p \xrightarrow{a} p'$ with $a \in A^r$ then $p \xrightarrow{h(a)} p'$.

The latter requirement corresponds to the fact that each must transition in an MTS is also a may transition.

The following proposition states that the translation $\mathcal{C}$ is correct.

Proposition 7. *A relation R is a refinement between P and Q iff R is a covariant-contravariant simulation between $\mathcal{C}(P)$ and $\mathcal{C}(Q)$.*

Proof. We prove the two implications separately.

($\Rightarrow$) Assume that $p\ R\ q$. If $p \xrightarrow{\mathrm{cv}(a)} p'$ in $\mathcal{C}(P)$ then, by construction, $p \xrightarrow{a}_{\Box} p'$ in P. Since R is a refinement, there is some q' in Q with $q \xrightarrow{a}_{\Box} q'$ and $p'\ R\ q'$. Since $q \xrightarrow{\mathrm{cv}(a)} q'$ is in $\mathcal{C}(Q)$ by construction, we are done. Now, assume that $q \xrightarrow{\mathrm{ct}(a)} q'$ in $\mathcal{C}(Q)$. Then $q \xrightarrow{a}_{\Diamond} q'$ in Q and, since R is a refinement, $p \xrightarrow{a}_{\Diamond} p'$ in P for some p' with $p'\ R\ q'$. By construction, $p \xrightarrow{\mathrm{ct}(a)} p'$ is in $\mathcal{C}(P)$ and we are done.

($\Leftarrow$) Assume that $p\ R\ q$. If $q \xrightarrow{a}_{\Diamond} q'$ in Q then $q \xrightarrow{\mathrm{ct}(a)} q'$ in $\mathcal{C}(Q)$ and, since R is a covariant-contravariant simulation, $p \xrightarrow{\mathrm{ct}(a)} p'$ for some p' in $\mathcal{C}(P)$ such that

$p' R q'$; hence $p \xrightarrow{a}_{\diamond} p'$ in P as required. Now, if $p \xrightarrow{a}_{\Box} p'$ in P then $p \overset{\mathrm{cv}(a)}{\rightarrow} p'$ in $\mathcal{C}(P)$. Since R is a covariant-contravariant simulation, there is some q' in $\mathcal{C}(Q)$ with $q \overset{\mathrm{cv}(a)}{\rightarrow} q'$ and $p' \; R \; q'$, and therefore $q \xrightarrow{a}_{\Box} q'$ in Q. □

Corollary 8. *Let P and Q be two MTSs with the same action set, and let $p \in P$ and $q \in Q$. Then $(P, p) \sqsubseteq (Q, q)$ iff $(\mathcal{C}(P), p) \lesssim_{cc} (\mathcal{C}(Q), q)$.*

Remark 4. It is easy to see that the mapping $\mathcal{C}$ is injective. Therefore, given an LTS P that is in the range of $\mathcal{C}$, we may write $\mathcal{C}^{-1}(P)$ for the unique MTS whose $\mathcal{C}$-image is P.

Again, we can also extend the translation $\mathcal{C}$ to also translate modal formulae. However, in this case, the change of alphabet requires a simple, but non-trivial, definition of the extension.

Definition 10. Let us extend $\mathcal{C}$ to translate formulae over the modal logic for modal transition systems with set of actions A to the modal logic that characterizes covariant-contravariant simulation with signature $A^r = \{\mathrm{cv}(a) \mid a \in A\}$, $A^l = \{\mathrm{ct}(a) \mid a \in A\}$ and $A^{bi} = \emptyset$.

- $\mathcal{C}(\bot) = \bot$.
- $\mathcal{C}(\top) = \top$.
- $\mathcal{C}(\varphi \wedge \psi) = \mathcal{C}(\varphi) \wedge \mathcal{C}(\psi)$.
- $\mathcal{C}(\varphi \vee \psi) = \mathcal{C}(\varphi) \vee \mathcal{C}(\psi)$.
- $\mathcal{C}(\langle a \rangle \varphi) = \langle \mathrm{cv}(a) \rangle \mathcal{C}(\varphi)$.
- $\mathcal{C}([a]\varphi) = [\mathrm{ct}(a)]\mathcal{C}(\varphi)$.

Proposition 9. *If P is an MTS and φ is a Boudol-Larsen modal formula, then for each $p \in P$:*

$$(P, p) \models \varphi \iff (\mathcal{C}(P), p) \models \mathcal{C}(\varphi).$$

Proof. By structural induction on φ, with the only non-trivial cases being those that correspond to the modal operators:

- $[a]\varphi$, with $a \in A$.

$$\begin{aligned}
(P, p) \models [a]\varphi &\iff (P, p') \models \varphi \text{ for all } p \xrightarrow{a}_{\diamond} p' \text{ in } P \\
&\overset{\mathrm{IH}}{\iff} (\mathcal{C}(P), p') \models \mathcal{C}(\varphi) \text{ for all } p \overset{\mathrm{ct}(a)}{\rightarrow} p' \text{ in } \mathcal{C}(P) \\
&\iff (\mathcal{C}(P), p) \models [\mathrm{ct}(a)]\mathcal{C}(\varphi)
\end{aligned}$$

- $\langle a \rangle \varphi$, with $a \in A$.

$$\begin{array}{rcl} (P,p) \models \langle a \rangle \varphi & \iff & (P,p') \models \varphi \text{ for some } p \xrightarrow{a}_{\Box} p' \text{ in } P \\ & \overset{\text{IH}}{\iff} & (\mathcal{C}(P),p') \models \mathcal{C}(\varphi) \text{ for some } p \overset{\text{cv}(a)}{\rightarrow} p' \text{ in } \mathcal{C}(P) \\ & \iff & (\mathcal{C}(P),p) \models \langle \text{cv}(a) \rangle \mathcal{C}(\varphi) \end{array}$$

□

Remark 5. In fact, it is very easy to see that the translations $\mathcal{M}$ and $\mathcal{C}$ also preserve, and reflect, the satisfaction of formulae in the extensions of the logics from Definitions 3 and 5 with infinite conjunctions and disjunctions.

It is natural to wonder whether it is possible to provide a version of Proposition 9 for formulae in covariant-contravariant modal logic over the signature $A^r = \{\text{cv}(a) \mid a \in A\}$, $A^l = \{\text{ct}(a) \mid a \in A\}$ and $A^{bi} = \emptyset$. To this end, let $\mathcal{C}^{-1}$ denote the inverse of $\mathcal{C}$ over Boudol-Larsen modal formulae defined in the obvious way. We then have that:

Proposition 10. *Let P be an MTS over the set of actions A, and let φ be a covariant-contravariant modal formula over the signature $A^r = \{\text{cv}(a) \mid a \in A\}$, $A^l = \{\text{ct}(a) \mid a \in A\}$ and $A^{bi} = \emptyset$. Then, for each $p \in P$.*

$$(P,p) \models \mathcal{C}^{-1}(\varphi) \iff (\mathcal{C}(P),p) \models \varphi.$$

Proof. By Proposition 9,

$$(P,p) \models \mathcal{C}^{-1}(\varphi) \iff (\mathcal{C}(P),p) \models \mathcal{C}(\mathcal{C}^{-1}(\varphi)).$$

The claim now follows since $\mathcal{C}(\mathcal{C}^{-1}(\varphi)) = \varphi$. □

The above observation is in contrast with the result we established earlier in Proposition 6. This may be taken to be a first indication that the translation from MTSs to LTSs, and the accompanying one for the associated modal logics, is "more natural" than the one from LTSs to MTSs provided in Section 3. We will explore this issue in more detail in Section 8.

5. Discussion of the previous translations

In Sections 3–4, we saw that it is possible to translate back and forth between the world of LTSs modulo the covariant-contravariant simulation preorder and MTSs modulo refinement. The translations we have presented preserve, and reflect, the preorders and the relevant modal formulae. There are, however, some interesting, and non-obvious, corollaries that one may infer from the translations.

To begin with, assume that P and Q are two LTSs with the same signature, with $A^{bi} \neq \emptyset$. Let $p \in P$ and $q \in Q$ be such that $(P,p) \lesssim_{cc} (Q,q)$. By Corollary 4, we know that this holds exactly when $(\mathcal{M}(P),p) \sqsubseteq (\mathcal{M}(Q),q)$.

Using Corollary 8, we therefore have that checking whether $(P, p) \lesssim_{cc} (Q, q)$ is equivalent to verifying whether $(\mathcal{C}(\mathcal{M}(P)), p) \lesssim_{cc} (\mathcal{C}(\mathcal{M}(Q)), q)$. Note now that A^{bi} is empty in the signature for the LTSs $\mathcal{C}(\mathcal{M}(P))$ and $\mathcal{C}(\mathcal{M}(Q))$. Therefore, checking whether two states are related by the covariant-contravariant simulation preorder can always be reduced to an equivalent check in a setting without bivariant actions.

It is also natural to wonder whether there is any relation between a state p in an LTS P and the equally-named state in $\mathcal{C}(\mathcal{M}(P))$. Similarly, one may wonder whether there is any relation between a state p in an MTS P and the equally-named state in $\mathcal{M}(\mathcal{C}(P))$. In both cases, we are faced with the difficulty arising from the fact that the transition systems resulting from the compositions of the two translations are over the alphabet $\{\mathrm{cv}(a), \mathrm{ct}(a) \mid a \in A\}$, whereas the original system P had transitions labelled by actions in A. In order to overcome this difficulty, we consider the renaming $\rho : \{\mathrm{cv}(a), \mathrm{ct}(a) \mid a \in A\} \to A$ that maps both $\mathrm{cv}(a)$ and $\mathrm{ct}(a)$ to a, for each $a \in A$. Besides, for any transition system P over the set of actions $\{\mathrm{cv}(a), \mathrm{ct}(a) \mid a \in A\}$, we write $\rho(P)$ for the transition system that is obtained from P by renaming the label of each transition in P as indicated by ρ. Then we have the following proposition:

Proposition 11.

1. *Let P be an MTS and $p \in P$. Then we have $(\rho(\mathcal{M}(\mathcal{C}(P))), p) \sqsubseteq (P, p)$.*
2. *Let P be an LTS and $p \in P$. Then we have $(P, p) \lesssim_{cc} (\rho(\mathcal{C}(\mathcal{M}(P))), p)$.*
3. *In general, $(P, p) \sqsubseteq (\rho(\mathcal{M}(\mathcal{C}(P))), p)$ does not hold for an arbitrary MTS P and any state $p \in P$; nor does $(\rho(\mathcal{C}(\mathcal{M}(P))), p) \lesssim_{cc} (P, p)$, for an arbitrary LTS P and any state $p \in P$.*

Proof. We limit ourselves to detailing the proof for the second statement and to offering counter-examples proving the third one. The proof of the first claim follows similar lines to the one for the second, and in fact is even simpler.

In order to prove the second claim, it suffices to show that the identity relation over P is a covariant-contravariant simulation between P and $\rho(\mathcal{C}(\mathcal{M}(P)))$. To this end, assume first that $p \xrightarrow{a} p'$ in P for some $a \in A^r \cup A^{bi}$. Then $p \xrightarrow{a}_{\Box} p'$ in $\mathcal{M}(P)$. Therefore, $p \xrightarrow{\mathrm{cv}(a)} p'$ in $\mathcal{C}(\mathcal{M}(P))$ and $p \xrightarrow{a} p'$ in $\rho(\mathcal{C}(\mathcal{M}(P)))$.

Assume now that $p \xrightarrow{a} p'$ in $\rho(\mathcal{C}(\mathcal{M}(P)))$ for some $a \in A^l \cup A^{bi}$. This means that either $p \xrightarrow{\mathrm{cv}(a)} p'$ or $p \xrightarrow{\mathrm{ct}(a)} p'$ in $\mathcal{C}(\mathcal{M}(P))$. We consider these two possibilities separately.

- Suppose that $p \xrightarrow{\mathrm{cv}(a)} p'$ in $\mathcal{C}(\mathcal{M}(P))$. Then $p \xrightarrow{a}_{\Box} p'$ in $\mathcal{M}(P)$. This means that $p \xrightarrow{a} p'$ in P and $a \in A^r \cup A^{bi}$. By our assumption, it must be the case that $a \in A^{bi}$, and we are done.

- Suppose that $p \xrightarrow{\mathrm{ct}(a)} p'$ in $\mathcal{C}(\mathcal{M}(P))$. Then $p \xrightarrow{a}_{\Diamond} p'$ in $\mathcal{M}(P)$. Since $a \in A^l \cup A^{bi}$ by our assumption, we have that $p' \neq u$ in $\mathcal{M}(P)$, because u can only be reached via A^r-labelled may transitions. Therefore, $p' \in P$ and $p \xrightarrow{a} p'$.

This completes the proof of the second claim.

We now argue that, in general, $(P, p) \sqsubseteq (\rho(\mathcal{M}(\mathcal{C}(P))), p)$ does not hold for an MTS P and a state $p \in P$. Let P be the MTS over the alphabet $A = \{a\}$, with p as its only state and with no transitions. State p has an outgoing a-labelled may transition in $\rho(\mathcal{M}(\mathcal{C}(P)))$, which cannot be matched by p in P. Therefore, $(P, p) \not\sqsubseteq (\rho(\mathcal{M}(\mathcal{C}(P))), p)$.

To complete the proof we now argue that, in general, $(\rho(\mathcal{C}(\mathcal{M}(P))), p) \lesssim_{cc} (P, p)$ does not hold for an LTS P and a state $p \in P$. Let P be an LTS with $A^r = \{a\}$, p as its only state, and with no transitions. The sets A^l and A^{bi} can be arbitrary and play no role in the counter-example. Then it is immediate to see that state p has a transition $p \xrightarrow{a} u$ in $\rho(\mathcal{C}(\mathcal{M}(P)))$, but this transition cannot be matched by p in P. □

We shall now present a result on the relationships between the translations $\mathcal{M}$ and $\mathcal{C}$ for LTSs without bivariant actions.

Definition 11. Let P be an LTS with its alphabet partitioned into A^r and A^l. Then the LTS $\overline{P}$ is that obtained from P by simply renaming every $a \in A^r$ as $\mathrm{cv}(a)$ and every $a \in A^l$ as $\mathrm{ct}(a)$.

Proposition 12. *Let P be an LTS over an alphabet $A^r \cup A^l$ and let Q be an MTS over the same alphabet. Then the following statements hold.*

1. *If a relation R is a covariant-contravariant simulation between $\overline{P}$ and $\mathcal{C}(Q)$, then R is a refinement between $\mathcal{M}(P)$ and Q.*
2. *If $(\overline{P}, p) \lesssim_{cc} (\mathcal{C}(Q), q)$ then $(\mathcal{M}(P), p) \sqsubseteq (Q, q)$, for all states $p \in P$ and $q \in Q$.*
3. *The converse implication of the above statement fails.*

Proof. We limit ourselves to detailing a proof of the first statement and to offering a counter-example showing the third. The second statement is an immediate corollary of the first.

To prove the first statement, assume that $p\ R\ q$ and that R is a covariant-contravariant simulation between $\overline{P}$ and $\mathcal{C}(Q)$. If $q \xrightarrow{a}_{\diamond} q'$ in Q then $q \xrightarrow{\mathrm{ct}(a)} q'$ in $\mathcal{C}(Q)$. Since R is a covariant-contravariant simulation between $\overline{P}$ and $\mathcal{C}(Q)$, there is some p' in $\overline{P}$ with $p \xrightarrow{\mathrm{ct}(a)} p'$ and $p'\ R\ q'$. Therefore, $p \xrightarrow{a} p'$ in P with $a \in A^l$, and $p \xrightarrow{a}_{\diamond} p'$ in $\mathcal{M}(P)$ with $p'\ R\ q'$, as required. Now, if $p \xrightarrow{a}_{\Box} p'$ in $\mathcal{M}(P)$ then $p \xrightarrow{a} p'$ in P with $a \in A^r$ and $p \xrightarrow{\mathrm{cv}(a)} p'$ in $\overline{P}$. Since R is a covariant-contravariant simulation between $\overline{P}$ and $\mathcal{C}(Q)$, there is some q' in $\mathcal{C}(Q)$ with $q \xrightarrow{\mathrm{cv}(a)} q'$ and $p'\ R\ q'$, and therefore $q \xrightarrow{a}_{\Box} q'$ in Q, as required.

To see that the converse implication of the second statement in the proposition fails in general, let P be an LTS with $A^r = \{a\}$, with p as its only state and with no transitions. In this case A^l can be arbitrary and plays no role in the counter-example. Let Q be a one-state MTS with the transition $q \xrightarrow{a}_{\diamond} q$. Then we have $(\mathcal{M}(P), p) \sqsubseteq (Q, q)$. On the other hand, $(\overline{P}, p) \not\lesssim_{cc} (\mathcal{C}(Q), q)$, because $q \xrightarrow{\mathrm{ct}(a)} q$ in $\mathcal{C}(Q)$ and $\mathrm{ct}(a)$ is a contravariant action, whereas the LTS $\overline{P}$ has no transitions. □

6. Characteristic formulae for processes

In this section, we show that the translation $\mathcal{C}$ can be used to transfer characteristic formulae from the setting of MTSs modulo refinement to that of LTSs modulo the covariant-contravariant simulation preorder. For consistency with the developments in [6], we focus on characteristic formulae for finite, "essentially loop-free" structures. Following [6, 9], we consider two signatures: the first generates terms describing a family of MTSs, and the second generates terms denoting a family of LTSs.

Definition 12 ([6]). Given a set of actions A, the set $\mathcal{T}_M(A)$ of MTS process terms is given by

$$t ::= 0 \mid \omega \mid a.t \mid a!t \mid t + t.$$

where $a \in A$.

We define the 'universal MTS' associated with $\mathcal{T}_M(A)$ as follows:

- Its set of states is just $\mathcal{T}_M(A)$.
- For each term $a.t$ we have the transition $a.t \xrightarrow{a}_{\diamond} t$; besides, for each $a \in A$, we have $\omega \xrightarrow{a}_{\diamond} \omega$.
- For each term $a!t$, and $\circ \in \{\Box, \diamond\}$ we have $a!t \xrightarrow{a}_{\circ} t$.
- For each term $t_1 + t_2$, $a \in A$ and $\circ \in \{\Box, \diamond\}$ we have $t_1 + t_2 \xrightarrow{a}_{\circ} t'$, if and only if, we have $t_i \xrightarrow{a}_{\circ} t'$ for some $i \in \{1, 2\}$.

Note that ω denotes the MTS U from Example 1 and is the only source of loops in the MTS we have just described. So, abstracting from the self-loops at the leaves labelled with ω, terms in $\mathcal{T}_M(A)$ may be viewed as describing finite synchronization trees, in the sense of Milner.

Definition 13. Let $(A^r, A^l, \emptyset)$ be a signature and let $A = A^r \cup A^l$. The set $\mathcal{T}_L(A)$ of LTS process terms is given by

$$t ::= 0 \mid \omega \mid a.t \mid t + t,$$

where $a \in A$.

We define the 'universal LTS' associated with $\mathcal{T}_L(A)$ as follows:

- Its set of states is just $\mathcal{T}_L(A)$.
- For each term at we have the transition $at \xrightarrow{a} t$; besides, for each $a \in A^l$, we have $\omega \xrightarrow{a} \omega$.
- For each term $t_1 + t_2$ and each $a \in A$, we have $t_1 + t_2 \xrightarrow{a} t'$, if and only if, we have $t_i \xrightarrow{a} t'$ for some $i \in \{1, 2\}$.

The translation $\mathcal{C}$ from MTSs over the alphabet A to LTSs over the signature $(\{\mathrm{cv}(a) \mid a \in A\}, \{\mathrm{ct}(a) \mid a \in A\}, \emptyset)$ can be extended to terms in $\mathcal{T}_M(A)$ yielding terms in $\mathcal{T}_L(\{\mathrm{cv}(a), \mathrm{ct}(a) \mid a \in A\})$ as the unique homomorphism that is the identity over constants and satisfies the following equalities:

$$\begin{aligned}\mathcal{C}(a!t) &= \mathrm{cv}(a).\mathcal{C}(t) + \mathrm{ct}(a).\mathcal{C}(t) \quad \text{and} \\ \mathcal{C}(a.t) &= \mathrm{ct}(a).\mathcal{C}(t).\end{aligned}$$

Then we have the following results:

Lemma 13. *Let t be an MTS term. Then the following statements hold:*

1. *If $t \xrightarrow{a}_{\Box} t'$ for some MTS term t' then $\mathcal{C}(t) \xrightarrow{\mathrm{cv}(a)} \mathcal{C}(t')$.*
2. *If $t \xrightarrow{a}_{\Diamond} t'$ for some MTS term t' then $\mathcal{C}(t) \xrightarrow{\mathrm{ct}(a)} \mathcal{C}(t')$.*
3. *If $\mathcal{C}(t) \xrightarrow{\mathrm{cv}(a)} u$ for some LTS term u then $t \xrightarrow{a}_{\Box} t'$ for some MTS term t' such that $u = \mathcal{C}(t')$.*
4. *If $\mathcal{C}(t) \xrightarrow{\mathrm{ct}(a)} u$ for some LTS term u then $t \xrightarrow{a}_{\Diamond} t'$ for some MTS term t' such that $u = \mathcal{C}(t')$.*

PROOF. The first two statements can be proven by induction on the proof of the relevant transition. The third and the fourth statement can be easily shown by induction on the structure of t. ◻

It is not hard to see that the LTS associated with $\mathcal{C}(t)$, where t is an MTS term, is the LTS one obtains by considering the MTS for term t, defined as in Definition 12, and applying the translation $\mathcal{C}$ from Definition 9 to it. Therefore, the following result follows essentially from Proposition 9. (One can also give a simple proof of this result using Lemma 13 above.)

Proposition 14. *For an MTS term t and a modal formula φ,*

$$t \models \varphi \iff \mathcal{C}(t) \models \mathcal{C}(\varphi).$$

The above result can be used to transfer characteristic formulae for MTS terms modulo refinement to characteristic formulae for their image LTS terms via $\mathcal{C}$.

We begin by recalling the definition of characteristic formulae for MTS terms modulo refinement from [6, 16].

Definition 14 ([6, 16]). For each term $t \in \mathcal{T}_M(A)$, the characteristic formula $\chi(t)$ is defined as follows:

$$\chi(t) = \bigwedge_{\phi \in \delta(t)} \phi \wedge \bigwedge_{a \in A} [a]\gamma_a(t), \tag{1}$$

where the set of formulae $\delta(t)$ and the formulae $\gamma_a(t)$ are given inductively thus

1. $\delta(0) = \emptyset$ and $\gamma_a(0) = \bot$,
2. $\delta(\omega) = \emptyset$ and $\gamma_a(\omega) = \top$,
3. $\delta(a.t) = \emptyset$, $\gamma_a(a.t) = \gamma_a(t)$ and $\gamma_b(a.t) = \bot$ $(b \neq a)$,
4. $\delta(a!t) = \{\langle a \rangle \chi(t)\}$ and $\gamma_b(a!t) = \gamma_b(a.t)$, for each $b \in A$, and
5. $\delta(t_1 + t_2) = \delta(t_1) \cup \delta(t_2)$ and $\gamma_a(t_1 + t_2) = \gamma_a(t_1) \vee \gamma_a(t_2)$.

As usual, an empty conjunction stands for $\top$.

The correctness of the above construction was proved by Larsen in [16].

Proposition 15. *Let $t, t' \in \mathcal{T}_M(A)$. Then $t \sqsubseteq t'$ iff $t' \models \chi(t)$.*

Note that the formula $\chi(\omega)$ is logically equivalent to $\top$. Moreover, for each term $t \in \mathcal{T}_M(A)$, we have that, up to logical equivalence,

$$\bigwedge_{\phi \in \delta(t)} \phi = \bigwedge \{\langle a \rangle \chi(t') \mid t \xrightarrow{a}_{\Box} t'\}.$$

Consider now the second conjunction in the formula (1). If t can perform an a-labelled may transition leading to a term that is equivalent to ω with respect to the kernel of $\sqsubseteq$, then, up to logical equivalence,

$$[a]\gamma_a(t) = \top.$$

For each term t, let A_t be the subset of A consisting of all the actions a such that each a-labelled may transition from t leads to a term that is *not* equivalent to ω with respect to the kernel of $\sqsubseteq$. Then, up to logical equivalence,

$$\bigwedge_{a \in A} [a]\gamma_a(t) = \bigwedge_{a \in A_t} [a] \bigvee \{\chi(t') \mid t \xrightarrow{a}_{\Diamond} t'\}.$$

In summary, working up to logical equivalence, we can rewrite the formula (1) thus:

$$\bigwedge \{\langle a \rangle \chi(t') \mid t \xrightarrow{a}_{\Box} t'\} \vee \bigwedge_{a \in A_t} [a] \bigvee \{\chi(t') \mid t \xrightarrow{a}_{\Diamond} t'\}.$$

Proposition 16. *$\mathcal{C}(\chi(t))$ is a characteristic formula for $\mathcal{C}(t)$, for each $t \in \mathcal{T}_M(A)$.*

Proof. By Propositions 15 and 14, $\mathcal{C}(t) \models \mathcal{C}(\chi(t))$. Now, assume that $s \models \mathcal{C}(\chi(t))$ for some $s \in \mathcal{T}_L(\{\mathrm{cv}(a), ct(a) \mid a \in A\})$. We shall show that $\mathcal{C}(t) \lesssim_{cc} s$. (Observe, in passing, that, since the map $\mathcal{C}$ is not surjective, the term s might not be the image of any MTS term.) To this end, it suffices to show that the relation

$$R = \{(\mathcal{C}(t), s) \mid s \models \mathcal{C}(\chi(t)), s \in \mathcal{T}_L(\{\mathrm{cv}(a), ct(a) \mid a \in A\}), t \in \mathcal{T}_M(A)\}$$

is a covariant-contravariant simulation.

To see this, note, first of all, that, in the light of the above discussion,

$$\mathcal{C}(\chi(t)) = \bigwedge\{\langle \mathrm{cv}(a)\rangle \mathcal{C}(\chi(t')) \mid t \xrightarrow{a}_{\Box} t'\} \vee \bigwedge_{a \in A_t} [\mathrm{ct}(a)] \bigvee \{\mathcal{C}(\chi(t')) \mid t \xrightarrow{a}_{\Diamond} t'\}.$$

The claim can now be easily shown using Lemma 13 and the fact that $\mathcal{C}(\omega) = \omega \lesssim_{cc} s'$, for each $s' \in \mathcal{T}_L(\{\mathrm{cv}(a), ct(a) \mid a \in A\})$. □

This last result can be used as an alternative to [2, Lemma 2] to prove the existence of characteristic formulae for LTS terms that are in the range of $\mathcal{C}$. Indeed, for those terms, the characteristic formula derived using the above proposition coincides with the one offered by the direct construction given in the above-cited reference.

7. Partial bisimulation

The partial bisimulation preorder has been proposed in [4] as a suitable behavioural relation over LTSs for studying the theory of supervisory control [20] in a concurrency-theoretic framework. Formally, the notion of partial bisimulation is defined over LTSs with a set of actions A and a so-called *bisimulation set* $B \subseteq A$. The LTSs considered in [4] also include a termination predicate $\downarrow$ over states. For the sake of simplicity, and since its role is orthogonal to our aims in this paper, instead of extending MTSs and their refinements and/or covariant-contravariant simulations with such a predicate, we simply omit this predicate in what follows.

Definition 15. A *partial bisimulation with bisimulation set* B between two LTSs P and Q is a relation $R \subseteq P \times Q$ such that, whenever $p\ R\ q$:

- For all $a \in A$, if $p \xrightarrow{a} p'$ then there exists some $q \xrightarrow{a} q'$ with $p'\ R\ q'$.
- For all $b \in B$, if $q \xrightarrow{b} q'$ then there exists some $p \xrightarrow{b} p'$ with $p'\ R\ q'$.

We write $p \lesssim_B q$ if $p\ R\ q$ for some partial bisimulation with bisimulation set B.

It is easy to see that partial bisimulation with bisimulation set B is a particular case of covariant-contravariant simulation.

Proposition 17. *Let P be an LTS. A relation R is a partial bisimulation with bisimulation set B iff it is a covariant-contravariant simulation for the same LTS when it is seen as a covariant-contravariant LTS with signature $A^r = A \setminus B$, $A^l = \emptyset$ and $A^{bi} = B$. As a consequence we have $p \lesssim_B q$ iff $p \lesssim_{cc} q$, for each $p, q \in P$.*

Proof. Immediate from the definitions. □

Remark 6. Note that, in the light of the discussion in Section 5, after having changed the signature of the LTS P in the manner described in the statement of the above result, checking whether $p \lesssim_B q$ holds in P can always be reduced to verifying whether $p \lesssim_{cc} q$ holds in $\mathcal{C}(\mathcal{M}(P))$. This check does *not* involve any bivariant action.

As a corollary of the above proposition, we immediately obtain the following result, that indicates us that, instead of the modal logic used in [4] to characterize the partial bisimulation preorder with bisimulation set B, one can use the simpler, negation-free logic for the covariant-contravariant simulation preorder.

Corollary 18. *Let p, q be states in some image-finite LTS. Then $p \lesssim_B q$ iff the collection of formulae in Definition 5 over the signature $A^r = A \setminus B$, $A^l = \emptyset$ and $A^{bi} = B$ satisfied by p is included in the collection of formulae satisfied by q.*

Note also that, as a corollary of Proposition 17, the translations of LTSs and formulae defined in Section 3 can be applied to embed LTSs modulo the partial bisimulation preorder into modal transition systems modulo refinement. In this case, however, there is an easier alternative transformation that does not require the extra state u.

Definition 16. Let P be an LTS over a set of actions A with a bisimulation set $B \subseteq A$. Then the MTS $\mathcal{N}(P)$ is constructed as follows:

- The set of states is that of P.
- For each transition $p \xrightarrow{a} p'$ in P, we add a transition $p \xrightarrow{a}_\diamond p'$ in $\mathcal{N}(P)$.
- For each transition $p \xrightarrow{b} p'$ in P with $b \in B$, we add a transition $p \xrightarrow{b}_\Box p'$ in $\mathcal{N}(P)$.
- There are no other transitions in $\mathcal{N}(P)$.

Proposition 19. *R is a partial bisimulation with bisimulation set B between P and Q iff R^{-1} is a refinement between $\mathcal{N}(Q)$ and $\mathcal{N}(P)$.*

Proof. ($\Rightarrow$) Assume that R is a partial bisimulation with bisimulation set B and suppose that $q\ R^{-1}\ p$. If $p \xrightarrow{a}_\diamond p'$ in $\mathcal{N}(P)$ then $p \xrightarrow{a} p'$ in P. Since R is a partial bisimulation, there is some $q \xrightarrow{a} q'$ in Q with $p'\ R\ q'$ and, by construction, $q \xrightarrow{a}_\diamond q'$ in $\mathcal{N}(Q)$ with $q'\ R^{-1}\ p'$. Now, if $q \xrightarrow{a}_\Box q'$ in $\mathcal{N}(Q)$ then $q \xrightarrow{a} q'$ in Q with $a \in B$. Since R is a partial bisimulation and $p\ R\ q$, there is some $p \xrightarrow{a} p'$ in P with $p'\ R\ q'$ and hence $p \xrightarrow{a}_\Box p'$ in $\mathcal{N}(P)$, as required.

($\Leftarrow$) Analogous. □

Remark 7. In the special case $B = \emptyset$, the partial bisimulation preorder is just the standard simulation preorder. Therefore, for the LTS defined by the term 0 we have, $0 \lesssim_B p$ for each state p in any LTS P. Since $B = \emptyset$, all the modal transition systems $\mathcal{N}(P)$ that result from the translation of an LTS P will have no must transitions; for such modal transition systems, $\mathcal{N}(P) \sqsubseteq 0$ always holds. Indeed, in that case $\sqsubseteq$ coincides with the inverse of the simulation preorder over MTSs.

The drawback of the direct transformation presented in Definition 16, as compared to that in Section 3, is that it does not preserve the satisfiability of modal formulae. The problem lies in the fact that, while the existential modality $\langle a \rangle$ allows any transition with $a \in A$ in the partial bisimulation framework, it requires a must transition in the setting of MTSs.

As we have seen, it is easy to express partial bisimulations as a special case of covariant-contravariant simulations. It is therefore natural to wonder whether the converse also holds. We shall present some indications that the partial bisimulation framework is strictly less expressive than both modal refinements and covariant-contravariant simulations.

Let us assume, by way of example, that the set of actions A is partitioned into $A^r = \{a\}$ and $A^l = \{b\}$—so the set of bivariant actions is empty. In this setting, there cannot be a translation $\mathcal{T}$ from LTSs modulo $\lesssim_{cc}$ into LTSs modulo $\lesssim_B$ that satisfies the following natural conditions (by abuse of notation, we identify an LTS P with a specific state p):

1. For all p and q, $p \lesssim_{cc} q \iff \mathcal{T}(p) \lesssim_B \mathcal{T}(q)$.
2. $\mathcal{T}$ is a homomorphism with respect to $+$, that is, $\mathcal{T}(p+q) = \mathcal{T}(p) + T(q)$, where $+$ denotes the standard notion of nondeterministic composition of LTSs from CCS [18]. (Intuitively, this compositionality requirement states that the translation only uses 'local information'.)
3. There is an n such that $\mathcal{T}(b^n)$ is not simulation equivalent to $\mathcal{T}(0)$, where b^n denotes an LTS consisting of n consecutive b-labelled transitions.

Indeed, observe that, by condition 2,

$$\mathcal{T}(p) = \mathcal{T}(p+0) = \mathcal{T}(p) + \mathcal{T}(0) \quad \text{for each } p,$$

and therefore $\mathcal{T}(p) + \mathcal{T}(0) \lesssim_B \mathcal{T}(p)$. This means that $\mathcal{T}(0) \lesssim \mathcal{T}(p)$ for each p, where $\lesssim$ is the simulation preorder. In particular, $\mathcal{T}(0) \lesssim \mathcal{T}(\perp)$ where $\perp$ is the process consisting of a b-labelled loop with one state, which is the least element with respect to $\lesssim_{cc}$.

Note now that $\perp \lesssim_{cc} b^{n+1} \lesssim_{cc} b^n \lesssim_{cc} 0$ for each $n > 0$. Therefore, by condition 1,

$$\mathcal{T}(\perp) \lesssim_B \mathcal{T}(b^{n+1}) \lesssim_B \mathcal{T}(b^n) \lesssim_B \mathcal{T}(0) \quad \text{for each } n > 0.$$

Hence,

$$\mathcal{T}(\perp) \lesssim \mathcal{T}(b^n) \lesssim \mathcal{T}(0) \lesssim \mathcal{T}(\perp) \quad \text{for each } n > 0.$$

This yields that, for each $n > 0$, $\mathcal{T}(b^n)$ is simulation equivalent to $\mathcal{T}(0)$, which contradicts condition 3. (Note that we have only used the soundness of the transformation $\mathcal{T}$.)

This is clearly indicating that any $\mathcal{T}$ that is compositional with respect to $+$ and is sound, in the sense of condition 1, would have to be very odd indeed, if it exists at all. Modulo simulation equivalence, such a translation would have to conflate a non-well-founded descending chain of LTSs into a single point, modulo simulation equivalence.

We end this section with a companion result.

Proposition 20. *Assume that $a \in A^r$ and $b \in A^l$. Suppose furthermore that $B = \emptyset$. Then there is no translation $\mathcal{T}$ from LTSs modulo $\lesssim_{cc}$ into LTSs modulo $\lesssim_B$ that satisfies conditions 1 and 2 above.*

Proof. Assume, towards a contradiction, that $\mathcal{T}$ is a translation from LTSs modulo $\lesssim_{cc}$ into LTSs modulo $\lesssim_B$ that satisfies the conditions in the statement of the proposition. Recall that, when B is empty, $\lesssim_B$ is the simulation preorder (see Remark 7). Therefore, using condition 2, for each p and q, we have that

$$\mathcal{T}(p) \lesssim_B \mathcal{T}(p) + \mathcal{T}(q) = \mathcal{T}(p + q).$$

This means, in particular, that $\mathcal{T}(a) \lesssim_B \mathcal{T}(a + b)$. By condition 1, it follows that $a \lesssim_{cc} a + b$. This is, however, false since b is in A^l. Therefore $\mathcal{T}$ cannot exist. □

8. Institutions and institution morphisms

In order to gain more insight into the relationships between modal transition systems modulo refinement and labelled transition systems modulo the covariant-contravariant simulation preorder, we will now study their connections at a more abstract level, in the context of institutions [12]. When compared at the level of institutions it turns out that the correspondence between these models is, in a sense, not one-to-one.

Definition 17. The institution $\mathcal{I}_{cc} = (\mathbf{Sign}_{cc}, sen_{cc}, \mathbf{Mod}_{cc}, \models_{cc})$, associated with the logic for the covariant-contravariant simulation preorder, is defined as follows.

- $\mathbf{Sign}_{cc}$ has as objects triples (A, B, C) of pairwise disjoint sets and morphisms $f : A \cup B \cup C \longrightarrow A' \cup B' \cup C'$ with $f(A) \subseteq A'$, $f(B) \subseteq B'$, and $f(C) \subseteq C'$.
- $sen_{cc}(A, B, C)$ is the set of formulae in the logic characterizing the covariant-contravariant simulation preorder, with A the set of covariant actions, B the set of contravariant actions, and C the set of bivariant actions. For each signature morphism f and formula φ, the formula $sen(f)(\varphi)$ is obtained from φ by replacing each action a with $f(a)$.
- $\mathbf{Mod}_{cc}(A, B, C)$ is the category of LTSs over the set of actions $A \cup B \cup C$, with a distinguished state; a morphism from (P, p) to (Q, q) is a covariant-contravariant simulation R such that $(p, q) \in R$.

  Now, if $f : A \cup B \cup C \longrightarrow A' \cup B' \cup C'$ is a signature morphism, then

  $$\mathbf{Mod}_{cc}(f) : \mathbf{Mod}_{cc}(A', B', C') \longrightarrow \mathbf{Mod}_{cc}(A, B, C)$$

  maps P to $P|_f$ and $R : P \longrightarrow Q$ to $R_f : P|_f \longrightarrow Q|_f$, where:

  - The set of states of $P|_f$ is the same as that of P, and the distinguished state remains the same.

– $p \stackrel{a}{\to} p'$ in $P|_f$ if $p \stackrel{f(a)}{\to} p'$ in P.

– $R|_f$ coincides with R.

- $(P,s) \models_{cc} \varphi$ if $(P,s) \models \varphi$ using the notion of satisfaction associated with the logic for the covariant-contravariant simulation preorder given in Definition 5.

Proposition 21. *$\mathcal{I}_{cc}$ is an institution.*

Proof. It is a routine exercise to check that all defined notions are indeed categories and functors. As for the satisfaction condition, if $f : A \cup B \cup C \longrightarrow A' \cup B' \cup C'$ in $\mathbf{Sign}_{cc}$, $(P',s) \in \mathbf{Mod}_{cc}(A',B',C')$, and $\varphi \in sen_{cc}(A,B,C)$, then

$$(P',s) \models_{cc} sen_{cc}(f)(\varphi) \iff \mathbf{Mod}_{cc}(f)(P',s) \models_{cc} \varphi$$

can be proved by structural induction on φ. We consider the possible forms φ may have.

- $\top$ and $\bot$ are trivial.
- For $\varphi_1 \wedge \varphi_2$:

$$\begin{aligned} (P',s) \models_{cc} sen_{cc}(f)(\varphi_1 \wedge \varphi_2) &\iff (P',s) \models_{cc} sen_{cc}(f)(\varphi_1) \wedge sen_{cc}(f)(\varphi_2) \\ &\stackrel{\text{IH}}{\iff} (P'|_f,s) \models_{cc} \varphi_1 \text{ and } (P'|_f,s) \models_{cc} \varphi_2 \\ &\iff (P'|_f,s) \models_{cc} \varphi_1 \wedge \varphi_2. \end{aligned}$$

- Analogously for $\varphi_1 \vee \varphi_2$.
- For $\langle a\rangle\varphi$, with $a \in A \cup C$:

$$\begin{aligned} &(P',s) \models_{cc} sen_{cc}(f)(\langle a\rangle\varphi) \\ &\iff \quad (P',s) \models_{cc} \langle f(a)\rangle sen_{cc}(f)(\varphi) \\ &\iff \quad \text{there is } s \stackrel{f(a)}{\to} p \text{ in } P' \text{ with } (P',p) \models_{cc} sen_{cc}(f)(\varphi) \\ &\stackrel{\text{def } P'|_f,\ \text{IH}}{\iff} \quad \text{there is } s \stackrel{a}{\to} p \text{ in } P'|_f \text{ with } (P'|_f,p) \models_{cc} \varphi \\ &\iff \quad (P'|_f,s) \models_{cc} \langle a\rangle\varphi. \end{aligned}$$

- For $[a]\varphi$, with $a \in B \cup C$:

$$\begin{aligned} &(P',s) \models_{cc} sen_{cc}(f)([a]\varphi) \\ &\iff \quad (P',s) \models_{cc} [f(a)] sen_{cc}(f)(\varphi) \\ &\iff \quad (P',p) \models_{cc} sen_{cc}(f)(\varphi) \text{ for all } s \stackrel{f(a)}{\to} p \text{ in } P' \\ &\stackrel{\text{def } P'|_f,\ \text{IH}}{\iff} \quad (P'|_f,p) \models_{cc} \varphi \text{ for all } s \stackrel{a}{\to} p \text{ in } P'|_f \\ &\iff \quad (P'|_f,s) \models_{cc} [a]\varphi. \end{aligned}$$

This completes the proof. $\square$

Definition 18. The institution $\mathcal{I}_{mts} = (\mathbf{Sign}_{mts}, sen_{mts}, \mathbf{Mod}_{mts}, \models_{mts})$, associated with the logic for refinement over modal transition systems, is defined as follows.

- $\mathbf{Sign}_{mts}$ is the category of sets.
- $sen_{mts}(A)$ is the set of formulae over A in the logic presented in Definition 3. The formula $sen_{mts}(f)(\varphi)$ is obtained from φ by replacing each action a with $f(a)$.
- $\mathbf{Mod}_{mts}(A)$ is the category of MTSs over the set of labels A, with a distinguished state. A morphism from (M, m) to (N, n) is a refinement R such that $(m, n) \in R$.

  If $f : A \longrightarrow B$ in $\mathbf{Sign}_{mts}$, then $\mathbf{Mod}_{mts}(f) : \mathbf{Mod}_{mts}(B) \longrightarrow \mathbf{Mod}_{mts}(A)$ maps an MTS M to $M|_f$ and a morphism R to $R|_f$, where:

  - $M|_f$ has the same set of states as M and the same distinguished state.
  - $p \xrightarrow{a}_\diamond p'$ in $M|_f$ if $p \xrightarrow{f(a)}_\diamond p'$ in M.
  - $p \xrightarrow{a}_\Box p'$ in $M|_f$ if $p \xrightarrow{f(a)}_\Box p'$ in M.
  - $R|_f$ coincides with R.
- $\models_{mts}$ is the notion of satisfaction presented in Definition 3.

Proposition 22. *$\mathcal{I}_{mts}$ is an institution.*

Proof. Again, let us just prove the satisfaction condition

$$(M', s) \models_{mts} sen_{mts}(f)(\varphi) \iff \mathbf{Mod}_{mts}(f)(M', s) \models_{mts} \varphi,$$

for $f : A \longrightarrow B$ in $\mathbf{Sign}_{mts}$, $(M', s) \in \mathbf{Mod}_{mts}(B)$, and $\varphi \in sen_{mts}(A)$, by induction on φ. We consider the possible forms φ may have.

- $\top$ and $\bot$ are trivial.
- For $\varphi_1 \wedge \varphi_2$:

$$\begin{aligned}
&(M', s) \models_{mts} sen_{mts}(f)(\varphi_1 \wedge \varphi_2) \\
&\quad\iff (M', s) \models_{mts} sen_{mts}(f)(\varphi_1) \wedge sen_{mts}(f)(\varphi_2) \\
&\quad\overset{\text{IH}}{\iff} (M'|_f, s) \models_{mts} \varphi_1 \text{ and } (M'|_f, s) \models_{mts} \varphi_2 \\
&\quad\iff (M'|_f, s) \models_{mts} \varphi_1 \wedge \varphi_2.
\end{aligned}$$

- Analogously for $\varphi_1 \vee \varphi_2$.

- For $\langle a\rangle\varphi$:

$$
\begin{array}{ll}
(M',s) \models_{mts} sen_{mts}(f)(\langle a\rangle\varphi) & \\
\iff & (M',s) \models_{mts} \langle f(a)\rangle sen_{mts}(f)(\varphi) \\
\iff & \text{there is } s \xrightarrow{f(a)}_{\square} p \text{ in } M' \text{ with } (M',p) \models_{mts} sen_{mts}(f)(\varphi) \\
\overset{\text{def } M'|_f,\ \text{IH}}{\iff} & \text{there is } s \xrightarrow{a}_{\square} p \text{ in } M'|_f \text{ with } (M'|_f,p) \models_{mts} \varphi \\
\iff & (M'|_f,s) \models_{mts} \langle a\rangle\varphi.
\end{array}
$$

- For $[a]\varphi$:

$$
\begin{array}{ll}
(M',s) \models_{mts} sen_{mts}(f)([a]\varphi) & \\
\iff & (M',s) \models_{mts} [f(a)]sen_{mts}(f)(\varphi) \\
\iff & (M',p) \models_{mts} sen_{mts}(f)(\varphi) \text{ for all } s \xrightarrow{f(a)}_{\diamond} p \text{ in } M' \\
\overset{\text{def } M'|_f,\ \text{IH}}{\iff} & (M'|_f,p) \models_{mts} \varphi \text{ for all } s \xrightarrow{a}_{\diamond} p \text{ in } M'|_f \\
\iff & (M'|_f,s) \models_{mts} [a]\varphi.
\end{array}
$$

□

As the following result shows, one can translate $\mathcal{I}_{mts}$ into $\mathcal{I}_{cc}$ using an institution morphism [13]. (The intuition for institution morphisms is that they are truth preserving translations from one logical system into another.)

Proposition 23. $(\Phi, \alpha, \beta) : \mathcal{I}_{mts} \longrightarrow \mathcal{I}_{cc}$ *is an institution morphism, where:*

- $\Phi : \mathbf{Sign}_{mts} \longrightarrow \mathbf{Sign}_{cc}$ *maps* A *to the triple* $(\mathrm{cv}(A), \mathrm{ct}(A), \emptyset)$*, with:*
  - $\mathrm{cv}(A) = \{\mathrm{cv}(a) \mid a \in A\}$ *and*
  - $\mathrm{ct}(A) = \{\mathrm{ct}(a) \mid a \in A\}$.

  For $f : A \longrightarrow B$*, we define* $\Phi(f)(\mathrm{cv}(a)) = \mathrm{cv}(f(a))$ *and* $\Phi(f)(\mathrm{ct}(a)) = \mathrm{ct}(f(a))$.
- *The natural transformation* $\alpha : sen_{cc} \circ \Phi \Rightarrow sen_{mts}$ *translates a formula* φ *in* $sen_{cc}(\mathrm{cv}(A), \mathrm{ct}(A), \emptyset)$ *as follows:*
  - $\alpha(\top) = \top$, $\alpha(\bot) = \bot$.
  - $\alpha(\varphi_1 \wedge \varphi_2) = \alpha(\varphi_1) \wedge \alpha(\varphi_2)$.
  - $\alpha(\varphi_1 \vee \varphi_2) = \alpha(\varphi_1) \vee \alpha(\varphi_2)$.
  - $\alpha(\langle \mathrm{cv}(a)\rangle\varphi) = \langle a\rangle\alpha(\varphi)$.
  - $\alpha([\mathrm{ct}(a)]\varphi) = [a]\alpha(\varphi)$.
- *The natural transformation* $\beta : \mathbf{Mod}_{mts} \Rightarrow \mathbf{Mod}_{cc} \circ \Phi$ *maps an MTS* (M, s) *in* $\mathbf{Mod}_{mts}(A)$ *to* $(\mathcal{C}(M), s)$*, and a morphism* R *to itself.*

PROOF. For A in $\mathbf{Sign}_{mts}$, (M, s) in $\mathbf{Mod}_{mts}(A)$, and φ in $sen_{cc}(\Phi(A))$, we prove the satisfaction condition

$$(M, s) \models_{mts} \alpha(\varphi) \iff \beta(M, s) \models_{cc} \varphi$$

by induction on φ. The only non-trivial cases correspond to formulae of the form $\langle \mathrm{cv}(a) \rangle \varphi$ and $[\mathrm{ct}(a)]\varphi$.

- For $\langle \mathrm{cv}(a) \rangle \varphi$, we reason thus:

$$\begin{aligned}
(M, s) \models \alpha(\langle \mathrm{cv}(a) \rangle \varphi) &\iff (M, s) \models \langle a \rangle \alpha(\varphi) \\
&\iff \text{there is } s \xrightarrow{a}_{\Box} p \text{ in } M \text{ with } (M, p) \models \alpha(\varphi) \\
&\overset{\mathrm{IH}}{\iff} \text{there is } s \xrightarrow{\mathrm{cv}(a)} p \text{ in } \mathcal{C}(M) \text{ with } (\mathcal{C}(M), p) \models \varphi \\
&\iff (\mathcal{C}(M), s) \models \langle \mathrm{cv}(a) \rangle \varphi.
\end{aligned}$$

- For $[\mathrm{ct}(a)]\varphi$, we argue as follows:

$$\begin{aligned}
(M, s) \models \alpha([\mathrm{ct}(a)]\varphi) &\iff (M, s) \models [a]\alpha(\varphi) \\
&\iff (M, p) \models \alpha(\varphi) \text{ for all } s \xrightarrow{a}_{\Diamond} p \text{ in } M \\
&\overset{\mathrm{IH}}{\iff} (\mathcal{C}(M), p) \models \varphi \text{ for all } s \xrightarrow{\mathrm{ct}(a)} p \text{ in } \mathcal{C}(M) \\
&\iff (\mathcal{C}(M), s) \models [\mathrm{ct}(a)]\varphi.
\end{aligned}$$

This completes the proof. □

The import of the above result is that MTSs modulo refinement and their accompanying modal logic can be 'translated in a truth preserving fashion' into LTSs modulo the covariant-contravariant simulation preorder and their companion modal logic. It is natural to ask oneself whether one can consider $\mathcal{I}_{mts}$ a 'subinstitution' of $\mathcal{I}_{cc}$. There are several related notions of subinstitution that have in common the requirement that the functor β, which is used to translate the models between the institutions, is an equivalence of categories.

Recall that an object in a category is *weakly final* if any other object has at least one arrow into it.

Proposition 24. $\mathbf{Mod}_{cc}(A, B, \emptyset)$ *has weakly final objects but* $\mathbf{Mod}_{mts}(A)$ *does not.*

PROOF. First, consider the pair (F, s) where F is the LTS with a single state s and transitions $s \xrightarrow{a} s$ for every $a \in A$. (Note that, if A is empty, then (F, s) is just the LTS 0.) It is immediate to check that (F, s) is a weakly final object of $\mathbf{Mod}_{cc}(A, B, \emptyset)$.

Now, assume that (F', s') is weakly final in $\mathbf{Mod}_{mts}(A)$ and consider the following two MTSs:

- (M, m), with m the only state in M and transitions $m \xrightarrow{a}_{\Box} m$ (and $m \xrightarrow{a}_{\Diamond} m$) for every $a \in A$.

- (N, n), with n the only state in N and no transitions.

The existence of a morphism, that is a refinement, from (M, m) to (F', s') implies that, for every $a \in A$, there must be transitions of the form $s' \xrightarrow{a}_{\Box} s'_a$ in F' for some s'_a; therefore, there are also transitions $s \xrightarrow{a}_{\Diamond} s'_a$. But then, the morphism from (N, n) to (F', s') requires the existence of transitions $n \xrightarrow{a}_{\Diamond} n$ in N, which do not exist by the definition of N. Hence, there is no weakly final object in $\mathbf{Mod}_{mts}(A)$. $\Box$

In other words, in the absence of bivariant actions, there is a universal implementation in the setting of LTSs modulo the covariant-contravariant simulation preorder. Within that framework, there is also a universal specification, namely the LTS (I, s) where I is the LTS with a single state s and transitions $s \xrightarrow{b} s$ for every $b \in B$. On the other hand, there is a universal specification with respect to modal refinements, namely the MTS U from Example 1, but no universal implementation.

Proposition 25. *There cannot exist an embedding (Φ, α, β) from $\mathcal{I}_{mts}$ into $\mathcal{I}_{cc}$ such that $\Phi(A)$ does not have bivariant actions for some A.*

Proof. If such an embedding existed then β_A, which is the natural transformation translating MTSs into LTSs and refinement relations into covariant-contravariant simulations, would be an equivalence between $\mathbf{Mod}_{mts}(A)$ and $\mathbf{Mod}_{cc}(\Phi(A))$. Since equivalences of categories preserve weakly final objects, the result follows from Proposition 24. $\Box$

We will now argue that $\mathcal{I}_{mts}$ cannot be embedded into $\mathcal{I}_{cc}$ even in the presence of bivariant actions. Recall that an object in a category is *weakly initial* if there is at least one arrow from it into any other object.

Proposition 26. *$\mathbf{Mod}_{mts}(A)$ has weakly initial objects but $\mathbf{Mod}_{cc}(A, B, C)$ does not if $C \neq \emptyset$.*

Proof. Consider the MTS (I, s) defined by $s \xrightarrow{a}_{\Diamond} s$ for all $a \in A$. We have already seen that it is weakly initial.

Now, assume that (I', s') is weakly initial in $\mathbf{Mod}_{cc}(A, B, C)$ and let $c \in C$. We define the following LTSs:

- (P, p) with $p \xrightarrow{c} p$, and
- (Q, q) with a single state q and no transitions.

A morphism from (I', s') to (P, p) requires a transition $s' \xrightarrow{c} s''$ in I' for some s''. But then, a morphism from (I', s') to (Q, q) requires a transition $q \xrightarrow{c} q$, which does not exist by definition. Therefore, (I', s') cannot exist. $\Box$

Proposition 27. *There cannot exist an embedding (Φ, α, β) from $\mathcal{I}_{mts}$ into $\mathcal{I}_{cc}$ such that $\Phi(A)$ has bivariant actions for some A.*

PROOF. Such an embedding β_A, if it existed, would be an equivalence of categories between $\mathbf{Mod}_{mts}(A)$ and $\mathbf{Mod}_{cc}(\Phi(A))$. This cannot hold by Proposition 26 because equivalences of categories preserve weakly initial objects. $\square$

A natural question to ask is whether there is an embedding from $\mathcal{I}_{cc}$ into $\mathcal{I}_{mts}$. The following proposition answers this question negatively.

Proposition 28. *There exists no embedding from $\mathcal{I}_{cc}$ into $\mathcal{I}_{mts}$.*

PROOF. If such an embedding (Φ, α, β) existed, $\beta_{(A,B,\emptyset)}$ would be an equivalence between $\mathbf{Mod}_{cc}(A, B, \emptyset)$ and $\mathbf{Mod}_{mts}(\Phi(A, B, \emptyset))$, which is not possible by Proposition 24 because equivalences preserve weakly final objects. $\square$

In [1] we conjectured that there is not even an institution morphism from $\mathcal{I}_{cc}$ to $\mathcal{I}_{mts}$; we now make this claim precise.

If we are not concerned about how contrived this morphism can be, then a trivial one can indeed be defined. Let Φ map any signature to the singleton set $\{1\}$, β map any LTS to a MTS with a single state s and transitions $s \xrightarrow{1}_\diamond s$ and $s \xrightarrow{1}_\Box s$, and α be recursively defined by $\alpha([1]\varphi) = \alpha(\varphi)$, $\alpha(\langle 1\rangle\varphi) = \alpha(\varphi)$, and as expected in the remaining cases. It is then a simple exercise to check that (Φ, α, β) satisfies the conditions to be an institution morphism, however trivial and artificial it may be.

Taking Proposition 23 as a model, and recalling the good properties of the function $\mathcal{M}$ studied in Section 3, a "natural" morphism from $\mathcal{I}_{cc}$ to $\mathcal{I}_{mts}$ would be expected to satisfy $\beta(M, s) = (\mathcal{M}(M), s)$. We now argue that such a morphism cannot exist.

Assume that $(A, B, C) \in \mathbf{Sign}_{cc}$, let $a \in A$ be any covariant action, and $[a]\bot$ a Boudol-Larsen modal formula: how should $\alpha([a]\bot)$ be defined? By the requirements of institution morphisms, the following equivalence must hold for all LTS M:

$$(M, s) \models_{cc} \alpha([a]\bot) \iff \beta(M, s) \models_{mts} [a]\bot.$$

The right-hand side is true iff $(\mathcal{M}(M), s') \models_{mts} \bot$ for all $s \xrightarrow{a}_\diamond s'$ in $\mathcal{M}(M)$ which, by construction, only holds if there is no s' in M with $s \xrightarrow{a} s'$ in M. Therefore, $\alpha([a]\bot)$ has to be such that:

- $(M, s) \models_{cc} \alpha([a]\bot)$ if there is no $s \xrightarrow{a} s'$ in M, but
- $(M, s) \not\models_{cc} \alpha([a]\bot)$ if there is $s \xrightarrow{a} s'$ in M.

The immediate candidate would be $[a]\bot$ itself, now considered as a covariant-contravariant modal formula, but this is not possible since in this framework the modality $[\_]$ requires a contravariant action. Actually, no such formula can be defined which means that no institution morphism with $\beta(M, s) = (\mathcal{M}(M), s)$ can exist.

Note also that, although not necessary for our negative results above, we could have associated a more general institution $\mathcal{I}'_{cc}$ to the logic for the covariant-contravariant simulation preorder as follows:

- $\mathbf{Sign}'_{cc}$ has as objects triples (A, B, C) of pairwise disjoint sets and morphisms are relations $R \subseteq (A \times A') \cup (B \times B') \cup (C \times C')$.

- $sen'_{cc}(A, B, C)$ is the set of formulae in the logic characterizing the covariant-contravariant simulation preorder, with A the set of covariant actions, B the set of contravariant actions, and C the set of bivariant actions. For each morphism R and formula φ, the formula $sen'_{cc}(R)(\varphi)$ is obtained from φ by "replacing" each action a with every a' such that aRa'. More precisely, $sen'_{cc}(R)(\varphi)$ is defined recursively so that $\langle a\rangle\varphi'$ becomes $\bigvee_{aRa'}\langle a'\rangle sen'_{cc}(R)(\varphi')$ and $[b]\varphi'$ becomes $\bigwedge_{bRb'}[b']sen'_{cc}(R)(\varphi')$.

- $\mathbf{Mod}'_{cc}(A, B, C)$ is the category of LTSs over the set of actions $A \cup B \cup C$, with a distinguished state; a morphism from (P, p) to (Q, q) is a covariant-contravariant simulation S such that $(p, q) \in S$.

  Now, if $R : (A, B, C) \longrightarrow (A', B', C')$ is a $\mathbf{Sign}'_{cc}$-signature morphism, then

  $$\mathbf{Mod}'_{cc}(R) : \mathbf{Mod}'_{cc}(A', B', C') \longrightarrow \mathbf{Mod}'_{cc}(A, B, C)$$

  maps P to $R(P)$ and a simulation $S : P \longrightarrow Q$ to $R(S) : R(P) \longrightarrow R(Q)$, where:

  – The set of states of $R(P)$ is the same as that of P, and the distinguished state remains the same.

  – $p \xrightarrow{a} p'$ in $R(P)$ if aRa' and $p \xrightarrow{a'} p'$ in P.

  – $R(S)$ coincides with S.

- $(P, s) \models'_{cc} \varphi$ if $(P, s) \models \varphi$ using the notion of satisfaction associated with the logic for the covariant-contravariant simulation preorder given in Definition 5.

That is, signature morphisms become arbitrary relations that "preserve" the modality of the actions.

Obviously, the institution $\mathcal{I}_{mts}$ could be subjected to an analogous generalization; then, it would be a simple exercise to translate to this new setting the results proved in Propositions 21–28.

9. Conclusions and future work

In this paper we have studied the relationships between three notions of behavioural preorders that have been proposed in the literature: refinement over modal transition systems, and the covariant-contravariant simulation and the partial bisimulation preorders over labelled transition systems. We have provided mutual translations between modal transition systems and labelled transition systems that preserve, and reflect, refinement and the covariant-contravariant simulation preorder, as well as the the modal properties that can be expressed in the logics that characterize those preorders. We have also offered a translation from labelled transition systems modulo the partial bisimulation preorder into the same model modulo the covariant-contravariant simulation preorder, together with some evidence that the former model is less expressive than the latter. Finally, in order to gain more insight into the relationships between modal transition systems modulo refinement and labelled transition systems modulo the covariant-contravariant simulation preorder, we have also phrased and studied their connections in the context of institutions.

The work presented in the study opens several interesting avenues for future research. Here we limit ourselves to mentioning a few research directions that we plan to pursue in future work.

First of all, it would be interesting to study the relationships between the LTS-based models we have considered in this article and variations on the MTS model surveyed in, for instance, [3]. In particular, the third author recently contributed in [10] to the comparison of several refinement settings, including modal and mixed transition systems. The developments in that paper offer a different approach to the comparison and application of the formalisms studied in this article.

In [9], three of the authors gave a ground-complete axiomatization of the covariant-contravariant simulation preorder over the language BCCS [18]. It would be interesting to see whether the translations between MTSs and LTSs we have provided in this paper can be used to lift that axiomatization result, as well as results on the nonexistence of finite (in)equational axiomatizations, to the setting of modal transition systems modulo refinement, using the BCCS-like syntax for MTSs given in [6] and used in Section 6 of this paper. We also intend to study whether our translations can be used to obtain characteristic-formula constructions [6, 14, 21] for one model from extant results on the existence of characteristic formulae for the other. In the setting of the finite LTSs that are the image of MTS terms via $\mathcal{C}$, this has been achieved in Section 6 of this study.

The existence of characteristic formulae allows one to reduce checking the existence of a behavioural relation between two processes to a model checking question. Conversely, the main result from [6] offers a complete characterization of the model checking questions of the form $(M, m) \models \varphi$, where M is an MTS and φ is a formula in the logic for MTSs considered in this paper, that can be reduced to checking for the existence of a refinement between (M_φ, m_φ) and (M, m), where (M_φ, m_φ) is an MTS with a distinguished state that 'graphically represents' the formula φ. In [2], we offered a characterization of the logical specifications that can be 'graphically represented' by LTSs modulo the covariant-contravariant simulation preorder and partial bisimilarity. This result applies directly to LTSs whose signature contains no bivariant actions. Such a characterization may shed further light on the relative expressive power of the two formalisms and may give further evidence of the fact that LTSs modulo the covariant-contravariant simulation preorder are, in some suitable formal sense, more expressive than LTSs modulo partial bisimilarity.

Last, but not least, the development of the notion of partial bisimulation in [4, 5] has been motivated by the desire to develop a process-algebraic model within which one can study topics in the field of *supervisory control* [20]. Recently, MTSs have been used as a suitable model for the specification of service-oriented applications, and results on the supervisory control of systems whose specification is given in that formalism have been presented in, e.g., [7, 11]. It is a very interesting area for future research to study whether the mutual translations between MTSs modulo refinement and LTSs modulo the covariant-contravariant simulation preorder can be used to transfer results on supervisory control from MTSs to LTSs. One may also wish to investigate directly the adaptation of the supervisory control theory of Ramadge and Wonham to the enforcement of specifications given in terms of LTSs modulo the covariant-contravariant simulation preorder.

References

[1] L. Aceto, I. Fábregas, D. de Frutos Escrig, A. Ingólfsdóttir, and M. Palomino. Relating modal refinements, covariant-contravariant simulations and partial bisimulations. In F. Arbab and M. Sirjani, editors, *Fundamentals of Software Engineering, FSEN 2011*, Lecture Notes in Computer Science. Springer, 2011. To appear.

[2] L. Aceto, I. Fábregas, D. de Frutos Escrig, A. Ingólfsdóttir, and M. Palomino. Graphical representation of covariant-contravariant modal formulae. In B. Luttik and F. Valencia, editors, *Proceedings of EXPRESS 2011, the 18th International Workshop on Expressiveness in Concurrency*, volume 64 of Electronic Proceedings in Theoretical Computer Science, pages 1–15, 2011.

[3] A. Antonik, M. Huth, K. G. Larsen, U. Nyman, and A. Wąsowski. 20 years of modal and mixed specifications. *Bull. Eur. Assoc. Theor. Comput. Sci. EATCS*, 95:94–129, 2008.

[4] J. Baeten, D. van Beek, B. Luttik, J. Markovski, and J. Rooda. Partial bisimulation. SE Report 2010-04, Systems Engineering Group, Department of Mechanical Engineering, Eindhoven University of Technology, 2010.

[5] J. Baeten, D. van Beek, A. van Hulst, and J. Markovski. A process algebra for supervisory coordination. In L. Aceto and M. Mousavi, editors, *Proceedings First International Workshop on Process Algebra and Coordination, PACO 2011*, volume 60 of Electronic Proceedings in Theoretical Computer Science, pages 36–55, 2011.

[6] G. Boudol and K. G. Larsen. Graphical versus logical specifications. *Theoretical Comput. Sci.*, 106(1):3–20, 1992.

[7] P. Darondeau, J. Dubreil, and H. Marchand. Supervisory control for modal specifications of services. In *Proceedings of WODES 2010, August 30–September 1, 2010, Berlin, Germany*, 2010. To appear.

[8] I. Fábregas, D. de Frutos-Escrig, and M. Palomino. Non-strongly stable orders also define interesting simulation relations. In A. Kurz, M. Lenisa,

and A. Tarlecki, editors, *Algebra and Coalgebra in Computer Science, Third conference CALCO 2009, Udine, Italy, September 7-10, 2009. Proceedings*, volume 5728 of *Lecture Notes in Computer Science*, pages 221–235. Springer-Verlag, 2009.

[9] I. Fábregas, D. de Frutos-Escrig, and M. Palomino. Logics for contravariant simulations. In J. Hatcliff and E. Zucca, editors, *Formal Techniques for Distributed Systems, Joint 12th IFIP WG 6.1 International Conference, FMOODS 2010 and 30th IFIP WG 6.1 International Conference, FORTE 2010, Amsterdam, The Netherlands, June 7-9, 2010. Proceedings*, volume 6117 of *Lecture Notes in Computer Science*, pages 224–231. Springer-Verlag, 2010.

[10] H. Fecher, D. de Frutos-Escrig, G. Lüttgen, and H. Schmidt. On the expressiveness of refinement settings. In F. Arbab and M. Sirjani, editors, *Fundamentals of Software Engineering, Third IPM International Conference, FSEN 2009, Kish Island, Iran, April 15-17, 2009, Revised Selected Papers*, volume 5961 of *Lecture Notes in Computer Science*, pages 276–291. Springer-Verlag, 2009.

[11] G. Feuillade and S. Pinchinat. Modal specifications for the control theory of discrete event systems. *Discrete Event Dynamical Systems*, 17:211–232, 2007.

[12] J. A. Goguen and R. M. Burstall. Institutions: Abstract model theory for specification and programming. *J. ACM*, 39(1):95–146, 1992.

[13] J. A. Goguen and G. Rosu, Institution morphisms. *Formal Aspects of Computing* 13:274–307, 2002.

[14] S. Graf and J. Sifakis. A modal characterization of observational congruence on finite terms of CCS. *Information and Control*, 68(1–3):125–145, Jan./Feb./Mar. 1986.

[15] J. Hughes and B. Jacobs. Simulations in coalgebra. *Theoretical Comput. Sci.*, 327(1–2):71–108, 2004.

[16] K. G. Larsen. Modal specifications. In J. Sifakis, editor, *Automatic Verification Methods for Finite State Systems*, volume 407 of *Lecture Notes in Computer Science*, pages 232–246. Springer-Verlag, 1989.

[17] K. G. Larsen and B. Thomsen. A modal process logic. In *Proceedings* 3^{th} *Annual Symposium on Logic in Computer Science,* Edinburgh, pages 203–210. IEEE Computer Society Press, 1988.

[18] R. Milner. *Communication and Concurrency.* Prentice-Hall International, Englewood Cliffs, 1989.

[19] D. Park. Concurrency and automata on infinite sequences. In P. Deussen, editor, *5th GI Conference,* Karlsruhe, Germany, volume 104 of *Lecture Notes in Computer Science*, pages 167–183. Springer-Verlag, 1981.

[20] P. Ramadge and W. Wonham. Supervisory control of a class of discrete event processes. *SIAM Journal of Control and Optimization*, 25:206–230, 1987.

[21] B. Steffen and A. Ingólfsdóttir. Characteristic formulae for processes with divergence. *Information and Computation*, 110(1):149–163, 1994.