It is not possible to have a true understanding of a programming language without a mental model of its semantics, i.e., “how the language works”. (Matthew Hennessy)
In Computer Science, we use formal languages to communicate with machines (programming languages) and describe expected properties of computations (specification languages).

Like natural languages, the languages we use have

1. a syntax and
2. a semantics.

Question
How are those described in CS?
Fact of (Computer Science) Life

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Syntax

Formally specified using, e.g., BNF notation. Example?

**CCS**

- nil 0
- prefixing \( at \)
- choice \( t + u \)
- parallel \( t \parallel u \)

where \( a \) is an action drawn from a non-empty, finite set \( A \).

Benefits: Too many to mention! For instance, compiler technology was revolutionized and went from art to science, in the sense of Knuth.

State of Play (Syntax)

The syntax of every language under the sky is formally specified. Uncontroversial!
Operational Semantics: What the program does

Meaning of a program ≈ Execution on an idealized machine. How is this specified?

Plotkin’s answer: Use logic! Define the semantics by using inference rules.

Operational Semantics for CCS

Given by transitions between terms of the form \( t \xrightarrow{a} u \). These associate a loop-free finite automaton with each term. How?

\[
\begin{align*}
ax \xrightarrow{a} x \\
\begin{array}{c}
\frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'} \\
\frac{x \xrightarrow{a} x'}{x || y \xrightarrow{a} x'||y} \\
\frac{x \xrightarrow{a} x', y \xrightarrow{a} y'}{x || y \xrightarrow{\tau} x'||y'}
\end{array}
\end{align*}
\]
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\frac{}{ax \xrightarrow{a} x} & \quad \quad \frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'} & \quad \quad \frac{x \xrightarrow{a} x'}{x||y \xrightarrow{a} x'||y} & \quad \quad \frac{x \xrightarrow{a} x', y \xrightarrow{\bar{a}} y'}{x||y \xrightarrow{\tau} x'||y'}
\end{align*}
\]
The Role of Equalities Between Programs

**Tenet:** Designers of languages often have expected algebraic properties of language constructs in mind when defining a language. Examples?

\[
\begin{align*}
\text{x; skip} & = x & \text{x + 0} & = x \\
\text{x + y} & = y + x & (\text{x + y}) + z & = \text{x + (y + z)} \\
\text{deadlock}; x & = \text{deadlock} & \text{x + x} & = x
\end{align*}
\]

How can we ensure the validity of such laws of programming?

1. **A posteriori verification:** Give the semantics and then use it to prove that the laws are valid in the semantic model.

2. “Intelligent design”: Give syntactic templates for the inference rules used in defining the operational semantics for certain operators that guarantee the validity of the laws by design!
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    x; \text{skip} &= x \\
    x + y &= y + x \\
    (x + y) + z &= x + (y + z) \\
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    x + x &= x
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2. **“Intelligent design”:** Give syntactic templates for the inference rules used in defining the operational semantics for certain operators that guarantee the validity of the laws by design!
Rule formats pave the way for a tool-set that can mechanically prove algebraic properties without involving user interaction.

Rule formats may serve as a guideline for language designers who want to ensure, a priori, that the constructs under design enjoy certain basic algebraic properties.

Trade-off: Generality vs. ease of application. Logic is an experimental science!

This Talk: A Rule Format for Unit Elements

A rule format guaranteeing that certain constants are left- or right-unit elements for a set of binary operators.

Technical Question: How can we guarantee the validity of equations like

\[ f(c, x) = x \quad \text{and} \quad f(x, c) = x? \]
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Intuition

Assumption: For each constant $c$, we assume that each $c$-defining inference rule is an axiom of the form $c \xrightarrow{l} p$ for some label $l$ and term $p$ without variables.

Question: When is $c$ a left unit for a binary operation $f$?

Wish List

1. **Desideratum 1**: $f(c, p)$ should be able to mimic the behaviour of $p$ for each program $p$. argument.
2. **Desideratum 2**: $f(c, p)$ can only mimic the behaviour of $p$.

How do we ensure those properties syntactically in a way that allows us to handle examples from the literature (and more)?
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**Wish List**

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How do we ensure those properties syntactically in a way that allows us to handle examples from the literature (and more)?
Example 1

Consider the binary operators $f_i$, $i \geq 0$, with rules

$$x_1 \xrightarrow{a} y_1$$

$$f_i(x_0, x_1) \xrightarrow{a} f_{i+1}(x_0, y_1).$$

Any program is a left unit for each $f_i$. Why?

**Lesson 1:** We may need to consider possibly infinite sets of operators. Such sets cannot be defined inductively.
Example 2

Consider the following operations;

\[
\begin{align*}
  y \xrightarrow{a} y' \\
  x \xrightarrow{a} x' \\
  f(x, y) \xrightarrow{a} g(y', x) \\
  g(x, y) \xrightarrow{a} f(y, x')
\end{align*}
\]

Any program is a left unit for \( f \) and a right unit for \( g \). Why?

**Lesson 2:** We may need to consider left and right units at the same time.
Given a “language specification”, the sets $L$ and $R$ of pairs of binary function symbols and constants are the largest sets satisfying the following constraints.

Constraint 1. $L$ (Implements: $f(c, p)$ should be able to mimic the behaviour of $p$ for each program $p$.) For each $(f, c) \in L$ and each action label $a$, there exists at least one rule of the following form:

$$\{ x_0 \xrightarrow{a_i} y_i \mid i \in I \} \cup \{ x_0 \xrightarrow{a_j} \mid j \in J \} \cup \{ x_1 \xrightarrow{a} y_1 \} \cup \{ x_1 \xrightarrow{a} t' \},$$

where (1) the variables are all pairwise distinct, (2) the axioms for $c$ “satisfy the premises involving $x_0$” and (3) “$y_1$ can be proved equal to a suitable instantiation of $t'$” using the laws $f(e, x) = x$ when $(f, e) \in L$ and $g(x, e') = x$ when $(g, e') \in R$. 
Constraint 2. Let (Implements: \( f(c, p) \) can only mimic the behaviour of \( p \).) Each \( f \)-defining rule has the following form:

\[
\Phi
\]

\[
f(t_0, t_1) \xrightarrow{a} t'
\]

where, for each closed substitution \( \sigma \) such that \( \sigma(t_0) \equiv c \),

1. either there exists some \( t_1 \xrightarrow{a} t'' \in \Phi \) with \( \sigma(t') \) “behaving like” \( \sigma(t'') \), or

2. there exists a premise \( \phi \in \Phi \) with \( t_0 \) as its source that cannot be met by \( c \).

The constraints for \( R \) are symmetric.
Consider a language with “operational semantics given by inference rules”. Assume that $L$ and $R$ are the sets defined as given previously. For each $(f, c) \in L$, the equation $f(c, x) = x$ is valid “up to any reasonable notion of equivalence over programs”. Symmetrically, for each $(f, c) \in R$, the equation $f(x, c) = x$ is valid “up to any reasonable notion of equivalence over programs”.

The proof relies on the construction of suitable bisimulations.

Cool, but can it be (easily) applied to examples from the literature?
Main Theorem
Consider a language with “operational semantics given by inference rules”. Assume that $L$ and $R$ are the sets defined as given previously. For each $(f, c) \in L$, the equation $f(c, x) = x$ is valid “up to any reasonable notion of equivalence over programs”. Symmetrically, for each $(f, c) \in R$, the equation $f(x, c) = x$ is valid “up to any reasonable notion of equivalence over programs”.

The proof relies on the construction of suitable bisimulations.
Cool, but can it be (easily) applied to examples from the literature?
Examples I

Non-deterministic Choice

Recall the rules for CCS $+$:

\[
\begin{align*}
x & \xrightarrow{a} x' \\
\end{align*}
\]

\[
\begin{align*}
y & \xrightarrow{a} y' \\
\end{align*}
\]

\[
\begin{align*}
x + y & \xrightarrow{a} x' \\
x + y & \xrightarrow{a} y' \\
\end{align*}
\]

The sets $R = L = \{(+, 0)\}$ meet the constraints. Therefore, our theorem yields the soundness of the well known equations:

\[
0 + x = x = x + 0.
\]
The sets $L = R = \{(\parallel_a, \text{RUN}_a)\}$ meet the constraints. Therefore, our theorem yields the soundness of the well known equations:

$$\text{RUN}_a \parallel_a x = x = x \parallel_a \text{RUN}_a.$$
Left Merge and Interleaving Parallel Composition

The following rules describe the operational semantics of the classic left merge and interleaving parallel composition operators.

\[
\begin{align*}
x & \xrightarrow{a} x' \\
\hline
x \parallel y & \xrightarrow{a} x' \parallel y \\
\end{align*}
\]

\[
\begin{align*}
x & \xrightarrow{a} x' \\
\hline
x \parallel y & \xrightarrow{a} x' \parallel y \\
\end{align*}
\]

\[
\begin{align*}
y & \xrightarrow{a} y' \\
\hline
x \parallel y & \xrightarrow{a} x \parallel y' \\
\end{align*}
\]

The sets \( L = \{ (\parallel, 0) \} \) and \( R = \{ (\parallel, 0), (\parallel, 0) \} \) meet the constraints. Therefore, our theorem yields the soundness of the well known equations:

\[
\begin{align*}
0 \parallel x &= x, &
x \parallel 0 &= x & \text{and} &
x \parallel 0 &= x.
\end{align*}
\]

Note that the pair \((\parallel, 0)\) cannot be added to \(L\) while preserving the constraints! Indeed, \(0\) is not a left unit for the left merge operator \(\parallel\).
Examples III

Left Merge and Interleaving Parallel Composition

The following rules describe the operational semantics of the classic left merge and interleaving parallel composition operators.

\[
\begin{align*}
  x \xrightarrow{a} x' & \quad \quad \quad x \parallel y \xrightarrow{a} x' \parallel y \\
  x \parallel y \xrightarrow{a} x' & \quad \quad \quad x \parallel y \xrightarrow{a} x' \parallel y \\
  y \xrightarrow{a} y' & \quad \quad \quad x \parallel y \xrightarrow{a} x \parallel y'
\end{align*}
\]

The sets \( L = \{(\parallel, 0)\} \) and \( R = \{(\parallel, 0), (\parallel, 0)\} \) meet the constraints. Therefore, our theorem yields the soundness of the well known equations:

\[
0 \parallel x = x, \quad x \parallel 0 = x \quad \text{and} \quad x \parallel 0 = x.
\]

Note that the pair \((\parallel, 0)\) cannot be added to \( L \) while preserving the constraints! Indeed, \( 0 \) is not a left unit for the left merge operator \( \parallel \).
Further Results

1. We have extended the rule format so that it applies to operational specifications with predicates (such as termination) as first class objects.

2. We have applied the format to more examples, such as CSP external choice, LOTOS-like disrupt, sequential composition and Plotkin’s fair parallel composition operators.

3. In joint work with Matteo Cimini (Reykjavik University), we have applied our ideas to obtain a rule format for zero elements to guarantee the validity of equations like

\[ \text{deadlock}; x = \text{deadlock}. \]
Read the paper *A Rule Format for Unit Elements* by Anna Ingolfsdottir, MohammadReza Mousavi, Michel Reniers and yours truly! (To appear in the Proceedings of SOFSEM 2010, Springer-Verlag, 2010.)

Thanks and Shameless Self-Promotion

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Thank You!
Any Questions?