

Non-Strongly Stable Orders and Simulation Relations

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Motivation

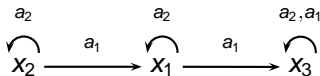
- We present two notions of simulation (those in Ignacio's talk) which can be defined as coalgebraic simulations.
 - Covariant-contravariant simulation: I/O Automata.
 - Conformance simulation: reducing non-determinism.
- In order to define them in a proper way we need an order with good enough properties.

Coalgebraic Simulations

- Generalize coalgebraic bisimulations by means of arbitrary preorder relations.
- Very general notion; perhaps, **too general**: the induced similarity relation needs not be transitive.
- In [HughesJacobs04] **stability** is also required, which is guaranteed by a stronger condition (“right-stability”).
 - We have shown that it induces a “natural” direction in the induced simulation order.
 - However, the symmetric “left-stability” also guarantees stability.
 - Other, more elaborated “combinations” of right and left stable orders also do the work.

Coalgebras

- For a functor F , an F -coalgebra is a function $c : X \rightarrow FX$, so that $x \in X$ is a state and $c(x)$ the set of successors of x .
- Choosing F we can obtain different structures:
 - ▶ $\mathcal{P}(X)^A$ for labelled transitions systems.



★ $X = \{x_1, x_2, x_3\}$.

★ $c : X \rightarrow \mathcal{P}(X)^{\{a_1, a_2\}}$

$$c(x_1) : \{a_1, a_2\} \rightarrow \mathcal{P}(X)$$

$$c(x_1)(a_1) = \{x_3\}$$

$$c(x_1)(a_2) = \{x_1\}$$

- ▶ $\mathcal{P}(AP) \times \mathcal{P}(X)$ for Kripke structures.

Bisimulations

- A functor $F : \mathbf{Sets} \rightarrow \mathbf{Sets}$ can be lifted to $\mathbf{Rel}(F) : \mathbf{Rel} \rightarrow \mathbf{Rel}$:

$$\mathbf{Rel}(F)(R) = \{ \langle u, v \rangle \in FX_1 \times FX_2 \mid \exists w \in F(R). F(r_1)(w) = u, F(r_2)(w) = v \}$$

If $R \subseteq X \times Y$ then $\mathbf{Rel}(F)(R) \subseteq FX \times FY$.

- A **bisimulation** for $c : X \rightarrow FX$ and $d : Y \rightarrow FY$ is a relation $R \subseteq X \times Y$ “closed under c and d ”:

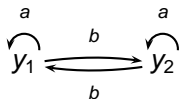
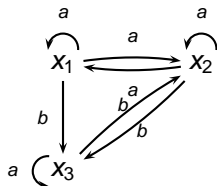
$$\text{if } (x, y) \in R \text{ then } (c(x), d(y)) \in \mathbf{Rel}(F)(R)$$

If states x and y are related, so are their successors $c(x)$ and $d(y)$.

Example: labelled transition systems

- Unfolding $\text{Rel}(\mathcal{P}(\text{id})^A)(R) \subseteq \mathcal{P}(X)^A \times \mathcal{P}(Y)^A$:

$$\text{Rel}(\mathcal{P}(\text{id})^A)(R) = \{(f, g) \mid \text{for all } a \in A, \\ \forall u \in f(a). \exists v \in g(a). uRv \wedge \\ \forall v \in g(a). \exists u \in f(a). uRv\}$$



$$c(x_1)(a) = \{x_1, x_2\}$$

$$c(x_1)(b) = \{x_3\}$$

$$d(y_2)(a) = \{y_2\}$$

$$d(y_2)(b) = \{y_1\}$$

- $R = \{(x_1, y_2), (x_2, y_2), (x_3, y_1)\}$.
- $(c(x_1), d(y_2)) \in \text{Rel}(F)(R)$ but $(c(x_2), d(y_1)) \notin \text{Rel}(F)(R)$.

Simulations

- An **order \sqsubseteq on F** is given by a collection $\sqsubseteq_X \subseteq FX \times FX$ that is functorial (roughly, it must be preserved by renaming).
- A **\sqsubseteq -simulation** for $c : X \rightarrow FX$ and $d : Y \rightarrow FY$ is a relation $R \subseteq X \times Y$ such that

$$\text{if } (x, y) \in R \text{ then } (c(x), d(y)) \in \text{Rel}_{\sqsubseteq}(F)(R),$$

that is,

$$c(x) \sqsubseteq_X u \text{ Rel}(F)(R) v \sqsubseteq_Y d(y),$$

for some u and v .

- Bisimulations are simulations for the identity order.

Stability

- \sqsubseteq for F is **stable** if $\text{Rel}_{\sqsubseteq}(F)$ commutes with substitution:
 - Given $f : X \rightarrow Z$ and $g : Y \rightarrow W$,

$$\text{Rel}_{\sqsubseteq}(F)((f \times g)^{-1}(R)) = (Ff \times Fg)^{-1}(\text{Rel}_{\sqsubseteq}(F)(R))$$

- Stable orders give rise to nice simulations.
- \sqsubseteq is **right-stable** if $(id \times Ff)^{-1} \sqsubseteq_Y \subseteq \coprod_{Ff \times id} \sqsubseteq_X$.
- Right-stability is equivalent to
 - \sqsubseteq stable and
 - $\text{Rel}(F)(R) \circ \sqsubseteq_X \subseteq \sqsubseteq_Y \circ \text{Rel}(F)(R)$.
- If F is right-stable,

$$\sqsubseteq_Y \circ \text{Rel}(F)(R) \circ \sqsubseteq_X = \sqsubseteq_Y \circ \text{Rel}(F)(R)$$

Plain Simulation

- Labelled transition systems (lts) are coalgebras for the functor $FX = \mathcal{P}(X)^A$.
- “Classical” simulations coincide with coalgebraic simulations for the order \sqsubseteq :
 - ▶ given $f, g \in FX = \mathcal{P}(X)^A$, that is, $f, g : A \rightarrow \mathcal{P}(X)$
 $f \sqsubseteq g$ if for all $a \in A$, $f(a) \subseteq g(a)$.
- It is right-stable.

Plain Simulation

- As a consequence of the right-stability \sqsubseteq -simulations can be characterized as the $(\sqsubseteq_Y \circ \text{Rel}(F)(R))$ -coalgebras.
 - ▶ The use of \sqsubseteq_X at the lhs can be replaced by that of \sqsubseteq_Y at the rhs:
 - \sqsubseteq_X “adds new successors to $c(x)$ ”.
 - \sqsubseteq_Y “removes successors of $d(y)$ ”.
 - If q simulates p , by removing the exceeding part of q we obtain q'' “bisimilar” to p .

$$p \text{ Rel}(F)(R) q'' \sqsubseteq q$$

Anti-simulations

- Anti-simulations are \supseteq -simulations for $FX = \mathcal{P}(X)^A$, that is,

$$f \sqsubseteq g \Leftrightarrow f(a) \supseteq g(a) \text{ for all } a \in A.$$

- c “simulates” d if and only if d “is simulated by” c .
- The order \supseteq is **not right-stable**.
- However, it is stable.

Left-stability

- F with \sqsubseteq is **left-stable** if for all $f : X \rightarrow Y$,

$$(Ff \times id)^{-1} \sqsubseteq_Y \subseteq \coprod_{id \times Ff} \sqsubseteq_X .$$

- Anti-simulation is left-stable.
- F with \sqsubseteq is stable iff it is stable with the inverse order \sqsubseteq^{op} .

Relating (Trivially) Left-stable and Right-stable Orders

- An order \sqsubseteq is left-stable iff \sqsubseteq^{op} is right-stable.
 - Both right-stability and left-stability give a **natural** direction to simulation relations.
- Left-stable orders have the same structural properties as right-stable ones.
 - ▶ \sqsubseteq -similarity is transitive, etc.
- The composition of right (resp. left)-stable orders gives us a new right (resp. left)-stable order.

Covariant-contravariant simulations

- Given an alphabet Act , we will consider a partition $\{Act^r, Act^l, Act^{bi}\}$ of Act .
- An (Act^r, Act^l) -**simulation** for $c : X \longrightarrow \mathcal{P}(X)^{Act}$ and $d : Y \longrightarrow \mathcal{P}(Y)^{Act}$ is a relation S such that $\forall (x, y) \in S$:
 - ▶ $\forall a \in Act^r \cup Act^{bi}, \forall x \xrightarrow{a} x' \exists y \xrightarrow{a} y'$ with $(x', y') \in S$.
 - ▶ $\forall a \in Act^l \cup Act^{bi}, \forall y \xrightarrow{a} y' \exists x \xrightarrow{a} x'$ with $(x', y') \in S$.

Covariant-contravariant simulations

- (Act^r, Act^l) -simulations can be defined as the coalgebraic simulations for the order $Act^r \sqsubseteq_{Act^l} \subseteq \mathcal{P}(X)^A \times \mathcal{P}(X)^A$.
- If $f, g : Act \longrightarrow \mathcal{P}(X)$, then $f \sqsubseteq_{Act^r \sqsubseteq_{Act^l}} g \Leftrightarrow$:
 - ▶ for all $a \in Act^r \cup Act^{bi}$, $f(a) \subseteq g(a)$, and
 - ▶ for all $a \in Act^l \cup Act^{bi}$, $f(a) \supseteq g(a)$.
- $Act^r \sqsubseteq_{Act^l}$ is stable.
 - ▶ It can be “decomposed” as a product of both right-stable and left-stable orders.
 - ▶ However, it is neither right-stable nor left-stable.

Conformance simulations

- They behave as plain simulations allowing the extension of the set of actions offered by a process:

$$a < a + b$$

- But a process can also be “improved” by reducing the nondeterminism in it.

$$ap + aq < ap$$

- A **conformance simulation** between $c : X \longrightarrow \mathcal{P}(X)^A$ and $d : Y \longrightarrow \mathcal{P}(Y)^A$, is a relation R such that if pRq then

- ▶ $\forall a \in A, p \xrightarrow{a} \Rightarrow q \xrightarrow{a}$.
- ▶ $\forall a \in A (q \xrightarrow{a} q' \wedge p \xrightarrow{a}) \Rightarrow p \xrightarrow{a} p' \text{ and } p'Rq'$.

Conformance simulations

- Conformance simulations can be defined as the coalgebraic simulations for the order $\sqsubseteq^{Conf} \subseteq \mathcal{P}(X)^A \times \mathcal{P}(X)^A$.
- If $f, g : A \longrightarrow \mathcal{P}X$, then $f \sqsubseteq_X^{Conf} g \Leftrightarrow$
 - ▶ Either $f(a) = \emptyset$, or
 - ▶ $f(a) \supseteq g(a)$ and $g(a) \neq \emptyset$.
- \sqsubseteq^{Conf} is stable.
 - ▶ However, it is neither right-stable nor left-stable.

Side stable orders

- In the proof of stability of the order for covariant-contravariant simulations, each subset of the partition of Act is dealt with separately.
- An order \sqsubseteq defined over F^A may be split into a family of orders \sqsubseteq^a over F .
- An order \sqsubseteq over a functor F^A is **action-distributive** if there exists a family of orders \sqsubseteq^a on F such that:

$$f \sqsubseteq g \iff f(a) \sqsubseteq^a g(a)$$

for all $a \in A$. We write $\sqsubseteq = \prod_{a \in A} \sqsubseteq^a$.

Side stable orders

- A **side stable order** is an action-distributive order such that each component is either right-stable or left-stable.
- If $\sqsubseteq = \prod_{a \in A} \sqsubseteq^a$ and each \sqsubseteq^a is stable, then \sqsubseteq is also stable.
 - ▶ Side stable orders are stable.
- By separating the right and the left-stable components we obtain $\sqsubseteq = (\sqsubseteq^{\bar{r}} \cup \sqsubseteq^l)^*$.
- The covariant-contravariant order $_{Act^r} \sqsubseteq_{Act^l}$ is side stable.

Composition of Right-stable and Left-stable Orders

- \sqsubseteq^{Conf} is also an action-distributive order, but not side stable.
- The distributivity of \sqsubseteq^{Conf} leads to its decomposition as a right-stable and a left-stable order that commute with each other.
- Given \sqsubseteq^r that is right-stable on F and \sqsubseteq^l that is left-stable, and commute with each other, their composition defines a stable order on F .
- Moreover, the coalgebraic simulations for $\sqsubseteq = \sqsubseteq^r \circ \sqsubseteq^l$ can be characterized as the $(\sqsubseteq^r \circ \text{Rel}(F)(R) \circ \sqsubseteq^l)$ -coalgebras.

Logical Characterizations

- We are interested in finding modal logics that characterize these two notions of simulations.
- A first approach is to build them from scratch taking, for example, plain simulations as models.
- A second way is to follow the general categorical constructions developed by Corina Cîrstea.

Logical Characterizations: the Categorical Way

- First, the adequate order has to be identified. Actually, Cîrstea's construction follows an alternative presentation of coalgebraic simulations.
- The language of the logic is the initial algebra of a suitable functor.
- The “semantics” of the logic is defined by means of another functor.
- Under certain conditions (a colimit needs to exist), the “semantics” induces a logic that characterizes similarity for the simulation.
- We were able to check that the logics obtained for our simulations using these two methods coincide.

Summary

- Two interesting notions of coalgebraic simulations which are not strongly stable.
 - ▶ Both can be factorized into the composition of a right and a left-stable component, and so are proved to be stable.
 - ▶ Witness that “strong” stability is, well, too strong.
- Right-stability is an asymmetric property.
 - ▶ We can use it to get a natural orientation for the simulation orders.
 - ▶ Its dualization leads to left-stability, with the same good properties.
 - ▶ By combining both right-stable and left-stable orders in several ways we can still preserve stability.
- These simulations can be endowed with a modal logic that characterizes them.
 - ▶ Ad-hoc manner.
 - ▶ Categorically.