Non-Strongly Stable Orders and Simulation Relations

Ignacio Fábregas David de Frutos Escrig **Miguel Palomino**

Departamento de Sistemas Informáticos y Computación, UCM

ICE-TCS Workshop on Logic and Concurrency
(Reykjavik, September 15th, 2010)

Motivation

- We present two notions of simulation (those in Ignacio's talk) which can be defined as coalgebraic simulations.
  - Covariant-contravariant simulation: I/O Automata.
  - Conformance simulation: reducing non-determinism.
- In order to define them in a proper way we need an order with good enough properties.

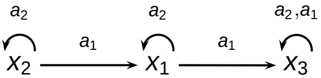
Coalgebraic Simulations

- Generalize coalgebraic bisimulations by means of arbitrary preorder relations.
- Very general notion; perhaps, too general: the induced similarity relation needs not be transitive.
- In [HughesJacobs04] stability is also required, which is guaranted by a stronger condition (“right-stability”).
  - We have shown that it induces a “natural” direction in the induced simulation order.
  - However, the symmetric “left-stability” also guarentees stability.
  - Other, more ellaborated “combinations” of right and left stable orders also do the work.

Coalgebras

- For a functor F, an F-coalgebra is a function $c : X \longrightarrow FX$, so that $x \in X$ is a state and $c(x)$ the set of succesors of x.

- Choosing F we can obtain different structures:
  - $\mathcal{P}(X)^A$ for labelled transitions systems.



    - ★ $X = \{x_1, x_2, x_3\}$.
    - ★ $c : X \longrightarrow \mathcal{P}(X)^{\{a_1, a_2\}}$

$$c(x_1) : \{a_1, a_2\} \longrightarrow \mathcal{P}(X)$$
$$c(x_1)(a_1) = \{x_3\}$$
$$c(x_1)(a_2) = \{x_1\}$$

  - $\mathcal{P}(AP) \times \mathcal{P}(X)$ for Kripke structures.

Bisimulations

- A functor F : **Sets** $\rightarrow$ **Sets** can be lifted to $\mathrm{Rel}(F)$: **Rel** $\rightarrow$ **Rel**:

$$\begin{array}{rcl}\mathrm{Rel}(F)(R) & = & \{\langle u, v\rangle \in FX_1 \times FX_2 \mid \\ & & \exists w \in F(R).F(r_1)(w) = u, F(r_2)(w) = v\}\end{array}$$

  If $R \subseteq X \times Y$ then $\mathrm{Rel}(F)(R) \subseteq FX \times FY$.

- A bisimulation for $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ is a relation $R \subseteq X \times Y$ “closed under c and d”:

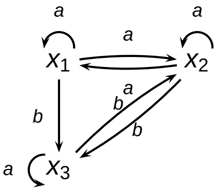
$$\text{if } (x, y) \in R \text{ then } (c(x), d(y)) \in \mathrm{Rel}(F)(R)$$

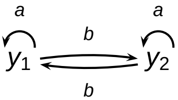
  If states x and y are related, so are their successors $c(x)$ and $d(y)$.

Example: labelled transition systems

- Unfolding $\mathrm{Rel}(\mathcal{P}(id)^A)(R) \subseteq \mathcal{P}(X)^A \times \mathcal{P}(Y)^A$:

$$\begin{array}{rcl} \mathrm{Rel}(\mathcal{P}(id)^A)(R) & = & \{(f,g) \mid \text{for all } a \in A, \\ & & \quad \forall u \in f(a).\, \exists v \in g(a).\, uRv \wedge \\ & & \quad \forall v \in g(a).\, \exists u \in f(a).\, uRv\}\} \end{array}$$





$$\begin{array}{l} c(x_1)(a) = \{x_1, x_2\} \\ c(x_1)(b) = \{x_3\} \\ \\ d(y_2)(a) = \{y_2\} \\ d(y_2)(b) = \{y_1\} \end{array}$$

- $R = \{(x_1, y_2), (x_2, y_2), (x_3, y_1)\}$.
- $(c(x_1), d(y_2)) \in \mathrm{Rel}(F)(R)$ but $(c(x_2), d(y_1)) \notin \mathrm{Rel}(F)(R)$.

Simulations

- An order $\sqsubseteq$ on F is given by a collection $\sqsubseteq_X \subseteq FX \times FX$ that is functorial (roughly, it must be preserved by renaming).
- A $\sqsubseteq$-simulation for $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ is a relation $R \subseteq X \times Y$ such that

$$\text{if } (x, y) \in R \text{ then } (c(x), d(y)) \in \mathrm{Rel}_{\sqsubseteq}(F)(R),$$

  that is,

$$c(x) \sqsubseteq_X u \ \mathrm{Rel}(F)(R)\ v \sqsubseteq_Y d(y),$$

  for some u and v.
- Bisimulations are simulations for the identity order.

Stability

- $\sqsubseteq$ for F is stable if $\mathrm{Rel}_{\sqsubseteq}(F)$ commutes with substitution:
  - Given $f : X \longrightarrow Z$ and $g : Y \longrightarrow W$,

$$\mathrm{Rel}_{\sqsubseteq}(F)((f \times g)^{-1}(R)) \;=\; (Ff \times Fg)^{-1}(\mathrm{Rel}_{\sqsubseteq}(F)(R))$$

- Stable orders give rise to nice simulations.
- $\sqsubseteq$ is right-stable if $(id \times Ff)^{-1} \sqsubseteq_Y \;\subseteq\; \coprod_{Ff \times id} \sqsubseteq_X$.
- Right-stability is equivalent to
  - $\sqsubseteq$ stable and
  - $\mathrm{Rel}(F)(R) \circ \sqsubseteq_X \;\subseteq\; \sqsubseteq_Y \circ \mathrm{Rel}(F)(R)$.
- If F is right-stable,

$$\sqsubseteq_Y \circ \mathrm{Rel}(F)(R) \circ \sqsubseteq_X \;=\; \sqsubseteq_Y \circ \mathrm{Rel}(F)(R)$$

Plain Simulation

- Labelled transition systems (lts) are coalgebras for the functor $FX = \mathcal{P}(X)^A$.
- "Classical" simulations coincide with coalgebraic simulations for the order $\subseteq$:
  - given $f, g \in FX = \mathcal{P}(X)^A$, that is, $f, g : A \longrightarrow \mathcal{P}(X)$
    $f \sqsubseteq g$ if for all $a \in A, f(a) \subseteq g(a)$.
- It is right-stable.

Plain Simulation

- As a consequence of the right-stability $\subseteq$-simulations can be characterized as the $(\subseteq_Y \circ \mathrm{Rel}(F)(R))$-coalgebras.
  - The use of $\subseteq_X$ at the lhs can be replaced by that of $\subseteq_Y$ at the rhs:
    - $\subseteq_X$ "adds new successors to $c(x)$".
    - $\subseteq_Y$ "removes successors of $d(y)$".
    - If q simulates p, by removing the exceeding part of q we obtain q'' "bisimilar" to p.

$$p \ \mathrm{Rel}(F)(R) \ q'' \subseteq q$$

Anti-simulations

- Anti-simulations are $\supseteq$-simulations for $FX = \mathcal{P}(X)^A$, that is,

$$f \sqsubseteq g \Leftrightarrow f(a) \supseteq g(a) \text{ for all } a \in A.$$

- c "simulates" d if and only if d "is simulated by" c.
- The order $\supseteq$ is not right-stable.
- However, it is stable.

Left-stability

- F with $\sqsubseteq$ is left-stable if for all $f : X \longrightarrow Y$,

$$(Ff \times id)^{-1} \sqsubseteq_Y \; \subseteq \; \coprod_{id \times Ff} \sqsubseteq_X .$$

- Anti-simulation is left-stable.
- F with $\sqsubseteq$ is stable iff it is stable with the inverse order $\sqsubseteq^{op}$.

Relating (Trivially) Left-stable and Right-stable Orders

- An order $\sqsubseteq$ is left-stable iff $\sqsubseteq^{op}$ is right-stable.
  - Both right-stability and left-stability give a natural direction to simulation relations.
- Left-stable orders have the same structural properties as right-stable ones.
  - $\subseteq$-similarity is transitive, etc.
- The composition of right (resp. left)-stable orders gives us a new right (resp. left)-stable order.

Covariant-contravariant simulations

- Given an alphabet Act, we will consider a partition $\{Act^r, Act^l, Act^{bi}\}$ of Act.
- An (Act^r, Act^l)**-simulation** for $c : X \longrightarrow \mathcal{P}(X)^{Act}$ and $d : Y \longrightarrow \mathcal{P}(Y)^{Act}$ is a relation S such that $\forall (x, y) \in S$:
  - $\forall a \in Act^r \cup Act^{bi}$, $\forall x \xrightarrow{a} x'$ $\exists y \xrightarrow{a} y'$ with $(x', y') \in S$.
  - $\forall a \in Act^l \cup Act^{bi}$, $\forall y \xrightarrow{a} y'$ $\exists x \xrightarrow{a} x'$ with $(x', y') \in S$.

Covariant-contravariant simulations

- (Act^r, Act^l)-simulations can be defined as the coalgebraic simulations for the order ${}_{Act^r}\sqsubseteq_{Act^l} \subseteq \mathcal{P}(X)^A \times \mathcal{P}(X)^A$.
- If $f, g : Act \longrightarrow \mathcal{P}(X)$, then $f \; {}_{Act^r}\sqsubseteq_{Act^l} \; g \;\; \Leftrightarrow$:
  - for all $a \in Act^r \cup Act^{bi}, \;\; f(a) \subseteq g(a)$, and
  - for all $a \in Act^l \cup Act^{bi}, \;\; f(a) \supseteq g(a)$.
- ${}_{Act^r}\sqsubseteq_{Act^l}$ is stable.
  - It can be “decomposed” as a product of both right-stable and left-stable orders.
  - However, it is neither right-stable nor left-stable.

Conformance simulations

- They behave as plain simulations allowing the extension of the set of actions offered by a process:

  $a < a + b$

- But a process can also be "improved" by reducing the nondeterminism in it.

  $ap + aq < ap$

- A **conformance simulation** between $c : X \longrightarrow \mathcal{P}(X)^A$ and $d : Y \longrightarrow \mathcal{P}(Y)^A$, is a relation R such that if pRq then
  - $\forall a \in A, p \xrightarrow{a} \Rightarrow q \xrightarrow{a}$.
  - $\forall a \in A\ (q \xrightarrow{a} q' \wedge p \xrightarrow{a}) \Rightarrow p \xrightarrow{a} p'$ and $p'Rq'$.

Conformance simulations

- Conformance simulations can be defined as the coalgebraic simulations for the order $\sqsubseteq^{Conf} \subseteq \mathcal{P}(X)^A \times \mathcal{P}(X)^A$.

- If $f, g : A \longrightarrow \mathcal{P}X$, then $f \sqsubseteq_X^{Conf} g \Leftrightarrow$
  - Either $f(a) = \emptyset$, or
  - $f(a) \supseteq g(a)$ and $g(a) \neq \emptyset$.

- $\sqsubseteq^{Conf}$ is stable.
  - However, it is neither right-stable nor left-stable.

Side stable orders

- In the proof of stability of the order for covariant-contravariant simulations, each subset of the partition of Act is dealt with separately.
- An order $\sqsubseteq$ defined over F^A may be split into a family of orders $\sqsubseteq^a$ over F.
- An order $\sqsubseteq$ over a functor F^A is action-distributive if there exists a family of orders $\sqsubseteq^a$ on F such that:

$$f \sqsubseteq g \iff f(a) \sqsubseteq^a g(a)$$

for all $a \in A$. We write $\sqsubseteq = \prod_{a \in A} \sqsubseteq^a$.

Side stable orders

- A side stable order is an action-distributive order such that each component is either right-stable or left-stable.
- If $\sqsubseteq \, = \prod_{a \in A} \sqsubseteq^a$ and each $\sqsubseteq^a$ is stable, then $\sqsubseteq$ is also stable.
  - Side stable orders are stable.
- By separating the right and the left-stable components we obtain $\sqsubseteq \, = (\sqsubseteq^{\bar{r}} \cup \sqsubseteq^{\bar{l}})^*$.
- The covariant-contravariant order ${}_{Act^r}\sqsubseteq_{Act^l}$ is side stable.

Composition of Right-stable and Left-stable Orders

- $\sqsubseteq^{Conf}$ is also an action-distributive order, but not side stable.
- The distributivity of $\sqsubseteq^{Conf}$ leads to its decomposition as a right-stable and a left-stable order that commute with each other.
- Given $\sqsubseteq^{r}$ that is right-stable on F and $\sqsubseteq^{l}$ that is left-stable, and commute with each other, their composition defines a stable order on F.
- Moreover, the coalgebraic simulations for $\sqsubseteq = \sqsubseteq^{r} \circ \sqsubseteq^{l}$ can be characterized as the $(\sqsubseteq^{r} \circ \mathrm{Rel}(F)(R) \circ \sqsubseteq^{l})$-coalgebras.

Logical Characterizations

- We are interested in finding modal logics that characterize these two notions of simulations.
- A first approach is to build them from scratch taking, for example, plain simulations as models.
- A second way is to follow the general categorical constructions developed by Corina Cîrstea.

Logical Characterizations: the Categorical Way

- First, the adequate order has to be identified. Actually, Cîrstea's construction follows an alternative presentation of coalgebraic simulations.
- The language of the logic is the initial algebra of a suitable functor.
- The "semantics" of the logic is defined by means of another functor.
- Under certain conditions (a colimit needs to exist), the "semantics" induces a logic that characterizes similarity for the simulation.
- We were able to check that the logics obtained for our simulations using these two methods coincide.

Summary

- Two interesting notions of coalgebraic simulations which are not strongly stable.
  - Both can be factorized into the composition of a right and a left-stable component, and so are proved to be stable.
  - Witness that “strong” stability is, well, too strong.
- Right-stability is an assymetric property.
  - We can use it to get a natural orientation for the simulation orders.
  - Its dualization leads to left-stability, with the same good properties.
  - By combining both right-stable and left-stable orders in several ways we can still preserve stability.
- These simulations can be endowed with a modal logic that characterizes them.
  - Ad-hoc manner.
  - Categorically.