

Coloring Powers of Planar Graphs*

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Abstract

We give nontrivial bounds for the *inductiveness* or *degeneracy* of power graphs G^k of a planar graph G . This implies bounds for the chromatic number as well, since the inductiveness naturally relates to a greedy algorithm for vertex coloring the given graph. The inductiveness moreover yields bounds for the *choosability* of the graph. We show that the inductiveness of a square of a planar graph G is at most $\lceil 9\Delta/5 \rceil$, for the maximum degree Δ sufficiently large, and that is sharp. In general, we show for a fixed integer $k \geq 1$, the inductiveness, the chromatic number, and the choosability of G^k to be $O(\Delta^{\lfloor k/2 \rfloor})$, which is tight.

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1 Introduction

The k -th power G^k of a graph G is defined on the same set of vertices as G , and has an edge between any pair of vertices of distance at most k in G . The topic of this paper is the coloring of power graphs, or equivalently coloring the underlying graphs so that vertices of distance at most k receive different colors. We focus on the planar case, long the center of attention for graph coloring.

We upper-bound the chromatic number and the *choosability*, see Definition 2.10, by the *inductiveness* of the graph G , which we denote here by $\text{ind}(G)$. This measure of G , also known as the *degeneracy*, the *coloring number*, and the *Szekeres-Wilf* number, is defined to be $\max_{H \subseteq G} \{\min_{v \in H} (d_H(v))\}$, where H runs through all induced subgraphs of G . Inductiveness leads to an ordering of the vertices, $\{v_1, \dots, v_n\}$, such that the number $d^+(v_i) = |\{v_j \in N_G(v_i) : j > i\}|$ of pre-neighbors of any v_i is at most $\text{ind}(G)$.

The problem of coloring squares of graphs has applications to frequency allocation. Transceivers in a radio network communicate using channels at given radio frequencies. Graph coloring formalizes this problem well when the constraint is that nearby pairs of transceivers cannot use the same channel due to interference. However, if two transceivers are using the same channel and both are adjacent to a third station, a clashing of signals is experienced at that third station. This can be avoided by additionally requiring all neighbors of a node to be assigned different colors, i.e. that vertices of distance at most two receive different colors. This is equivalent to coloring the square of the underlying network. Another application of this problem, from a completely different direction, is that of approximating certain Hessian matrices [12]. Observe that neighbors of a node in a graph form a clique in the square of the graph. Thus, the minimum number of colors needed to color any square graph is at least $\Delta + 1$, where $\Delta = \Delta(G)$ is the maximum degree of the original graph. As a result, the number of colors used by our algorithms on power graphs will necessarily be a function of Δ . We are particularly interested in the asymptotic behavior as Δ grows.

Coloring squares of graphs, in particular planar graphs, has been studied in the literature from two perspectives: in graph theory, focusing on bounding the number of colors needed, and in computer science, focusing on complexity and approximate algorithms. We attempt here to contribute to both of these directions. We first review graph-theoretic results on planar graphs in chronological order.

The first reference on coloring squares of planar graphs is by Wegner [19], who gave bounds on the clique number of such graphs. In particular, he gave an instance for which the clique number is at least $\lfloor 3\Delta/2 \rfloor + 1$ (which is largest possible), and conjectured this to be an upper bound on the chromatic number. He conjectured that

$$\chi(G^2) \leq \begin{cases} \Delta + 5 & \text{if } 4 \leq \Delta \leq 7; \\ \lfloor 3\Delta/2 \rfloor & \text{if } \Delta \geq 8. \end{cases}$$

Some work has been done on the case $\Delta = 3$, as listed in [8, Problem 2.18]. Ramanathan and Lloyd [16, 15] showed that $\text{ind}(G^2) \leq 9\Delta$, which is obtained by a minimum-degree greedy coloring algorithm. Krumke, Marathe and Ravi [10] generalized the bound to other classes of graphs, obtaining that $\text{ind}(G^2) \leq (2 \text{ind}(G) - 1)\Delta$.

Independent of the original version of this paper [1], there were at least two unrelated papers on bounding the chromatic number $\chi(G^2)$ of a square of a planar graph. van den Heuvel and McGuinness [6] showed that $\chi(G^2) \leq 2\Delta + 25$, using methods similar to those of the proof of

the 4-Color Theorem. And Jendrol' and Skupień [7] showed that $\chi(G^2) \leq 3\Delta + 9$, by bounding the inductiveness.

In the current paper, we show that for large values of Δ , squares of planar graphs are $\lceil 9\Delta/5 \rceil$ -inductive, implying a $\lceil 9\Delta/5 \rceil + 1$ -coloring. We show that this is sharp for all large values of Δ by constructing graphs attaining this inductiveness. For larger powers of a planar graph G , we obtain that G^k is $O(\Delta^{\lfloor k/2 \rfloor})$ -inductive, for any $k \geq 1$. This gives an asymptotically tight algorithmic bound for the chromatic number of the power graph.

McCormick [12] showed that the problem of coloring the power of a graph is NP-complete, for any fixed power, and a later proof was given by Lin and Skiena [11]. McCormick gave a greedy algorithm with a $O(\sqrt{n})$ -approximation for squares of general graphs. Heggernes and Telle [5] showed that determining if the square of a cubic graph can be colored with 4 colors or less is NP-complete, while it is easily determined if 3 colors suffice.

Ramanathan and Lloyd [16, 15] showed the problem of coloring squares of planar graphs to be NP-complete. Their bound mentioned earlier gave an algorithm with a performance ratio of 9, which was the best result known previous to [1]. The result of Krumke, Marathe and Ravi [10] yields in general a performance ratio of $2 \text{ind}(G) - 1$. They also gave a polynomial algorithm for graphs of both bounded treewidth and bounded degree, and used that to give a 2-approximation for bounded-degree planar graphs.

Sen and Huson [17] showed that coloring squares of unit-circle graphs is NP-complete, while a constant approximation algorithm was given in [18].

Zhou et al. [20] have in independent work given a polynomial algorithm for distance- d coloring partial- k trees, for any constants d and k . As indicated in Section 4, this implies a 2-approximation for distance- d coloring planar graphs for any d . Their algorithm, however, has a large polynomial complexity.

Our contributions give several approximation results. Combining the bound for squares of large-degree planar graphs with previous results for bounded-degree graphs, we obtain a 2-approximation for coloring that holds for all values of Δ . By itself, our bound gives a 1.8 *asymptotic* approximate coloring, as the chromatic number of the square goes to infinity. For higher powers of planar graphs, we obtain the first constant factor approximation for coloring cubes of planar graphs. However, the real strength of the current bounds are in giving absolute bounds on the number of colors used by the algorithm, as opposed to relative approximations, and thus implicitly bounding the number of colors used by an optimal solution.

Note the fine distinction between coloring the power graph G^k , and finding a *distance- k* coloring of G . The resulting coloring is naturally the same. However, in the latter case, the original graph is given. While it is easy to compute the power graph G^k from G , Motwani and Sudan [13] showed that it is NP-hard to compute the k -th root G of a graph G^k . All of the algorithms presented in this paper work without knowledge of the underlying root graph.

The rest of the paper is organized as follows. We bound the inductiveness of squares of planar graphs in Section 2, and general powers of planar graphs in Section 3. We consider the implications of these bounds to approximate colorings of powers of planar graphs in Section 4.

Notation The degree of a vertex v within a graph G is denoted by $d_G(v)$ or simply by $d(v)$ when there is no danger of ambiguity. The maximum degree of G is denoted by $\Delta = \Delta(G)$. For a vertex v denote by $d_k(v)$ the degree of v in G^k . The distance between two vertices u and v in a graph is the number of edges on the shortest path from u to v , and is denoted by $d_G(u, v)$. Let $G[W]$ denote the subgraph of G induced by vertex subset W . Let $N(v) = N_G(v)$ be the set of neighbors of v in G , and $N[v] = N_G[v]$ be the closed neighborhood of v in G given by $N[v] = N(v) \cup \{v\}$. The common closed neighborhood of u and v in G , denoted $N[uv]$ or

$N_G[uv]$, is given by $N[uv] = N[u] \cap N[v]$.

2 Squares of Planar Graphs

We start with a look at the main technique we use to derive bounds on the inductiveness of a square graph (and more generally, power graphs). The argument that is used, e.g., to show that planar graphs are 5-inductive is the following. Euler's formula states that in a planar graph G , $|E(G)| \leq 3|V(G)| - 2$ (see [4, p.74]). Thus, G contains a vertex of degree at most 5. Place one such node first in the inductive ordering, and remove it from the graph. Now the remaining graph is planar, so inductively we obtain a 5-inductive ordering.

The upper bound of 5 on the minimum degree of a planar graph also implies that squares of planar graphs are of minimum degree at most 5Δ . That would seem to imply a 5Δ -ordering of the square graph. However, when a vertex is deleted from the graph, its incident edges are deleted as well, so that vertices originally distance two apart may become much further apart in the remaining graph. An example of this is shown in Figure 1. Namely, the problem is that an induced subgraph does not preserve the paths of length two between vertices within the subgraph. The upshot is that degrees in the remaining graph do not adequately characterize degrees in the remaining part of the square of the graph. Our solution is to replace the *deletion* of a vertex by the *contraction* of an incident edge.

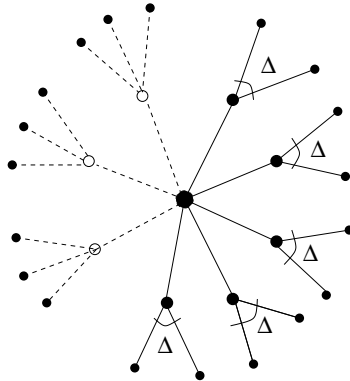


Figure 1: After the removal of nodes from a graph, a vertex can have vastly more of its original distance-2 neighbors remaining than of its neighbors. After the deletion of the three white vertices, the center node has five neighbors but $5\Delta + 9$ of its remaining distance-2 neighbors.

The *contraction* of an edge uv in graph G is the operation of collapsing the vertices u and v into a new vertex, giving the simple graph G/uv defined by $V(G/uv) = V(G) \setminus \{v\}$ and $E(G/uv) = \{ww' \in E(G) : w, w' \neq v\} \cup \{uw : vw \in E(G)\}$. Note that if G is planar, then G/uv is also planar. This is a property of various classes of graphs that are *closed under minor operations*. By the classic theorems of Kuratowski and Wagner (see [4, p.85]), planar graphs are precisely those graphs for which repeated contractions do not yield supergraphs of K_5 or $K_{3,3}$. Minor-closedness holds for various other classes of graphs, e.g. partial- k trees, but not d -inductive graphs in general.

Since our bounds on the inductiveness are functions of Δ , it is imperative that the contraction operations do not increase the maximum degree.

Definition 2.1 *An edge uv is mergeable if $|N[u] \cup N[v]| \leq \Delta + 2$.*

The contraction of a mergeable uv in G yields a simple planar graph G/uv whose maximum degree stays at most Δ . Also, by the property of edge contractions, the new distance function is dominated by the one on G (i.e. distances in G/uv are at most those in G). Thus, to show that a square graph G^2 is $f(\Delta)$ -inductive, we want to show the existence of a mergeable edge uv with $d_2(v) \leq f(\Delta)$. We state this as a general proposition.

Proposition 2.2 *Let \mathcal{G} be a class of graphs closed under edge contractions, and let f be a non-decreasing function. Suppose every graph G in \mathcal{G} contains a mergeable edge uv with $d_2(v) \leq f(\Delta)$. Then, the square of each G in \mathcal{G} is $f(\Delta)$ -inductive.*

2.1 Example applications of the contraction technique

We first illustrate the technique on simpler examples. Consider a minor-closed class of graphs that are 2-inductive (e.g. *partial-2 trees* or *series-parallel graphs*).

Theorem 2.3 *Squares of partial-2 trees are 2Δ -inductive.*

Proof. We inductively choose a vertex of degree at most 2 in the graph and contract one of its incident edges. In this case, either of its incident edges is mergeable, as the degree of each of its remaining neighbors does not increase. At most 2Δ vertices are within distance at most 2 of the selected vertex. Thus we obtain a 2Δ -inductive ordering of the square graph. \square

Our second example yields a bound on the inductiveness of planar graphs of small degree that improves on the 9Δ -bound of [16] for 5-inductive graphs.

Theorem 2.4 *If G is a planar graph with $\Delta(G) \geq 9$, then $\text{ind}(G^2) \leq 4\Delta(G) + 4$.*

Proof. We consider a maximal supergraph G' of G , and apply a theorem of Kotzig [9] (see also [7]). The theorem states that a maximal planar graph G' contains an edge uv such that $d_{G'}(u) + d_{G'}(v) \leq 13$, and further that $d_{G'}(u) + d_{G'}(v) \leq 11$ unless $d_{G'}(u) = 3$. We may assume $d_{G'}(u) \leq d_{G'}(v)$.

We claim that uv is mergeable when $\Delta \geq 9$, and that $d_2(v) \leq 4\Delta + 4$ (within G). By Proposition 2.2, this yields the theorem. We show this by considering two cases. Observe first that since G' is maximal, u and v share two common neighbors a and b in G' , and also that $N_G[w] \subseteq N_{G'}[w]$ for any node w .

Case when $d_{G'}(u) = 3$: Thus, $N_{G'}[u] = \{u, v, a, b\} \subseteq N_{G'}[v]$. Then, the union of the closed neighborhoods of u and v in G satisfies

$$N_G[u] \cup N_G[v] \subseteq N_{G'}[u] \cup N_{G'}[v] = N_{G'}[v].$$

Hence, $|N_G[u] \cup N_G[v]| \leq d_{G'}(v) + 1 \leq 11$. So, the edge uv is mergeable when $\Delta \geq 9$.

The number of distance-2 neighbors of u in G is at most the sum of the degrees of a , b , and v , not counting the possible edges from v to a and b , or at most $2\Delta + 8$.

Case when $4 \leq d_{G'}(u) \leq 5$: Recall that the closed neighborhoods of u and v in G' share the four nodes a, b, u and v . Thus,

$$|N_G[u] \cup N_G[v]| \leq |N_{G'}[u] \cup N_{G'}[v]| = |N_{G'}[u]| + |N_{G'}[v]| - 4 = d_{G'}(u) + d_{G'}(v) - 2 \leq 9.$$

Thus, uv is mergeable when $\Delta \geq 7$.

When counting the number of distance-2 neighbors of u in G , each of the neighbors of u contributes at most Δ of them, while v contributes itself along with those of its neighbors not among $\{u, a, b\}$. Thus,

$$d_2(u) \leq (d(u) - 1)\Delta + [1 + (11 - d(u) - 3)] \leq 4\Delta + 4.$$

□

Jendrol' and Skupień [7] have recently given a refinement of Kotzig's result, obtaining a bound of $3\Delta + 8$ on the inductiveness of the square a planar graph G with $\Delta(G) \geq 8$.

2.2 Sharp upper bound for large degree graphs

We now turn to the main result of this section, which is that when G is planar and $\Delta = \Delta(G)$ is large enough, then G^2 is $\lceil 9\Delta/5 \rceil$ -inductive. The following lemma is the key to this result.

Lemma 2.5 *Let G be a simple planar graph of maximum degree $\Delta \geq 48$. Then there exists a mergeable edge vw in G with $d_2(v) \leq \max(\lceil 9\Delta/5 \rceil, \Delta + 600)$.*

Proof. We assume that we have a fixed planar embedding of G , i.e. that G is a plane graph. Let $V_h = \{v \in V(G) : d(v) \geq 26\}$ and $V_l = V(G) \setminus V_h$.

If there is a vertex $v \in V_l$ with at most one neighbor in V_h , then $d_2(v) \leq 1 \cdot \Delta + 24 \cdot 25 = \Delta + 600$. Select any incident edge vw to a low degree neighbor w of v , and notice that the contracted edge would result in a node of degree at most $(25 - 1) + (25 - 1) = 48$. Since $\Delta \geq 48$, vw satisfies the claim of the lemma. Hence, for the rest of this proof, we assume the contrary, that every vertex in V_l has at least two neighbors in V_h .

Call a cycle of four vertices in G *forbidden*, if exactly two opposite vertices of the cycle are in V_h and the enclosed region formed by the cycle in the plane properly contains at least one vertex in V_h .

If G contains a forbidden 4-cycle then let G' be the subgraph of G induced by the region bounded by a minimal such 4-cycle. (Here, minimal means that no other 4-cycle is inside). If G contains no such cycle then let G' be G .

Consider now the multigraph H with vertex set $V(H) = V_h \cap V(G')$ and with colored edges defined as follows. For each edge uw in $E(G')$ with both $u, w \in V_h$ connect u and w with a red edge. For each vertex $v \in V_l$ adjacent to u and $w \in V_h$ in G' and to no other vertex in V_h , connect u and w in H with a green edge. Finally, for $v \in V_l$ adjacent to $u_1, u_2, \dots, u_t \in V_h$ in G' in a clockwise order for $t \geq 3$, connect u_1 to u_2 , u_2 to u_3, \dots, u_{t-1} to u_t and u_t to u_1 with blue edges in H .

Since G is planar, we note that H is also a planar multigraph. Hence, we can assume we have a drawing of H in the plane such that

- The vertices of H have the same configuration as they have in the plane graph G .
- For every pair $\{u, w\}$ of vertices of H connected by green or blue edges, their order with respect to u and w is the same as the order of the corresponding vertices of V_l .

By our assumption there is no vertex in V_l with at most one neighbor in V_h in G and hence in G' . Therefore, the degree of a vertex in H is at least that in G' .

As reference, we show in Fig. 2 the common neighborhood in G of two vertices u and v , along with the the corresponding multigraph. Vertices in V_h are in black, blue vertices are grey, and green vertices are white. Here $N[uv]$ contains five nodes in addition to u and v , corresponding to two blue, and three green edges. Hence, in this figure we have in clockwise order w.r.t. the vertex u that x_1 is blue (grey in figure), since it has three black neighbors, the vertices x_2 , x_3 and x_4 are green (white in figure), since they each have two black neighbors, u and v , and x_5 is blue (grey in figure), since it has four black neighbors.

Let $v \in V(H)$ denote a vertex with at most five neighbors in H , such that v is not on the 4-cycle defining G' (if G' was so defined). Euler's formula for planar graphs implies that there

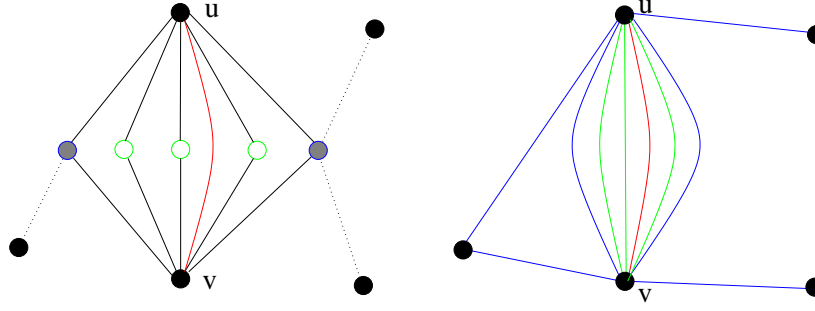


Figure 2: Example of a common neighborhood and the corresponding multigraph

are at least three vertices of $V(H) = V_h \cap V(G')$ with at most five neighbors in H . Hence, there is such a vertex that is not on the 4-cycle defining G' , as required. From now, let v denote such a vertex.

Claim 2.6 *Let $x \in N_H(v)$. There are at most two vertices in $V_l \cap N_{G'}[vx]$ that have neighbors outside $N_{G'}[vx] \cup \{v, x\}$.*

Assume the contrary that there are three vertices in $V_l \cap N_{G'}[vx]$ that have neighbors outside $N_{G'}[vx] \cup \{v, x\}$. Since G' is a plane graph, one of these three vertices, call it w , must be contained in the 4-cycle formed by v, x and the other two vertices of those three. If w has a neighbor in $(V_h \cap V(G')) \setminus N_{G'}[vx]$, then we have a smaller forbidden 4-cycle, contradicting our assumption. If w has a neighbor in $(V_l \cap V(G')) \setminus N_{G'}[vx]$, then by our assumption, that neighbor must have at least two neighbors in $V_h \cap V(G')$ that cannot be the vertices $\{v, x\}$. That would again yield a smaller forbidden 4-cycle, a contradiction. Hence, the claim.

From now, let u be the node in $V(H)$ with the largest neighborhood $N_{G'}[uv]$ in common with v in G' . When breaking ties, we prefer nodes that are not adjacent to v with a red edge.

Claim 2.7 *There is a vertex $w \in N_{G'}[uv]$ such that vw is mergeable and $N_G[N_G[w]] \subseteq N_G[v] \cup N_G[w]$.*

Observe that the selection criteria for u also serves to maximize the multiplicity m_{uv} of edges uv in H . Since $d_G(v) \geq 26$ and $d_H(v) \leq 5$, we have that $m_{uv} \geq \lceil 26/5 \rceil = 6$. Among these at least six edges, there is at most one red edge, and (by Claim 2.6) at most two edges (blue or green) that correspond to vertices of $V_l \cap N_{G'}[uv]$ with neighbors outside $N_{G'}[uv] \cup \{u, v\}$. Let w', w and w'' be nodes in V_l corresponding to the first three of the remaining edges in a clockwise order from v . By the planarity of G' , w must be properly enclosed in the cycle formed by $C = \{u, v, w', w''\}$. Hence, $N_G(w) = N_{G'}(w) \subseteq C$, and vw is mergeable. Further, since w' and w'' have no neighbors outside of $N[v] \cup N[u]$, all distance-2 neighbors of w are in $N[v] \cup N[u]$ as claimed.

To prove the lemma, it suffices to bound the distance-2 degree of either v or w . We split the argument into two cases, depending on whether there is a red edge incident to v in H or not.

Case I: There is no red edge incident on v . Then all of v 's neighbors are in V_l . Recall that each of them must have at least two high degree neighbors, thus each of them belongs to some $N_{G'}[vx]$ for some $x \in N_H(v)$. For each $x \in N_H(v)$, there are by Claim 2.6 at most two nodes in $N_{G'}[vx]$, excluding v and x , that have neighbors outside of $N_{G'}[vx]$. Since there are at most five nodes in $N_H(v)$, there are at most 10 neighbors of v that have neighbors outside of $N_{G'}[v] \cup N_H(v)$. Hence,

$$d_2(v) \leq \Delta + 10 \cdot 25 + 5 < \Delta + 600.$$

Case II: There is a red edge incident on v , say x_1v . Thus, $v \in N_{G'}[x_1v]$. Since each node in V_l is by assumption adjacent to at least two vertices in V_h , it holds that $\bigcup_{x \in N_H(v)} N_{G'}[xv] = N_{G'}[v]$. Then,

$$|N_G[uv]| = |N_{G'}[uv]| \geq |N_{G'}[v]|/|N_H(v)| \geq \lceil (d_{G'}(v) + 1)/5 \rceil.$$

By Claim 2.7, $N_G[w] \subseteq N_{G'}[uv]$, and since $x \mapsto x - \lceil x/5 \rceil$ is an increasing function, we have

$$\begin{aligned} d_2(w) + 1 &= |N_G[u] \cup N_G[v]| \\ &\leq |N_G[u]| + |N_G[v]| - |N_G[uv]| \\ &\leq (\Delta + 1) + (d_{G'}(v) + 1) - \lceil (d_{G'}(v) + 1)/5 \rceil \\ &\leq 2(\Delta + 1) - \lceil (\Delta + 1)/5 \rceil \\ &= \lceil 9\Delta/5 \rceil + 1. \end{aligned}$$

Together, the two cases establish that for at least one of the nodes v, w , we have that the distance-2 degree is at most $\max(\lceil 9\Delta/5 \rceil, \Delta + 600)$. \square

Our main result now follows from Lemma 2.5 and Proposition 2.2.

Theorem 2.8 *If G is a planar graph with $\Delta = \Delta(G) \geq 750$, then G^2 is $\lceil 9\Delta/5 \rceil$ -inductive.*

It turns out that $\lceil 9\Delta/5 \rceil$ is a *sharp* upper bound for the inductiveness, for all values of $\Delta \geq 750$.

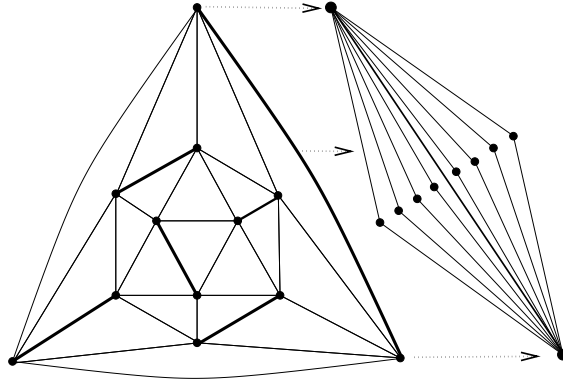


Figure 3: Icosahedron graph, and split edges

Observation 2.9 *For any $\Delta \geq 5$, there exists a planar graph G of maximum degree Δ such that G^2 is of minimum degree $\lceil 9\Delta/5 \rceil$.*

Proof. Let $\Delta \geq 5$ and $q = \lceil \Delta/5 \rceil + 1$. Then $\Delta = 5q - i$, where $q \geq 2$ and $i \in \{1, 2, 3, 4, 5\}$. Let H be a five-regular planar icosahedron graph that can be partitioned into five perfect matchings (see Fig. 3, where the edges of the first perfect matching are shown in bold). We construct from H a graph G as follows: To the first i perfect matchings we add $q - 2$ paths of length two, and we replace the remaining $5 - i$ perfect matchings with q paths of length two. Observe that there are two kinds of vertices in G , one kind has degree two and the other has degree Δ .

Consider a vertex w of degree two in G . If the neighbors of w of degree Δ are u and v , then there are precisely q vertices in $N[uv]$. Hence, the distance-two degree of w is given by

$$\begin{aligned} d_2(w) + 1 &= |N[u]| + |N[v]| - |N[uv]| \\ &= 2(\Delta + 1) - (\lceil \Delta/5 \rceil + 1) \\ &= \lceil 9\Delta/5 \rceil + 1. \end{aligned}$$

However, a vertex v of degree Δ is connected to $i \geq 1$ other vertices of degree Δ . Call one of them u . Note that every vertex in $N[v] \cup N[u]$ is of distance two or less from v , hence we have

$$d_2(v) + 1 \geq |N[v] \cup N[u]| = |N[u]| + |N[v]| - |N[uv]| = \lceil 9\Delta/5 \rceil + 1.$$

Therefore, the minimum degree of G^2 is precisely $\lceil 9\Delta/5 \rceil$, thereby completing our proof. \square

Recall the following definition of choosability given in [4].

Definition 2.10 *A graph G is k -choosable, if for every collection $\{S_v : v \in V(G)\}$ of lists of colors, $S_v \subseteq \{1, 2, 3, \dots\}$ where $|S_v| = k$ for every $v \in V(G)$, there is a color assignment*

$$c : V(G) \rightarrow \bigcup_{v \in V(G)} S_v,$$

such that

- $c(v) \in S_v$ for each $v \in V(G)$, and
- if $c(v) = c(u)$ then v and u are not neighbors in G .

The minimum such k is called the choosability of G and denoted by $\text{ch}(G)$.

We note that if a graph is k -choosable, then it is k -colorable. Also, by an easy induction, one can see that if a graph is k -inductive then it is $(k+1)$ -choosable. For any graph G we therefore have

$$\chi(G) \leq \text{ch}(G) \leq \text{ind}(G) + 1.$$

Hence, from Theorem 2.8 we have in particular the following corollary.

Corollary 2.11 *If G is a planar graph with $\Delta = \Delta(G) \geq 750$, then $\text{ch}(G^2) \leq \lceil 9\Delta/5 \rceil + 1$.*

3 General Powers of Planar Graphs

In this section we consider general powers G^k of planar graphs, and establish tight asymptotic bounds of the inductiveness of $\text{ind}(G^k)$. In fact we prove the following theorem, which in particular, improves the bound of $\chi(G^k)$ given in [7], where it is shown that $\chi(G^k)$ is bounded from above by a polynomial in Δ of degree $k-1$.

Theorem 3.1 *Let G be a planar graph with maximum degree Δ . For any fixed $k \geq 1$, G^k is $O(\Delta^{\lfloor k/2 \rfloor})$ -colorable. Also, there is a family of graphs that attains this bound. This bound is also asymptotically tight for the clique number, inductiveness, choosability, arboricity, and minimum degree of G^k .*

Let us first give a construction that matches the bound of the theorem. Given $k, \Delta \geq 1$, consider the tree T of height $\lfloor k/2 \rfloor$ where internal vertices have degree Δ . The number of vertices in T is

$$D_{\Delta,k} = 1 + \Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 + \dots + \Delta(\Delta - 1)^{\lfloor k/2 \rfloor - 1} = \frac{\Delta(\Delta - 1)^{\lfloor k/2 \rfloor} - 2}{\Delta - 2}.$$

Observe that T^k is a complete graph, thus $\chi(T^k) = D_{\Delta,k}$.

We now turn to proving the upper bound of the theorem. First we introduce some terminology.

Notation and Arboricity

A k -*path* is a path of length exactly k . A (k, \leq) -*path* is a path of length k or less. If u and v are vertices of a given graph, then a *walk* of length k from u to v is simply a sequence $(u_0, e_1, u_1, \dots, u_{k-1}, e_k, u_k)$, where $u_0 = u$, $u_k = v$ and each e_i has endvertices u_{i-1} and u_i . Note that in a walk, both vertices and edges may be repeated.

Definition 3.2 For a graph G , define its arboricity, denoted $\text{arb}(G)$, as the minimum number of forests needed to cover all the edges of the graph G .

By the Nash-Williams theorem [14] we have

$$\text{arb}(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil.$$

Arboricity is closely related to inductiveness.

Lemma 3.3 For any graph G , we have $\text{arb}(G) \leq \text{ind}(G) \leq 2 \text{arb}(G) - 1$.

Proof. Let q be $\text{ind}(G)$. We first show that $E(G)$ can be partitioned into q forests. Given a linear arrangement of the vertices, such that each vertex v_i has at most q later neighbors, we arbitrarily color the edges from v_i to later vertices with at most q colors. In this way, each color class is acyclic, since two edges of the same color cannot have the same first-labeled endpoint, and thus a forest. Therefore $\text{arb}(G) \leq q$, proving the first inequality.

For the second inequality, let $\text{ind}(G) = q$. Let H be a subgraph of G such that $\min_v (d_H(v)) = q$. Since $2|E(H)| = \sum_{v \in V(H)} d_H(v) \geq q|V(H)|$, we have $\text{arb}(G) > |E(H)|/|V(H)| \geq q/2$. Since $\text{arb}(G)$ is an integer, we have $q \leq 2 \text{arb}(G) - 1$, which completes our lemma. \square

Note that if G is planar we have that $\text{arb}(G) \leq 3$ by Euler's formula and the Nash-Williams theorem. Also we have that $\text{ind}(G) \leq 5$. Since there are planar graphs obtaining these values, Lemma 3.3 is tight for planar graphs.

From Theorem 2.4 and Lemma 3.3 we have in particular that $\text{arb}(G^2) \leq 4\Delta + 4$, if $\Delta \geq 9$.

Arboricity of power graphs.

We now want to find an upper bound of the arboricity of G^k in terms of Δ , where G is a planar graph. For a vertex set $U \subseteq V(G)$, let $E^k(U)$ be the edgeset of the subgraph of G^k induced by U . Then, the arboricity of G^k is

$$\text{arb}(G^k) = \max_{U \subseteq V(G)} \left\lceil \frac{|E^k(U)|}{|U| - 1} \right\rceil. \quad (1)$$

We will use this to bound $\text{arb}(G^k)$, but first we note the following.

Lemma 3.4 If G is a simple graph with $\text{arb}(G) = \alpha$, then the edges of G can be directed in such a way that for each vertex $v \in V(G)$, at most α directed edges are pointing from v .

Proof. Let F_1, \dots, F_α be the forests that cover the edges of G . For each subtree T of each F_i , direct its edges upward towards an arbitrarily chosen root r of T . In this way each F_i becomes a directed forest F_i^d in which every vertex, but the root, has outdegree one, and the root has outdegree zero. Hence, as G is the disjoint union of the forests F_i , the outdegree of each vertex in G is at most α . \square

Let G be a planar graph, and $U \subseteq V(G)$. Note that if two vertices of U are connected in G^k , then there is a (k, \leq) -path in G between them, and hence an i -walk between them, where $i \in \{k-1, k\}$.

Theorem 3.5 For any graph G , we have $\text{arb}(G^k) \leq 2^{k+1} \alpha^{\lceil k/2 \rceil} \Delta^{\lfloor k/2 \rfloor}$, where $\alpha = \text{arb}(G)$.

REMARK: The main idea of the proof below, of counting the i -walks directly, is due to the anonymous referees.

Proof. By Lemma 3.4 we can direct the edges of G in such a way that for each vertex $v \in V(G)$ there are at most α directed edges pointing from v .

Let $U \subseteq V(G)$. If $uv \in E^k(U)$, then there is an i -walk in G , where $i \in \{k-1, k\}$, either from u to v , or from v to u , that walks against at most $\lfloor i/2 \rfloor$ of the given directions of the edges. Assume in this case there is such an i -walk \vec{w} from u to v . There are $\sum_{j=0}^{\lfloor i/2 \rfloor} \binom{i}{j}$ possibilities of at most $\lfloor i/2 \rfloor$ edges in \vec{w} pointing against the walk. Also, for each vertex on \vec{w} , there are at most α choices of directed edges pointing from the vertex, and at most $\Delta \geq \alpha$ choices of directed edges pointing to the vertex. Hence, the number of possible such i -walks \vec{w} from u , with at most $\lfloor i/2 \rfloor$ against directions, is $\sum_{j=0}^{\lfloor i/2 \rfloor} \binom{i}{j} \alpha^{i-j} \Delta^j \leq (\sum_{j=0}^{\lfloor i/2 \rfloor} \binom{i}{j}) \alpha^{\lceil k/2 \rceil} \Delta^{\lfloor k/2 \rfloor}$. Hence,

$$|E^k(U)| \leq \left(\sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{k-1}{j} + \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{j} \right) \alpha^{\lceil k/2 \rceil} \Delta^{\lfloor k/2 \rfloor} |U| \leq 2^k \alpha^{\lceil k/2 \rceil} \Delta^{\lfloor k/2 \rfloor} |U|.$$

The theorem now follows from (1). □

Note that for a planar graph G we have $\text{arb}(G) \leq 3$. Also note that from the above proof, that for any $U \subseteq V(G)$ in a general graph, we have $2|E^k(U)| = \sum_{v \in U} d_{G[U]^k}(v)$, and hence there is a vertex v with $d_{G[U]^k}(v) \leq 2^{k+1} \alpha^{\lceil k/2 \rceil} \Delta^{\lfloor k/2 \rfloor}$. With this in mind we have the following.

Corollary 3.6 *For a planar graph G with $\Delta \geq 3$ we have*

$$\text{arb}(G^k), \text{ind}(G^k) \leq 2^{k+1} 3^{\lceil k/2 \rceil} \Delta^{\lfloor k/2 \rfloor}.$$

By Lemma 3.3 and Theorem 3.5 we have that for any planar graph G , the chromatic number, clique number, choosability and inductiveness is at most $2 \text{arb}(G)$, which completes the proof of Theorem 3.1.

Remarks

The original proof of Theorem 3.1, as found in [1] was different. Our argument was partly based on the following ‘‘expansion property’’ for planar graphs, which took the longest to prove, but is of interest in its own right: For a planar graph G and any subset $W \subseteq V(G)$ of vertices, there is a subset W' with $W \subseteq W' \subseteq V(G)$ and $|W'| \leq 10^{k-1}|W|$, such that if any two vertices in W are neighbors in G^k , then they are also neighbors in $G[W']^k$, the subgraphs of G^k induced by W' .

4 Approximation Algorithms

We can improve the best approximation factor known for coloring squares of planar graphs. Recall that since neighbors in G must be colored differently in G^2 , $\chi(G^2) \geq \Delta + 1$. Thus, for $\Delta \geq 750$, Theorem 2.8 yields a 1.8-approximation. Hence, we obtain an *asymptotic* ratio of 1.8.

For constant values of Δ , we can use a result of Krumke, Marathe and Ravi [10]. They stated a 3-approximation, but actually a 2-approximation easily follows from their approach which is based on an often-used decomposition due to Baker [2]. The complexity of their approach is equivalent to the complexity of coloring a partial $O(\Delta)$ -tree. Combined, we obtain a 2-approximation for any value of Δ .

Theorem 4.1 *The problem of coloring squares of planar graphs has a 2-approximation.*

Theorem 3.1 also immediately gives a $O(1)$ -approximation to coloring cubes of planar graphs. However, better factors are possible.

Zhou et al. [20] independently gave a polynomial algorithm for distance- d coloring partial k -trees, for any constant d and k . The complexity of their algorithm is $O(n(\alpha+1)^{2^{2(k+1)(d+2)+1}} + n^3)$, where $\alpha = O(\min(\Delta^{d/2}, n))$ is the number of colors needed. Since it is not indicated in [20], we show here how this results yields a 2-approximation for coloring G^d , for any constant d , when combined with the decomposition of Baker.

The technique of Baker [2] partitions the vertex set V of a planar graph into subsets V_1, V_2, \dots referred to as *layers*, such that all edges are between adjacent layers or within the same layer, i.e. if $u \in V_i$ and $uv \in E$, then $v \in V_{i-1} \cup V_i \cup V_{i+1}$. Now, let $V' = \cup_{i \bmod 2d < d} V_i$, $V'' = V - V'$, and G', G'' be the subgraphs induced by V' and V'' . Observe that both G' and G'' consist of a collection of disjoint subgraphs U_i , corresponding to $V_{di} \cup V_{di+1} \cup \dots \cup V_{d(i+1)-1}$. Further, notice that the subgraphs induced by the U_i will also be disjoint in G'^d and G''^d , since distance between any pair of nodes in different subgraphs U_i is at least $d+1$. Thus, G'^d can be computed by considering each U_i separately. Now, G^d restricted to U_i is a subgraph of the graph H_i^d , where $H_i = G[\cup_{j=di-(d-1)}^{d(i+1)-(d-2)} U_i]$. H_i is a $3d-2$ -outerplanar graph, which means that it is a partial $9d-8$ -tree by a result of Bodlaender [3]. Hence, we can compute the optimal coloring of each H_i in time $O(n^{2^{(9d-7)(d+2)+1}})$. Thus, we can solve G'^2 and G''^2 exactly, and in total, using at most twice the optimal number of colors.

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References

- [1] G. Agnarsson and M. M. Halldórsson. Coloring powers of planar graphs. In *Proc. 11th Ann. ACM-SIAM Symp. on Discrete Algorithms*, pages 654–662. ACM-SIAM, 2000.
- [2] B. S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *J. ACM*, 41:153–180, Jan. 1994.
- [3] H. L. Bodlaender. A partial k -arboretum of graphs with bounded treewidth. In *Theor. Comp. Sci*, volume 209, pages 1–45, 1998.
- [4] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer Verlag, 1997.
- [5] P. Heggenes and J. A. Telle. Partitioning graphs into generalized dominating sets. *Nordic J. Computing*, 5(2):128–143, Summer 1998.
- [6] J. van den Heuvel and S. McGuinness. Colouring the square of a planar graph. Research Report LSE-CDAM-99-06, Centre for Discrete and Applicable Mathematics, June 1999.

- [7] S. Jendrol' and Z. Skupieñ. Local structures in plane maps and distance colourings. *Disc. Math.*, 236:167–177, 2001.
- [8] T. R. Jensen and B. Toft. *Graph Coloring Problems*. Wiley Interscience, 1995. <http://www.imada.sdu.dk/Research/Graphcol/>.
- [9] A. Kotzig. Extremal polyhedral graphs. *Ann. New York Acad. Sci.*, 319:569–570, 1979.
- [10] S. O. Krumke, M. V. Marathe, and S. S. Ravi. Approximation algorithms for channel assignment in radio networks. In *Dial M for Mobility, 2nd International Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications*, Dallas, Texas, 1998.
- [11] Y.-L. Lin and S. Skiena. Algorithms for square roots of graphs. *SIAM J. Discrete Math.*, 8:99–118, 1995.
- [12] S. T. McCormick. Optimal approximation of sparse Hessians and its equivalence to a graph coloring problem. *Math. Programming*, 26(2):153–171, 1983.
- [13] R. Motwani and M. Sudan. Computing roots of graphs is hard. *Discrete Appl. Math.*, 54:81–88, 1994.
- [14] C. S. J. A. Nash-Williams. Decomposition of finite graphs into forests. *J. London Math. Soc.*, 39(12), 1964.
- [15] S. Ramanathan and E. L. Lloyd. On the complexity of distance-2 coloring. In *Proc. 4th Int. Conf. Comput. and Inform.*, pages 71–74, May 1992.
- [16] S. Ramanathan and E. L. Lloyd. Scheduling algorithms for multi-hop radio networks. *IEEE/ACM Trans. on Networking*, 1(2):166–172, April 1993.
- [17] A. Sen and M. L. Huson. A new model for scheduling packet radio networks. *Wireless Networks*, 3:71–82, 1997.
- [18] A. Sen and E. Malesinska. On approximation algorithms for radio network scheduling. Technical Report 96–08, Dept. Computer Science and Engineering, Arizona State University, 1996.
- [19] G. Wegner. Graphs with given diameter and a coloring problem. Technical report, University of Dortmund, 1977.
- [20] X. Zhou, Y. Kanari, and T. Nishizeki. Generalized vertex-colorings of partial k -trees. *IEICE Trans. Fundamentals*, E83-A:671–678, 2000.