Computing Wireless Capacity*

Magnús M. Halldórsson† Roger Wattenhofer‡

September 27, 2010

Abstract

In this paper we address two common questions in wireless communication from an algorithmic perspective, assuming a geometric path loss model: First, how long does it take to satisfy an arbitrary set of wireless communication requests? Second, given a set of arbitrary communication requests, how many of them can be transmitted concurrently. Our main results are efficient algorithms that find a constant approximation for the second problem, and a logarithmic approximation for the first problem. In addition we present a robustness result, showing that constant parameter and model changes will modify a result only by a constant.

1 Introduction

Despite the omnipresence of wireless networks, surprisingly little is known about their algorithmic complexity and efficiency: Designing and tuning a wireless network is a matter of experience, regardless whether it is a Wireless LAN in an office building, a GSM phone network, or a sensor network on a volcano.

We are interested in the fundamental communication limits of wireless networks. In particular, we would like to know what communication throughput can possibly be achieved. This question essentially boils down to spatial reuse, i.e., which devices can transmit concurrently, without interfering. More precisely, formulated as an optimization problem: Given a set of communication requests, how many of them can be scheduled concurrently, or how much time does it take to schedule all of them?

Evidently the answer to these questions depends on the wireless transmission model. In the past, computational research has focused on graph-based models, also known as protocol models. Unfortunately, graph-based models are too simplistic. Consider for instance a case of three wireless transmissions, every two of which can be scheduled

*A draft of this paper appeared in [16]. This work has been submitted to the IEEE for possible publication.

†School of Computer Science, Reykjavik University, 103 Reykjavik, Iceland. Work done while visiting Research Institute for Mathematical Sciences (RIMS) at Kyoto University. Supported in part by Icelandic Research Fund grant. Email: mmh@ru.is

‡Computer Engineering and Networks Laboratory, ETH Zurich, Switzerland. Email: wattenhofer@tik.ee.ethz.ch

concurrently without a conflict. In a graph-based model one will conclude that all three transmissions may be scheduled concurrently as well, while in reality this might not be the case since wireless signals sum up. Instead, it may be that two transmissions *together* generate too much interference, hindering the third receiver from correctly receiving the signal of its sender. This *many-to-many* relationship makes understanding wireless transmissions difficult – a model where interference sums up seems paramount to truly comprehending wireless communication. Similarly, a graph-based model oversimplifies wireless attenuation. In graph-based models the signal is "binary", as if there was an invisible wall at which the signal immediately drops. Not surprisingly, in reality the signal decreases gracefully with distance.

In contrast to the algorithmic ("CS") community which focuses on graph-based models, researchers in information, communication, or network theory ("EE") are working with wireless models that sum up interference and respect attenuation. One basic model is the signal-to-interference-plus-noise (SINR) model with pure geometric loss – we will formally introduce it in Section 3. The SINR model is reflecting the physical reality more precisely, it is therefore sometimes simply called the physical model. On the other hand, researchers in wireless communications are often not really looking for computational/algorithmic results. Instead, they propose heuristics that are evaluated by simulation. Analytical work is only done for special cases, e.g. the network has a grid structure, or traffic is random. However, these special cases do neither give insights into the complexity of the problem, nor do they give algorithmic results that may ultimately lead to new protocols.

The specific questions we are addressing in this paper are two classic questions in wireless communication: Given a set of arbitrary communication requests, (i) how many of them can be scheduled concurrently, and (ii) how long does it take to schedule all of them? These problems are at the heart of wireless communication, as they are the key to understanding the capacity of a wireless network. Our solution is hopefully pleasing to the EE community as it is using their models, and it is hopefully pleasing to the CS community because we make no restrictions on the input. We can solve the first problem asymptotically optimally. The solution of the first problem then directly leads to an understanding of the second problem. In particular it gives an approximation which is optimal up to a factor that is logarithmic in the number of requests.

Our third contribution is a proof of robustness of the SINR model with geometric path loss. One may argue that in reality path loss will not follow the perfect geometric pattern given by d^α. Instead, several external influences will have effects, e.g., antenna gain may be higher in some directions, obstacles may influence attenuation, or noise may be location dependent. We show that as long as influences are constant, results will only be affected by a constant. As such, the SINR model is *robust*. This result holds in a variety of settings, including power controlled transmissions.

2 Related Work

Most work in wireless scheduling in the physical (SINR) model is of heuristic nature, e.g. [3, 6]. Only after the work of Gupta and Kumar [13], analytical results became *en vogue*. The analytical results however are restricted to networks with a well-behaving topology and traffic pattern. On the one hand this restriction keeps the math involved tractable, on the other hand, it allows for presenting the results in a concise form, i.e., "the throughput capacity of a wireless network with X and Y is Z", where X and Y are some parameters defining the network, and Z is a function of the network size. This area of research has been exceptionally popular, with a multi-dimensional parameter space (e.g. node distribution, traffic pattern, transport layer, mobility), and consequently literally thousands of publications. The intrinsic problem with this line of research is that real networks often do not resemble the models studied here, so one cannot learn much about the capacity of a *real* network. Moreover, it is difficult to deduce protocols since the results are not algorithmic.

In contrast there is a body of algorithmic work, however, mostly on graph-based models. Studying wireless communication in graph-based models commonly implies studying some variants of independent set, matching, or coloring [22, 30]. Although these algorithms present extensive theoretical analysis, they are constrained to the limitations of a model that ultimately abstracts away the nature of wireless communication. The inefficiency of graph-based protocols in the SINR model is well documented and has been shown theoretically as well as experimentally [12, 23, 27].

Algorithmic work in the SINR model is fairly new; to the best of our knowledge it was started just four years ago [26]. In this paper Moscibroda et al. present an algorithm that successfully schedules a set of links (carefully chosen to strongly connect an arbitrary set of nodes) in polylogarithmic time, even in arbitrary worst-case networks. In contrast to our work the links themselves are *not* arbitrary, but do have structure that will simplify the problem. The results of this work has been improved [24, 21], and extended and applied to topology control [7, 28], sensor networks [24], combined scheduling and routing [5], ultra-wideband [17], or analog network coding [11], just to name a few. Apart from these papers, algorithmic SINR results also started popping up here and there, for instance in a game theoretic context or a distributed algorithms context, e.g., [1, 2, 4, 9, 18, 29].

So far there are only a few papers that tackle the general problem of scheduling arbitrary wireless links. Goussevskaia et al. give a simple proof that the problem is NP-complete [10], and test popular heuristics [25]. Both papers also present approximation algorithms, however, in both cases the approximation ratio may grow linearly with the network size.

In this paper we present the first results that provide approximation guarantees independent of the topology of the network. This paper fixes several minor plus one larger mistake (an erroneous claim on the scheduling complexity) from the preliminary conference version [16]. Our main contributions are:

- Given an arbitrary set of requests, we present a simple greedy algorithm that can will choose a subset of the requests that can be transmitted concurrently without violating the SINR constraints. This subset is guaranteed to be within a constant factor of the optimal subset.
- Furthermore, by applying that single-slot subroutine repeatedly we realize a $O(\log n)$ approximation for the problem of minimizing the number of time slots needed to schedule a given set of arbitrary requests.

- Finally, we are able to present a quite general robustness result, showing that constant parameter and model changes will modify the result only by a constant.
- All our results rely on a new definition to understand physical interference: affectance. This definition has been proved to be of general utility for analyzing algorithms in the SINR context, both for scheduling with fixed-but-different power assignments [20, 15] and in power controlled scheduling [14, 19, 15].

As in [16], we use several results and techniques of our previous work [8], giving a self-contained and simplified treatment.

In very recent work, Halldórsson and Mitra [15] have extended our results to obtain a constant factor approximation of Single-Slot Scheduling in general metrics, that holds for any fixed power assignment that is monotone and sub-linear. Their algorithm and basic analysis builds directly on ours. Kesselheim [19] has obtained a constant factor approximation for the Single-Slot Scheduling with power control, also building on some of the ideas developed here.

3 Notation and Model

Given is a set of links $\ell_1, \ell_2, \ldots, \ell_n$, where each link ℓ_v represents a communication request from a sender s_v to a receiver r_v. We assume the senders and receivers are points in the Euclidean plane; this can be extended to other metrics. The Euclidean distance between two points p and q is denoted $d(p, q)$. The asymmetric distance from link v to link w is the distance from v's sender to w's receiver, denoted $d_{vw} = d(s_v, r_w)$. The length of link ℓ_v is denoted $d_{vv} = d(s_v, r_v)$. We shall assume for simplicity of exposition that all links are of different length; this does not affect the results. We assume that each link has a unit-traffic demand, and model the case of non-unit traffic demands by replicating the links. We also assume that all nodes transmit with the same power level P. We show later how to extend the results to variable power levels, with a slight increase in the performance ratio.

We assume the *path loss radio propagation* model for the reception of signals, where the received signal from transmitter w at receiver v is $P_{wv} = P/d_{wv}^{\alpha}$ and $\alpha > 2$ denotes the path-loss exponent. When $w \neq v$, we write $I_{wv} = P_{wv}$. We adopt the *physical interference model*, in which a node r_v successfully receives a message from a sender s_v if and only if the following condition holds:

$$\frac{P_{vv}}{\sum_{\ell_w \in S \setminus \{\ell_v\}} I_{wv} + N} \geq \beta, \tag{1}$$

where N is the ambient noise, β denotes the minimum SINR (signal-to-interference-plus-noise-ratio) required for a message to be successfully received, and S is the set of concurrently scheduled links in the same channel or *slot*. We say that S is *SINR-feasible* if (1) is satisfied for each link in S.

The problems we treat are the following. In all cases are we given a set of links of arbitrary lengths. In the Scheduling problem, we want to partition the set of input links into minimum number of SINR-feasible sets, each referred to as a *slot*. In the

Single-Shot Scheduling problem, we seek the maximum cardinality subset of links that is SINR-feasible. Let χ denote the minimum number of slots in an SINR-feasible schedule.

We make crucial use of the following new definitions.

Definition 3.1 *The* relative interference *(RI) of link ℓ_w on link ℓ_v is the increase caused by ℓ_w in the inverse of the SINR at ℓ_v, namely $RI_w(v) = I_{wv}/P_{vv}$. For convenience, define $RI_v(v) = 0$. Let $c_v = \frac{1}{1-\beta N/P_{vv}}$ be a node-dependent constant that indicates the extent to which the ambient noise approaches the required signal at receiver r_v. The* affectance[1] *of link ℓ_v, caused by a set S of links, is the sum of the relative interferences of the links in S on ℓ_v, scaled by c_v, or*

$$a_S(\ell_v) = c_v \cdot \sum_{\ell_w \in S} RI_w(v).$$

*For a single link ℓ_w, we use the shorthand $a_w(\ell_v) = a_{\{\ell_w\}}(\ell_v)$. We define a p-*signal *set or schedule to be one where the affectance of any link is at most $1/p$.*

The constant c_v is monotone increasing with the length of the link: $d_{vv} \geq d_{ww}$ implies that $c_v \geq c_w$. Note that $c_v \leq 1$, with equality holding only in the absence of noise.

Observation 3.2 *The affectance function satisfies the following properties for a set S of links:*

1. (Range) *S is SINR-feasible if and only if, for all $\ell_v \in S$, $a_S(\ell_v) \leq 1/\beta$.*
2. (Additivity) *$a_S = a_{S_1} + a_{S_2}$, whenever (S_1, S_2) is a partition of S.*
3. (Distance bound) *$a_w(\ell_v) = c_v \cdot \left(\frac{d_{vv}}{d_{wv}}\right)^\alpha$, for any pair ℓ_w, ℓ_v in S.*

Note that the concepts of affectance and relative interference are equally useful in contexts of non-uniform power assignments. If P_v is the power of link ℓ_v, the affectance of link ℓ_v on ℓ_v is given by $a_w(\ell_v) = c_v \cdot \left(\frac{P_w/d_{wv}}{P_v/d_{vv}}\right)$.

4 Robustness of SINR

We present here properties of schedules in the SINR model, which double as tools for the algorithm designer. The results of this section apply equally to scheduling links of different powers. In the next subsection, we examine the desirable property of link dispersion, and how any schedule can be dispersed at a limited cost.

We now explore how signal requirements (in the value of β), or equivalently interference tolerance, affects schedule length. It is not *a priori* obvious that minor discrepancies

[1] Affectance is closely related to *affectedness*, defined in [8], but treats the effect of noise more accurately.

cause only minor changes in schedule length, but by showing that it is so, we can give our algorithms the advantage of being compared with a stricter optimal schedule. This also has implications regarding the robustness of SINR models with respect to perturbations in signal transmissions.

The pure geometric version of SINR given in (1) is an idealization of true physical characteristics. It assumes, e.g., perfectly isotropic radios, no obstructions, and a constant ambient noise level. That begs the question, why move algorithm analysis from analytically amenable graph-based models to a more realistic model if the latter isn't all that realistic? Fortunately, the fact that schedule lengths are relatively invariant to signal requirements shows that these concerns are largely unnecessary.

The following result on signal requirement applies also to throughput optimization.

Theorem 4.1 *There is a polynomial-time algorithm that takes a p-signal schedule and refines into a p'-signal schedule, for $p' > p$, increasing the number of slots by a factor of at most $\lceil 2p'/p \rceil^2$.*

Proof: Consider a p-signal schedule $\mathcal{S}$ and a slot S in $\mathcal{S}$. We partition S into a sequence $S_1, S_2, \ldots$ of sets. Order the links in S in decreasing order. For each link ℓ_v, assign ℓ_v to the first set S_j for which $a_{S_j}(\ell_v) \leq 1/2p'$, i.e. the accumulated affectance on ℓ_v among the previous, longer links in S_j is at most $1/2p'$. Since each link ℓ_v originally had affectance at most $1/p$, then by the additivity of affectance, the number of sets used is at most $\lceil \frac{1/p}{1/2p'} \rceil = \lceil \frac{2p'}{p} \rceil$.

We then repeat the same approach on each of the sets S_i, processing the links this time in increasing order. The number of sets is again $\lceil \frac{2p'}{p} \rceil$ for each S_i, or $\lceil \frac{2p'}{p} \rceil^2$ in total. In each final slot (set), the affectance on a link by shorter links in the same slot is at most $1/2p'$. In total, then, the affectance on each link is at most $2 \cdot 1/2p' = 1/p'$. □

This result applies in particular to optimal solutions. Let OPT_p be an optimal p-signal schedule and let χ_p be the number of slots in OPT_p. It is not *a priori* clear that a smooth relationship exists between χ_p and χ, for $p > 1$.

Corollary 4.2 $\chi_p \leq \lceil 2p/\beta \rceil^2 \chi$.

This has significant implications. One regards the validity of studying the pure SINR model. As asked in [8], "what if the signal is attenuated by a certain factor in one direction but by another factor in another direction?" A generalized physical model was introduced in [28] to allow for such a deviation.

Theorem 4.1 implies that scheduling is relatively robust under discrepancies in the SINR model. This validates the analytic study of the pure SINR model, in spite of its simplifying assumptions.

Corollary 4.3 *If a scheduling algorithm gives a ρ-approximation in the SINR model, it provides a $O(\theta^2\rho)$-approximation in variations in the SINR model with a discrepancy of up to a factor of θ in signal attenuation or ambient noise levels.*

This result can be contrasted with the strong $n^{1-\epsilon}$-approximation hardness of scheduling in an abstract (non-geometric) SINR model that allows for arbitrary distances between nodes [10]. Alternatively, Theorem 4.1 allows us to analyze algorithms under more relaxed situations than the optimal solutions that we compare to.

It is important to note that these results do not depend on the power assignment and apply equally well in the power-control setting. Also, they actually do not depend on the formula used to compute affectance or relative interference, and apply also in non-geometric and non-metric settings.

Remark: Note that the converse of Theorem 4.1 – that a schedule can be shortened by a constant factor so that the signal decreases only by a constant factor – does not hold. An easy example is found by duplicating a feasible set S by any number t of copies (possibly separating the nodes by a sufficiently small distance). Any attempt to use fewer than t slots results in an arbitrarily bad signal.

4.1 Dispersion properties

One desirable property of schedules is that links in the same slot be spatially well separated. This blurs the difference in position between sender and receiver of a link, since it affects distances only by a small constant. Intuitively, we want to measure nearness as a fraction of the lengths of the respective links. Given the affectance measure, it proves to be useful to define nearness somewhat less restrictively.

Definition 4.4 *Link ℓ_w is said to be q-*near *link ℓ_v, if $d_{wv} < q \cdot c_v^{1/\alpha} \cdot d_{vv}$. A set of links is q-*dispersed *if no (ordered) pairs of links in the set are q-near.*

Observation 3.2, item 3, states that link ℓ_w is q-near a link ℓ_v if and only if $a_w(\ell_v) > q^{-\alpha}$. This immediately gives the following strengthening of Lemma 4.2 in [8].

Lemma 4.5 *Fewer than q^α/β senders in an SINR-feasible set S are q-near to any given link $\ell_v \in S$.*

For constant q, α, any schedule can be made dispersed at a cost of a constant factor.

Lemma 4.6 *There is a polynomial-time algorithm that takes a SINR-feasible schedule and refines it into a q-dispersed schedule, increasing the number of slots by a factor of at most $\lceil (q+2)^\alpha \rceil$.*

Proof: Let S be a slot in the schedule. We show how to partition S into sets $S_1, S_2, \ldots, S_t$ that are q-dispersed, where $t \leq (q+2)^\alpha + 1$.

Process the links of S in increasing order of length, assigning each link ℓ_v "first-fit" to the first set S_j in which the receiver r_v is at least $\left(qc_v^{1/\alpha} + 2\right) \cdot d_{vv}$ away from any other link. Let ℓ_w be a link previously in S_j, and note that ℓ_w is shorter than ℓ_v. By the selection rule, $d_{wv} \geq \left(qc_v^{1/\alpha} + 2\right) \cdot d_{vv} > qc_v^{1/\alpha} \cdot d_{vv}$. Also,

$$d_{vw} \geq d_{wv} - d_{ww} - d_{vv} \geq \left(qc_v^{1/\alpha} + 1\right) d_{vv} - d_{ww} \geq qc_v^{1/\alpha} d_{ww} \geq qc_w^{1/\alpha} d_{ww}.$$

Since this holds for every pair in the same set, the schedule is q-dispersed. Suppose S_t is the last set used by the algorithm, and let ℓ_v be a link in it. Then, each S_i, for $i = 1, 2, \ldots, t-1$, contains a link whose sender is closer than $(qc_v^{1/\alpha} + 2) \cdot d_{vv} \leq (q+2)c_v^{1/\alpha} d_{vv}$ to r_v, i.e., is $(q+2)$-near to ℓ_v. By Lemma 4.5, $t-1 < (q+2)^\alpha/\beta$. Hence, $t \leq \lceil (q+2)^\alpha/\beta \rceil$. $\square$

Intuitively, there is a correlation between low affectance and high dispersion in schedules. The following result makes this connection clearer. The converse is, however, not true, since high interference can be caused by shorter far-away links.

Lemma 4.7 *A p-signal schedule is also $p^{1/\alpha}$-dispersed.*

Proof: Let ℓ_v and ℓ_w be an ordered pair of links in a slot S in a p-signal schedule. By definition, $a_w(\ell_v) \leq a_S(\ell_v) \leq 1/p$. By Observation 3.2, item 3, $d_{wv} \geq (c_v p)^{1/\alpha} \cdot d_{vv}$. $\square$

5 Approximation Algorithms

The algorithm we analyze is a simplified version of the algorithm of [8].

Let $C = 2^3 9 = 72$, $\tau = 2 + \max\left(2, \left((C+1)\beta\frac{\alpha-1}{\alpha-2}\right)^{\frac{1}{\alpha}}\right)$, and $c = 1/\tau^\alpha$.

```
A(c)
  sort the links ℓ_1, ℓ_2, ..., ℓ_n by non-decreasing order of length
  S ← ∅
  for v ← 1 to n do
     if (a_S(ℓ_v) ≤ c)
        add ℓ_v to S
  output S
```

It is rather surprising that a $O(1)$-approximation algorithm can be obtained in a single sweep. This should help in applying the ideas further, e.g., in distributed implementations. Note that recent research shows that such a single sweep is also feasible when using power control [19].

Simulation results in [8] also indicate very good practical performance, in relation to previous algorithms, and the simplification given here is likely to perform at least as well.

It is not immediate that algorithm **A** produces a feasible solution.

Lemma 5.1 *Algorithm **A** produces a $(\tau-2)$-dispersed solution.*

Proof: Let ℓ_w be a link in the set S output by algorithm **A**. Let N_w^- (N_w^+) be the set of links in S that are shorter (longer) than ℓ_w. Consider first a link $\ell_u \in N_w^-$. Since ℓ_w was added by the algorithm after adding ℓ_u, $a_u(\ell_w) \leq c = 1/\tau^\alpha$, which implies by Observation 3.2, item 3, that $d_{uw} \geq \tau c_w^{1/\alpha} d_{ww} > (\tau-2)c_w^{1/\alpha} d_{ww}$. Consider next a link $\ell_v \in N_w^+$. Since ℓ_v was added after ℓ_w, it holds that $a_w(\ell_v) \leq c = 1/\tau^\alpha$. So by Observation 3.2, $d_{wv} \geq \tau \cdot c_v^{1/\alpha} d_{vv}$. Again we use that $c_v \geq c_w$ whenever $d_{vv} \geq d_{ww}$. Then, using the triangular inequality,

$$d_{vw} = d(s_v, r_w) \geq d_{wv} - d_{vv} - d_{ww} \geq \left(\tau c_v^{1/\alpha} - 2\right) d_{vv} \geq (\tau-2)c_w^{1/\alpha} d_{ww}.$$

Since this holds for every ordered pair in S, we have that S is $(\tau-2)$-dispersed. □

The following appeared as part of Lemma 4.1 in [8].

Lemma 5.2 *Let S be a Z-dispersed feasible set, where $Z \geq 2$. Then, for any link ℓ_v in S, it holds that*

$$a_{S_v^+}(\ell_v) < \left(\frac{\alpha-1}{\alpha-2}C\right) Z^{-\alpha},$$

where S_v^+ is the set of links in S at least as long ℓ_v.

Proof: Let $z = Zc_v^{1/\alpha}$. Form discs D_w of radius $r = (z-1)d_{vv}/2$ around each sender s_w in S_v^+. We claim that these discs are disjoint. By the dispersion property, the distance from any sender $s_u \in S$ to any receiver $r_w \in S_v^+$, $w \neq u$, is at least $Zc_w^{1/\alpha} d_{ww} \geq z d_{ww}$, using that $c_w \geq c_v$ since $\ell_w \geq \ell_v$. It follows by the triangular inequality that the separation between two senders s_u, s_w in S is at least $(z-1)d_{ww} \geq (z-1)d_{vv} = r$, and hence the discs are disjoint.

We next partition the sender set in S_v^+ into concentric rings R_k of width $z \cdot d_{vv}$ around the receiver r_v. Each ring R_k contains all senders $s_w \in S_v^+$ satisfying $k(z \cdot d_{vv}) \leq d_{wv} \leq (k+1)(z \cdot d_{vv})$. We know that the first ring R_0 contains no sender. Consider all senders $s_w \in R_k$ for some integer $k > 0$. are contained in an annulus A_k centered at r_v of width $zd_{vv} + 2r = (2z-1)d_{vv}$ that has r added both to the inside and outside of R_k. The area of A_k is

$$\begin{aligned} A(A_k) &= \left[(d_{vv}(k+1)z + r)^2 - (d_{vv}kz - r)\right]\pi \\ &= (2k+1)d_{vv}^2 z(2z-1)\pi. \end{aligned}$$

Since discs D_w of area $A(D_w) = r^2\pi$ around senders in S_v^+ do not intersect, and the minimum distance between r_v and $s_w \in R_k, k > 0$ is $k(z \cdot d_{vv})$, we can use an area argument to bound the number of senders inside each ring. The total relative interference from senders in $R_k, k \geq 1$ on ℓ_v is bounded by

$$RI_{R_k}(\ell_v) \leq \sum_{s_w \in R_k} RI_{s_w}(\ell_v) \leq \frac{A(A_k)}{A(D_w)} \cdot \frac{1}{(kz)^\alpha} \leq \frac{(2k+1)}{k^\alpha} \cdot \frac{4}{z^\alpha} \frac{z(2z-1)}{(z-1)^2} \leq \frac{1}{k^{(\alpha-1)}} \cdot \frac{2^3 9}{z^\alpha}.$$

where the last inequality holds since $k \geq 1 \Rightarrow 2k+1 \leq 3k$ and $z \geq 2 \Rightarrow z-1 \geq z/2$ and $2z-1 \leq 3(z-1)$. Summing up the interferences over all rings yields

$$RI_{S_v^+}(\ell_v) < \sum_{k=1}^{\infty} RI_{R_k}(\ell_v) \leq \sum_{k=1}^{\infty} \frac{1}{k^{\alpha-1}} \cdot \frac{C}{z^\alpha} < \frac{\alpha-1}{\alpha-2} \cdot \frac{C}{z^\alpha},$$

where the last inequality holds since $\alpha > 2$. This results in affectance of

$$a_{S_v^+}(\ell_v) = \frac{\beta I_{S_v^+}(\ell_v)}{P_v(v)} < \frac{\alpha-1}{\alpha-2} \cdot \frac{C\beta}{z^\alpha} ,$$

as claimed. □

Theorem 5.3 *Algorithm **A** produces an SINR-feasible solution.*

Proof: Let ℓ_w be a link in the set S output by algorithm **A**. Let S_w^- (S_w^+) be the set of links in S that are shorter (longer) than ℓ_w. The links in S_w^- were processed before ℓ_w, so by the if-condition in the algorithm, $a_{S_w^-}(\ell_v) \le c$. Note that $c \le \frac{1}{(C+1)\beta}$. By Lemma 5.1, S is $\tau - 2$-dispersed, so by Lemma 5.2 and the definitions of τ and dispersion,

$$a_{S_w^+}(\ell_w) < \left(\frac{\alpha-1}{\alpha-2} C\right) \frac{1}{(\tau-2)^\alpha} \le \frac{C}{(C+1)\beta}.$$

Hence, the affectance of each link in S is at most $a_{S_w^-}(\ell_v) + a_{S_w^+}(\ell_v) \le 1/\beta$. □

5.1 Performance analysis

We use a geometric result from [8].

Definition 5.4 *Let $\mathcal{R}$ and $\mathcal{B}$ be disjoint pointsets in a metric space $(\mathcal{V}, d)$, referred to as the red and blue points, respectively. A point $b \in \mathcal{B}$ is* blue-dominant *if every ball $B_\delta(b)$ around b, comprised by points w such that $d(w,b) \le \delta$, contains $q \in \mathbb{Z}^+$ times more blue points than red points. Formally, $\forall \delta \in \mathbb{R}_0^+ : |B_\delta(b) \cap \mathcal{B}| > |B_\delta(b) \cap \mathcal{R}|$.*

For a red point $r \in \mathcal{R}$ and a set $G \subseteq \mathcal{B}$ of blue points, we say that G guards *r if for all $b \in \mathcal{B} \setminus G$, we have that $B_{d(b,r)}(b) \cap G \neq \emptyset$.*

Lemma 5.5 *(Blue-dominant centers lemma in 2D) Let $\mathcal{R}$ and $\mathcal{B}$ be disjoint sets of red and blue points in a 2-dimensional Euclidean space. If $|\mathcal{B}| > 5 \cdot |\mathcal{R}|$ then there exists at least one blue-dominant point in $\mathcal{B}$.*

Proof: Process the points in $\mathcal{R}$ in an arbitrary order while maintaining a subset $\mathcal{B}'$ of $\mathcal{B}$ as follows (initially, $\mathcal{B}' = \mathcal{B}$). For each $r \in \mathcal{R}$, we construct a guarding set $G(r) \subseteq \mathcal{B}'$ (guarding r relative to the current $\mathcal{B}'$) and remove $G(r)$ from $\mathcal{B}'$.

We claim that it is possible to construct a guarding set $G(r)$ of size at most 5. The procedure to construct $G(r)$ is as follows. Consider a red point r. Include a closest blue point $b_1 \in \mathcal{B}'$ in $G(r)$. Draw 5 sectors originating at r in the following manner. The first sector has 120° and is centered at b_1, the remaining 4 sectors have 60° each and evenly divide the remaining area around r. For each of these 4 sectors sec_j, include the closest blue point $b_j \in sec_j$ in $G(r)$ (if sec_j has no blue points from $\mathcal{B}'$, skip this sector). Now $G(r)$ has size at most 5 and we claim that it is guarding r. Suppose not. Then, there is a point $b^* \in \mathcal{B}' \setminus G(r)$ with $B_{d(b^*,r)}(r) \cap G(r) = \emptyset$. Suppose b^* is located in sec_j

and we selected blue point b_j from sec_j into $G(r)$. This means that $d(b^*, b_j) > d(b^*, r)$, which implies that the sector angle is larger than 60°. (Note that if $G(r)$ contains no point b_j from sector sec_j, then b^* would have been picked to guard r in that sector, also establishing a contradiction.)

After going through all the points in $\mathcal{R}$, the set $\mathcal{B}'$ is still nonempty by the assumption on the relative sizes of $\mathcal{R}$ and $\mathcal{B}$. We claim that every point in $\mathcal{B}'$ is now blue-dominant. This holds since (1) the guarding sets of point in $\mathcal{R}$ are pairwise disjoint and (2) every ball $B_\delta(b^*), b^* \in \mathcal{B}'$, that contains a red point r, contains also a blue point in $G(r)$. Hence, for every blue node $b^* \in \mathcal{B}'$, every ball $B_\delta(b^*)$ contains more blue points than red points ("more", since the center b^* is also blue). □

The following lemma builds on Lemma 4.5 of [8].

Lemma 5.6 *Let* $\nu = 2(3\tau/2)^\alpha$ *be a constant. Let* ALG *be the solution output by algorithm* $\boldsymbol{A}$ *on the given instance and* OPT_ν *be an optimal* ν*-signal solution. Then,* $|OPT_\nu| \leq 5|ALG|$.

Proof: Let $\mathcal{R} = \{s_w | \ell_w \in ALG \setminus OPT_\nu\}$ and $\mathcal{B} = \{s_v | \ell_v \in OPT_\nu \setminus ALG\}$ be the sets of senders in exactly one of ALG and OPT_ν; we call them red and blue points, respectively. Suppose the claim is false. It follows that $|\mathcal{B}| > 5|\mathcal{R}|$. By Lemma 5.5, there is a blue-dominant s_b in $\mathcal{B}$. We shall argue that the blue link $\ell_b = (s_b, r_b)$ would have been picked by our algorithm, which is a contradiction.

Consider any red point $s_x \in \mathcal{R}$. Let $D = d(s_x, s_b)$. Let s_y denote the guard for s_x w.r.t. s_b, i.e., the blue point that is closer to s_b than s_x is, i.e., within distance D from s_b. Note that by Lemma 4.7, OPT_ν is a s-dispersed set, where $s = \nu^{1/\alpha} \geq 3\tau/2 \geq 6$. Applying Definition 4.4, we know that $d_{xb} \geq s \cdot c_v^{1/\alpha} \cdot d_{bb}$. Using $c_v \geq 1$, we get $d_{xb} \geq 6d_{bb}$. The guarding property and the triangular inequality ensure that

$$d_{yb} \leq d(s_y, s_b) + d_{bb} \leq D + d_{bb} \leq d_{xb} + 2d_{bb} \leq \frac{4}{3} d_{xb} \ .$$

Thus,

$$RI_x(b) = \left(\frac{d_{bb}}{d_{xb}}\right)^\alpha \leq \left(\frac{4}{3} \cdot \frac{d_{bb}}{d_{yb}}\right)^\alpha = \left(\frac{4}{3}\right)^\alpha RI_y(b) \ .$$

Let t denote $\left(\frac{3}{4}\right)^\alpha$. This holds for any $s_x \in \mathcal{R}$, so the total interference that ℓ_b receives from the red senders (those in ALG) is at least t times that from the blue senders. Since ℓ_b is in OPT_ν, it is affected by at most $1/\nu$ by OPT_ν. Using that each node in OPT_ν participates in at most one guardset, we get that

$$a_{ALG \setminus OPT_\nu}(\ell_b) = c_b \cdot \sum_{s_x \in \mathcal{R}} RI_x(\ell_b) \leq c_b \sum_{\ell_g \in \mathcal{B}} t \cdot RI_g(b) = t \cdot a_{OPT \setminus ALG}(\ell_b) \leq t/\nu < c/2 \ .$$

Further, since OPT_ν is a ν-signal solution, $a_{ALG \cap OPT_\nu}(\ell_b) \leq 1/\nu < c/2$. Thus,

$$a_{ALG}(\ell_b) = a_{ALG \setminus OPT_\nu}(\ell_b) + a_{ALG \cap OPT_\nu}(\ell_b) < c,$$

which contradicts the assumption that ℓ_b was not selected by the algorithm. □

The following result is now immediate from Lemma 5.6 in combination with the correctness result in Theorem 5.3 and the signal-strengthening property of Theorem 4.1.

Theorem 5.7 *Algorithm **A** approximates the* Single-Shot Scheduling *problem within a constant factor.*

5.2 Scheduling approximation

Given the constant factor approximation for the Single-Slot Scheduling problem, it is now standard to argue that a $O(\log n)$-approximation for the Scheduling problem.

Theorem 5.8 *Repeated application of algorithm **A** yields a a $O(\log n)$-approximation for the* Scheduling *problem.*

Proof: Recall that χ is the minimum number of slots in a feasible solution, and let $\rho = O(1)$ be the performance guarantee of $\mathbf{A}$. Any subset S' of the input instance with N links contains a feasible set of size N/χ. Thus, Algorithm $\mathbf{A}$ applied to S' results in a feasible subset of size at least $N/(\rho\chi)$, with the number of remaining unscheduled links becoming at most $N(1 - 1/(\rho\chi))$. Starting with n links, the number of unscheduled links remaining after s iterations is at most $n(1 - 1/(\rho\chi))^s < ne^{-s/(\rho\chi)}$. Thus, when $s \geq \ln n \cdot \rho\chi$, less than one link remains unscheduled, that is, all the links have been scheduled. Hence, $\ln n \cdot \rho\chi$ slots suffice, for an approximation factor of $\rho \ln n$. □

Handling different transmission powers We can treat the case when links transmit with different powers in two different ways. Let P_{max} (P_{min}) be the maximum (minimum) power used by a link, respectively. By introducing a factor of P_{min}/P_{max} into the affectance threshold c, our algorithm still produces a feasible schedule, that is longer by a factor of at most P_{max}/P_{min}.

Alternatively, we can partition the instance into "power regimes", where each regime consists of links whose powers are equal up to a factor of 2. We schedule each power regime separately, obtaining an approximation factor of at most $\log P_{max}/P_{min}$, or at most the number of different power values.

If P_{max}/P_{min} cannot be bounded, and if more generally the number of power levels cannot be bounded, we refer to recent work of [15].

6 Conclusions

The main open question is to obtain a constant factor approximation to the scheduling problem, as originally (and erroneously) claimed in the conference version of this paper [16]. Additionally, various parameter combinations are still open, and deserve more research, e.g. multi-hop traffic, scheduling and routing, analog network coding, stochastic fading models beyond pure geometric gain, such as Rician or Rayleigh fading.

References

[1] V. Auletta, L. Moscardelli, P. Penna, and G. Persiano. Interference games in wireless networks. In *Proceedings of the 4th International Workshop On Internet And Network Economics (WINE)*, volume 5385 of *LNCS*, pages 278–285, 2008.

[2] C. Avin, Y. Emek, E. Kantor, Z. Lotker, D. Peleg, and L. Roditty. SNR diagrams: Towards algorithmically usable SINR models of wireless networks. In *Proceedings of the Twenty-Eighth Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, 2008.

[3] G. Brar, D. Blough, and P. Santi. Computationally Efficient Scheduling with the Physical Interference Model for Throughput Improvement in Wireless Mesh Networks. In *Proceedings of the 12th Annual International Conference on Mobile Computing and Networking (Mobicom)*, 2006.

[4] G. S. Brar, D. M. Blough, and P. Santi. The SCREAM approach for efficient distributed scheduling with physical interference in wireless mesh networks. In *Proceedings of the 28th IEEE International Conference on Distributed Computing Systems (ICDCS 2008)*, pages 214–224, 2008.

[5] D. Chafekar, V. Kumar, M. Marathe, S. Parthasarathy, and A. Srinivasan. Cross-layer Latency Minimization for Wireless Networks using SINR Constraints. In *Proceedings of the 8th ACM Int. Symposium on Mobile Ad Hoc Networking and Computing (MobiHoc)*, 2007.

[6] T. A. ElBatt and A. Ephremides. Joint scheduling and power control for wireless ad hoc networks. *IEEE Transactions on Wireless Communications*, 3(1):74–85, 2004.

[7] Y. Gao, J. C. Hou, and H. Nguyen. Topology control for maintaining network connectivity and maximizing network capacity under the physical model. In *Proceedings of the 27th Annual IEEE Conference on Computer Communications (INFOCOM)*, 2008.

[8] O. Goussevskaia, M. M. Halldórsson, R. Wattenhofer, and E. Welzl. Capacity of Arbitrary Wireless Networks. In *Proceedings of the 28th Annual IEEE Conference on Computer Communications (INFOCOM)*, April 2009.

[9] O. Goussevskaia, T. Moscibroda, and R. Wattenhofer. Local Broadcasting in the Physical Interference Model. In *Proceedings of the Fifth ACM SIGACT-SIGOPS International Workshop on Foundations of Mobile Computing*, August 2008.

[10] O. Goussevskaia, Y. A. Oswald, and R. Wattenhofer. Complexity in Geometric SINR. In *Proceedings of the Eigth ACM Int. Symposium on Mobile Ad Hoc Networking and Computing (MobiHoc)*, pages 100–109, 2007.

[11] O. Goussevskaia and R. Wattenhofer. Complexity of scheduling with analog network coding. In *Proceedings of the 1st ACM International Workshop on Foundations of Wireless Ad Hoc and Sensor Networking and Computing (FOWANC)*, May 2008.

[12] J. Gronkvist and A. Hansson. Comparison between graph-based and interference-based STDMA scheduling. In *Proceedings of the Second ACM Int. Symposium on Mobile Ad Hoc Networking and Computing (MobiHoc)*, pages 255–258, 2001.

[13] P. Gupta and P. R. Kumar. The Capacity of Wireless Networks. *IEEE Trans. Information Theory*, 46(2):388–404, 2000.

[14] M. M. Halldórsson. Wireless scheduling with power control. `www.ru.is/faculty/mmh/papers/esa09full.pdf`. Earlier version appears in ESA '09, September 2010.

[15] M. M. Halldórsson and P. Mitra. Wireless Capacity with Oblivious Power in General Metrics. In *Proceedings of the 22nd ACM-SIAM Symposium on Discrete Algorithms (SODA)*, January 2011.

[16] M. M. Halldórsson and R. Wattenhofer. Wireless Communication is in APX. In *Proceedings of the 37th International Conference on Algorithms, Languages and Programming (ICALP)*, July 2009.

[17] Q.-S. Hua and F. C. M. Lau. The scheduling and energy complexity of strong connectivity in ultra-wideband networks. In *Proceedings of the 9th International Symposium on Modeling Analysis and Simulation of Wireless and Mobile Systems (MSWiM)*, pages 282–290, 2006.

[18] B. Katz, M. Völker, and D. Wagner. Link scheduling in local interference models. In *Proceedings of the Fourth International Workshop on Algorithmic Aspects of Wireless Sensor Networks (ALGOSENSORS)*, pages 57–71, 2008.

[19] T. Kesselheim. A Constant-Factor Approximation for Wireless Capacity Maximization with Power Control in the SINR Model. In *Proceedings of the 22nd ACM-SIAM Symposium on Discrete Algorithms (SODA)*, January 2011.

[20] T. Kesselheim and B. Vöcking. Distributed contention resolution in wireless networks. In *Proceedings of the 24th International Symposium on Distributed Computing (DISC)*, pages 163–178, 2010.

[21] D. R. Kowalski and M. A. Rokicki. Connectivity problem in wireless networks. In *Proceedings of the 24th International Symposium on Distributed Computing (DISC), Boston, Massachusetts*, pages 344–358, 2010.

[22] V. Kumar, M. Marathe, S. Parthasarathy, and A. Srinivasan. Algorithmic Aspects of Capacity in Wireless Networks. In *Proceedings of the International Conference on Measurements and Modeling of Computer Systems (SIGMETRICS)*, 2005.

[23] R. Maheshwari, S. Jain, and S. R. Das. A measurement study of interference modeling and scheduling in low-power wireless networks. In *Proceedings of the 6th International Conference on Embedded Networked Sensor Systems (SenSys)*, pages 141–154, 2008.

[24] T. Moscibroda. The worst-case capacity of wireless sensor networks. In *Proceedings of the 6th International Conference on Information Processing in Sensor Networks (IPSN)*, pages 1–10, 2007.

[25] T. Moscibroda, Y. A. Oswald, and R. Wattenhofer. How optimal are wireless scheduling protocols? In *Proceedings of the 26th Annual IEEE Conference on Computer Communications (INFOCOM)*, pages 1433–1441, 2007.

[26] T. Moscibroda and R. Wattenhofer. The Complexity of Connectivity in Wireless Networks. In *Proceedings of the 25th Annual IEEE Conference on Computer Communications (INFOCOM)*, 2006.

[27] T. Moscibroda, R. Wattenhofer, and Y. Weber. Protocol Design Beyond Graph-Based Models. In *Proceedings of the Fifth Workshop on Hot Topics in Networks (HotNets)*, November 2006.

[28] T. Moscibroda, R. Wattenhofer, and A. Zollinger. Topology Control meets SINR: The Scheduling Complexity of Arbitrary Topologies. In *Proceedings of the 6^{th} ACM Int. Symposium on Mobile Ad Hoc Networking and Computing (MobiHoc)*, 2006.

[29] C. Scheideler, A. W. Richa, and P. Santi. An $O(\log n)$ dominating set protocol for wireless ad-hoc networks under the physical interference model. In *Proceedings of the 9^{th} ACM Int. Symposium on Mobile Ad Hoc Networking and Computing (MobiHoc)*, pages 91–100, 2008.

[30] G. Zussman, A. Brzezinski, and E. Modiano. Multihop Local Pooling for Distributed Throughput Maximization in Wireless Networks. In *Proceedings of the 27th Annual IEEE Conference on Computer Communications (INFOCOM)*, 2008.