Improved Approximation Results for the Stable Marriage Problem*

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Abstract

The stable marriage problem has recently been studied in its general setting, where both ties and incomplete lists are allowed. It is NP-hard to find a stable matching of maximum size, while any stable matching is a maximal matching and thus trivially we can obtain a 2-approximation algorithm.

In this paper, we give the first nontrivial result for approximation of factor less than two. Our algorithm achieves an approximation ratio of $2/(1+L^{-2})$ for instances in which only men have ties of length at most L. When both men and women are allowed to have ties, but the lengths are limited to two, then we show a ratio of $13/7(< 1.858)$. We also improve the lower bound on the approximation ratio to $21/19(> 1.1052)$.

Keywords: Stable marriage problem, Ties, Incomplete lists, Approximation algorithms

1 Introduction

An instance of the *stable marriage problem* consists of N men, N women and each person's preference list. A preference list records the order of preferences (allowing ties) over a subset of the members of the opposite sex. A matching is *stable* if there is no pair that favor each other more than their current partners (if any). A formal definition is given in the next section.

The problem of finding a stable matching of maximum size was recently proved to be NP-hard [19], which also holds for several restricted cases such as the case that all ties occur only in one sex, are of length two and every person's list contains at most one tie [24]. The hardness result has been further extended to APX-hardness [12]. Since a stable matching is a maximal matching, the sizes of any two stable matchings for an instance differ by a factor at most two. Hence, any stable matching is a 2-approximation; yet, there is no known approximation algorithm with approximation ratio strictly better than 2.

Our Contribution In this paper, we give nontrivial upper and lower bounds on approximating a maximum cardinality solution. On the negative side, we show that the problem is hard to approximate

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within a factor of $21/19 (> 1.1052)$. This is obtained via an approximability relation with Minimum Vertex Cover. If the strong conjecture of $(2-\epsilon)$-hardness for Minimum Vertex Cover holds, then our lower bound will be improved to 1.25.

On the positive side, we give an algorithm called ShiftBrk, which is based on the following simple idea. Suppose, for simplicity, that all ties are of the same length L (but the following results hold for the case where all ties are of length *at most* L). Then ShiftBrk first breaks all the ties into an arbitrary order and obtains a stable marriage instance *without ties*. Then we "shift" cyclically the order of all the originally tied women in each man's list simultaneously, creating L different instances. For each of them, we in turn apply the shift operation against the ties of women's lists, obtaining L^2 instances in total. We finally compute L^2 stable matchings for these L^2 instances in polynomial time, all of which are stable in the original instance [11], and select a largest solution. We prove the following: (i) ShiftBrk achieves an approximation ratio of $2/(1+L^{-2})$ (1.6 and 1.8 when $L=2$ and 3, respectively) if the given instance includes ties in only men's (or women's) lists. (Note that in this case, ShiftBrk constructs only L instances.) We also give a tight example for this analysis. (ii) When both men and women are allowed to include ties, but only of length two, in their preference lists, it achieves an approximation ratio of $13/7 (< 1.858)$. We conjecture that ShiftBrk also achieves a factor of less than two for general instances with $L \geq 3$.

Related Work The stable marriage problem has great practical significance. One of the best known applications is to assign medical students to hospitals based on the preferences of students over hospitals and vice versa, examples of which are NRMP in the US [11], CaRMS in Canada, and SPA in Scotland [17]. Another application, reported in [28], is the assignment of students to secondary schools in Singapore.

The stable marriage problem was first introduced by Gale and Shapley in 1962 [9]. In its original definition, each preference list must include all members of the opposite sex, and the preferences must form a total order. They proved that every instance admits a stable matching, and gave an $O(N^2)$-time algorithm to find one, which is called the Gale-Shapley algorithm. Even if ties are allowed in a list, it is easy to find a perfect stable matching using the Gale-Shapley algorithm [11]. If we allow persons to exclude unacceptable partners from the list, the stable matching may no longer be a perfect matching. However, it is well known that, in the absence of ties, all stable matchings for the same instance are of the same size [10]. Again, it is easy to find a stable matching by the Gale-Shapley algorithm [10]. Hence, the problem of finding a maximum stable matching is trivial in all these three variations, while the situation changes if both ties and incomplete lists are allowed, as mentioned before. The only known non-trivial approximation algorithms are a randomized one for restricted instances [13], and a $(2-c\frac{1}{\sqrt{N}})$-approximation algorithm where c is a constant such that $c \leq \frac{1}{4\sqrt{6}}$ [20].

When ties are allowed in the lists, there are two other notions of stability: super-stability and strong stability (in this context, the definition above is sometimes called *weak stability*). In both cases, there can be instances that do not have a stable matching but there are polynomial-time algorithms to find such matchings when they exist [16]. The book by Gusfield and Irving [11] gives a comprehensive survey of results as of the late 1980's. Research on stable matchings has been quite intensive recently, which includes studies on strong stability [18, 23], exchange-stability [5], and others [2, 8, 1].

There are several optimization problems that resemble MAX SMTI, where designing a 2-approximation algorithm is trivial but obtaining a $(2-\epsilon)$-approximation algorithm for a positive constant ϵ appears to be hard, such as Minimum Vertex Cover (MIN VC for short) and Minimum Maximal Matching (MIN MM for short). As with MAX SMTI, there are a lot of approximability results for restricted cases of these problems. For example, MIN VC is approximable within 7/6 if the maximum degree of an input graph is bounded by 3 [4], or within $2/(1+\epsilon)$ if every vertex has degree at least $\epsilon|V|$ [22]. For MIN MM, there is a $(2-1/d)$-approximation algorithm for regular graphs with degree d [29], and a PTAS for planar graphs [26]. For general inputs, $(2-o(1))$-approximation algorithms are known for MIN VC, namely, $2 - \frac{\log\log|V|}{2\log|V|}$ and $2 - (1-o(1))\frac{2\ln\ln|V|}{\ln|V|}$ [25, 3, 15]. Very recently, this has been improved to $2 - \Theta(\frac{1}{\sqrt{\log|V|}})$ [21].

2 Definitions and Notations

An instance of the stable marriage problem consists of N men, N women and each person's preference list. If a person p includes q on his/her preference list, then we say that q is *acceptable* to p. A *matching* is a set of disjoint pairs of a man and a woman (m, w) such that m is acceptable to w and vice versa. If m has a partner in M, i.e., $\exists w(m,w) \in M$, then m is said to be *matched* in M, and otherwise, *single*. If $(m, w) \in M$, we write $M(m) = w$ and $M(w) = m$. Given a matching M, a man m and a woman w are said to form a *blocking pair* for M (or simply "(m, w) *blocks* M") if all the following conditions are met:

1. m and w are not matched together in M but are acceptable to each other,
2. m is either single in M or prefers w to $M(m)$, and
3. w is either single in M or prefers m to $M(w)$.

A matching is called *stable* if it contains no blocking pair.

Let *SMTI* (Stable Marriage with Ties and Incomplete lists) denote the general stable marriage problem, and *MAX SMTI* be the problem of finding a stable matching of maximum size. *SMI* (Stable Marriage with Incomplete lists) is a restriction of SMTI, where ties are not allowed.

Throughout this paper, instances contain an equal number N of men and women. We may assume without loss of generality that acceptability is mutual, i.e., w is acceptable to m if and only if m is acceptable to w. If, for example, the preference list of a man m contains w_1, w_2 and w_3, in this order, we write

$$m : w_1\ w_2\ w_3.$$

Two or more persons tied in a list are given in parentheses, such as

$$m : w_1\ (w_2\ w_3).$$

If m strictly prefers w_i to w_j in an instance I, we write "$w_i \succ w_j$ in m's list in I", or simply, "$w_i \succ w_j$ in m's I-list".

Let M_1 and M_2 be two stable matchings for an SMTI instance $\hat{I}$. We sometimes use the bipartite graph G_{M_1,M_2} defined as follows: Each vertex of G_{M_1,M_2} is associated with a person in $\hat{I}$. We include an edge between vertices m and w if and only if m and w are matched in M_1 or M_2 (if they are matched in both, we include two edges between them; hence G_{M_1,M_2} is a multigraph). Observe that the degree of each vertex is then at most two, and each connected component of G_{M_1,M_2} is a simple path, a cycle or an isolated vertex.

A goodness measure of an approximation algorithm for a maximization problem is defined as usual: Let P be a maximization problem, and T be an approximation algorithm for P. T is said to be an $r(N)$-approximation algorithm if $opt(x)/T(x) \leq r(N)$ for all instances x of size N, where $opt(x)$ and $T(x)$ are the size of the optimal and T's solution for x, respectively. The problem P is said to be *approximable* within $r(N)$ if there is a polynomial-time $r(N)$-approximation algorithm for P. P is *NP-hard to approximate within* $r(N)$, if the existence of an $r(N)$-approximation algorithm for P implies P=NP.

3 Inapproximability Results

In this section, we obtain a lower bound on the approximation ratio of MAX SMTI using a reduction from the Minimum Vertex Cover problem (MVC for short). Let $G = (V, E)$ be a graph. A vertex cover C for G is a set of vertices in G such that every edge in E has at least one endpoint in C. MVC is to find, for a given graph G, a vertex cover with the minimum number of vertices, which is denoted by $VC(G)$. Dinur and Safra [7] gave an improved lower bound of $10\sqrt{5} - 21$ on the approximation ratio of MVC using the following proposition with $p = \frac{3-\sqrt{5}}{2} - \delta$ for arbitrarily small δ. We shall, however, see that the value $p = 1/3$ is optimal for our purposes.

Proposition 3.1 *[7] For any $\epsilon > 0$ and $p < \frac{3-\sqrt{5}}{2}$, the following holds: If there is a polynomial-time algorithm that, given a graph $G = (V, E)$, distinguishes between the following two cases, then P=NP.*

(1) $|VC(G)| \leq (1 - p + \epsilon)|V|$.
(2) $|VC(G)| > (1 - \max\{p^2, 4p^3 - 3p^4\} - \epsilon)|V|$.

For a MAX SMTI instance $\hat{I}$, let $OPT(\hat{I})$ be a maximum cardinality stable matching and $|OPT(\hat{I})|$ be its size.

Theorem 3.2 *For any $\epsilon > 0$ and $p < \frac{3-\sqrt{5}}{2}$, the following holds: If there is a polynomial-time algorithm that, given a MAX SMTI instance $\hat{I}$ of size N, distinguishes between the following two cases, then P=NP.*

(1) $|OPT(\hat{I})| \geq \frac{2+p-\epsilon}{3}N$.
(2) $|OPT(\hat{I})| < \frac{2+\max\{p^2, 4p^3-3p^4\}+\epsilon}{3}N$.

Proof. Given a graph $G = (V, E)$, we will construct, in polynomial time, an SMTI instance $\hat{I}(G)$ with N men and N women. Our reduction satisfies the following two conditions: (i) $N = 3|V|$, and (ii) $|OPT(\hat{I}(G))| = 3|V| - |VC(G)|$. Then, it is not hard to see that Proposition 3.1 implies Theorem 3.2.

Now we show the reduction. For each vertex v_i of G, we construct three men v_i^A, v_i^B and v_i^C, and three women v_i^a, v_i^b and v_i^c. Hence, there are $3|V|$ men and $3|V|$ women in total. Suppose that the vertex v_i is adjacent to d vertices $v_{i_1}, v_{i_2}, \cdots, v_{i_d}$. Then, preference lists of six people corresponding to v_i are as follows.

v_i^A:	v_i^a	v_i^a:	$v_i^B \ v_{i_1}^C \cdots v_{i_d}^C \ v_i^A$
v_i^B:	$(v_i^a \ v_i^b)$	v_i^b:	$v_i^B \ v_i^C$
v_i^C:	$v_i^b \ v_{i_1}^a \cdots v_{i_d}^a \ v_i^c$	v_i^c:	v_i^C

The order of persons in preference lists of v_i^C and v_i^a are determined as follows: $v_p^a \succ v_q^a$ in v_i^C's list if and only if $v_p^C \succ v_q^C$ in v_i^a's list. Clearly, this reduction can be performed in polynomial time. It is not hard to see that condition (i) holds.

We show that condition (ii) holds. Given a minimum vertex cover $VC(G)$ for G, we construct a stable matching M for $\hat{I}(G)$ as follows: For each vertex v_i, if $v_i \in VC(G)$, let $M(v_i^B) = v_i^a$, $M(v_i^C) = v_i^b$, and leave v_i^A and v_i^c single. If $v_i \notin VC(G)$, let $M(v_i^A) = v_i^a$, $M(v_i^B) = v_i^b$, and $M(v_i^C) = v_i^c$. Fig. 1 shows a part of M corresponding to v_i.

It is straightforward to verify that M is stable in $\hat{I}(G)$. It is easy to see that there is no blocking pair consisting of a man and a woman associated with the same vertex. Suppose there is a blocking pair associated with different vertices v_i and v_j. Then it must be (v_i^C, v_j^a) or (v_j^C, v_i^a), and without loss of generality, we assume that it is (v_i^C, v_j^a). Then, v_i^C and v_j^a are acceptable to each other, and so, by the construction of preference lists, v_i and v_j must be adjacent in G. As a result, either or both are contained in $VC(G)$. By the construction of the matching, this implies that either v_i^C or v_j^a is matched with a person at the top of his/her preference list, which is a contradiction. Hence, there is no blocking pair for M. Observe that $|M| = 2|VC(G)| + 3(|V| - |VC(G)|) = 3|V| - |VC(G)|$. Hence $|OPT(\hat{I}(G))| \geq |M| = 3|V| - |VC(G)|$.

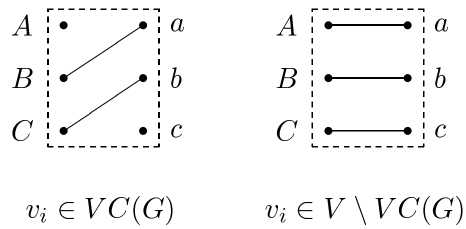

Figure 1: A part of matching M

Conversely, let M be a maximum stable matching for $\hat{I}(G)$. (We use M instead of $OPT(\hat{I}(G))$ for simplicity.) Consider a vertex $v_i \in V$ and the corresponding six persons. Note that v_i^B is matched in M, as otherwise (v_i^B, v_i^b) would block M. We consider two cases according to his partner.

Case (1) $M(v_i^B) = v_i^a$ Then, v_i^b is matched in M, as otherwise (v_i^C, v_i^b) blocks M. Since v_i^B is already matched with v_i^a, $M(v_i^b) = v_i^C$. Then, both v_i^A and v_i^c must be single in M. In this case, we say that "v_i causes a pattern 1". A diagrammatic representation of a pattern 1 is given in Fig. 2.

Case (2) $M(v_i^B) = v_i^b$ Then, v_i^a is matched in M, as otherwise (v_i^A, v_i^a) blocks M. Since v_i^B is already matched with v_i^b, there remain two cases: (a) $M(v_i^a) = v_i^A$ and (b) $M(v_i^a) = v_{i_j}^C$ for some j. Similarly, for v_i^C, there are two cases: (c) $M(v_i^C) = v_i^c$ and (d) $M(v_i^C) = v_{i_j}^a$ for some j. Hence, we have four cases in total. These cases are referred to as patterns 2 through 5 (see Fig. 2). For example, a combination of cases (b) and (c) corresponds to pattern 4.

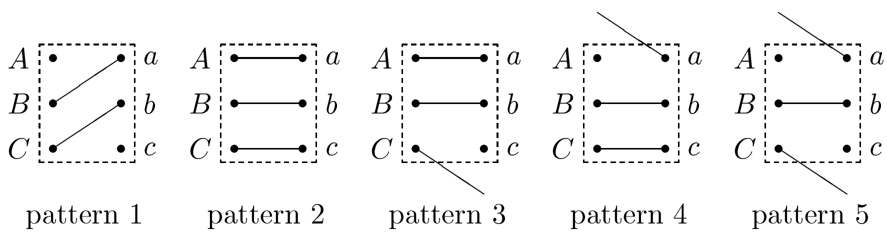

Figure 2: Five patterns caused by v_i

Lemma 3.3 *Each vertex causes a pattern 1, 2 or 5.*

Proof. Suppose that a vertex v causes a pattern 3. Then, there is a sequence of vertices $v_{i_1}(= v), v_{i_2}, \ldots, v_{i_\ell}(\ell \geq 2)$ such that $M(v_{i_1}^A) = v_{i_1}^a$, $M(v_{i_j}^C) = v_{i_{j+1}}^a$ $(1 \leq j \leq \ell - 1)$ and $M(v_{i_\ell}^C) = v_{i_\ell}^c$, so that v_{i_1} causes a pattern 3, v_{i_2} through $v_{i_{\ell-1}}$ cause a pattern 5, and v_{i_ℓ} causes a pattern 4. (If we assume that v causes a pattern 4, then by the same argument, we can show the existence of a sequence of vertices with the same property as above, namely, $v_{i_\ell}(= v), v_{i_{\ell-1}}, \ldots, v_{i_1}(\ell \geq 2)$ such that v_{i_ℓ} causes a pattern 4, $v_{i_{\ell-1}}$ through v_{i_2} cause a pattern 5, and v_{i_1} causes a pattern 3.)

First, consider the case of $\ell \geq 3$. We show that, for each $2 \leq j \leq \ell - 1$, $v_{i_{j+1}}^a \succ v_{i_{j-1}}^a$ in $v_{i_j}^C$'s list. We will prove this fact by induction.

Since $v_{i_1}^a$ is matched with $v_{i_1}^A$, the man at the tail of her list, $M(v_{i_2}^C)(= v_{i_3}^a) \succ v_{i_1}^a$ in $v_{i_2}^C$'s list; otherwise, $(v_{i_2}^C, v_{i_1}^a)$ blocks M. Hence, the statement is true for $j = 2$. Suppose that the statement is true for $j = k$, namely, $v_{i_{k+1}}^a \succ v_{i_{k-1}}^a$ in $v_{i_k}^C$'s list. By the construction of preference lists, $v_{i_{k+1}}^C \succ v_{i_{k-1}}^C$ in $v_{i_k}^a$'s list. Then, if $v_{i_k}^a \succ v_{i_{k+2}}^a$ in $v_{i_{k+1}}^C$'s list, $(v_{i_{k+1}}^C, v_{i_k}^a)$ blocks M. Hence, the statement is true for $j = k+1$.

Now, it turns out that $v_{i_\ell}^a \succ v_{i_{\ell-2}}^a$ in $v_{i_{\ell-1}}^C$'s list, which implies that $v_{i_\ell}^C \succ v_{i_{\ell-2}}^C$ in $v_{i_{\ell-1}}^a$'s list. Then, $(v_{i_\ell}^C, v_{i_{\ell-1}}^a)$ blocks M since $M(v_{i_\ell}^C) = v_{i_\ell}^c$, a contradiction.

It is straightforward to verify that, when $\ell = 2$, $(v_{i_2}^C, v_{i_1}^a)$ blocks M, a contradiction. □

By Lemma 3.3, each vertex v_i causes a pattern 1, 2 or 5. Construct the subset C of vertices in the following way: If v_i causes a pattern 1 or pattern 5, then let $v_i \in C$, otherwise, let $v_i \notin C$.

We show that C is actually a vertex cover for G. Suppose not. Then, there are two vertices v_i and v_j in $V \setminus C$ such that $(v_i, v_j) \in E$ and both of them cause pattern 2, i.e., $M(v_i^C) = v_i^c$ and $M(v_j^A) = v_j^a$. Then (v_i^C, v_j^a) blocks M, contradicting the stability of M. Hence, C is a vertex cover for G. It is easy to see that $|M|(= |OPT(\hat{I}(G))|) = 2|C| + 3(|V| - |C|) = 3|V| - |C|$. Thus $|VC(G)| \leq |C| = 3|V| - |OPT(\hat{I}(G))|$. Hence, condition (ii) holds. □

Theorem 3.2 implies the following inapproximability result:

Corollary 3.4 *MAX SMTI is NP-hard to approximate within a factor* $\frac{21}{19} - \delta$ *for any constant* $\delta > 0$.

Proof. By letting $p = \frac{1}{3}$ in Theorem 3.2, we know that the existence of a polynomial-time algorithm that distinguishes between the following two cases implies P=NP for an arbitrary small positive constant ϵ:

(1) $|OPT(\hat{I})| \geq \frac{21-\epsilon}{27} N$.
(2) $|OPT(\hat{I})| < \frac{19+\epsilon}{27} N$.

Now, suppose that there is a polynomial-time approximation algorithm T for MAX SMTI whose approximation ratio is at most $\frac{21}{19} - \delta$ for some δ. Then, consider the above statement with fixed constant ϵ such that $\epsilon < \frac{361\delta}{40-19\delta}$.

If an instance of the case (1) is given to T, it outputs a solution whose size is at least $\frac{21-\epsilon}{27} N \frac{1}{\frac{21}{19}-\delta}$. If an instance of the case (2) is given to T, it outputs a solution whose size is less than $\frac{19+\epsilon}{27} N$. It is easy to observe that $\frac{21-\epsilon}{27} N \frac{1}{\frac{21}{19}-\delta} > \frac{19+\epsilon}{27} N$ by the definition of ϵ. Hence, using T, we can distinguish between the cases (1) and (2), which implies P=NP. This completes the proof. □

Remark. Observe that Theorem 3.2 and Corollary 3.4 hold for the restricted case where ties occur only in the preference lists of one sex and are of length only two. Furthermore, each preference list is either totally ordered or consists of a single tied pair.

Remark. A long-standing conjecture states that MVC is hard to approximate within a factor of $2 - \epsilon$. We obtain a 1.25 lower bound for MAX SMTI, modulo this conjecture as follows.

Suppose that MAX SMTI has a polynomial-time r-approximation algorithm. Let $G = (V, E)$ be a graph such that $|VC(G)| \geq \frac{|V|}{2}$. (Due to Nemhauser and Trotter [27], approximability of MVC for general graphs is equivalent to approximability of MVC for graphs $G = (V, E)$ with this restriction.) Obtain a MAX SMTI instance $\hat{I}(G)$ using the reduction given in the proof of Theorem 3.2. We have already proved that $|OPT(\hat{I}(G))| = 3|V| - |VC(G)|$. Also, it is easy to see that given any stable matching M for $\hat{I}(G)$, we can obtain a vertex cover C for G with $|C| \leq 3|V| - |M|$. Combining these facts with $|VC(G)| \geq \frac{|V|}{2}$ and $|OPT(\hat{I}(G))|/|M| \leq r$, we have that $|C| \leq (6 - \frac{5}{r})|VC(G)|$. Hence, we can construct a polynomial-time $(6 - \frac{5}{r})$-approximation algorithm for MVC using a polynomial-time r-approximation algorithm for MAX SMTI.

4 Approximation Algorithm ShiftBrk

In this section, we give an approximation algorithm ShiftBrk for MAX SMTI. We define first some notation regarding ties.

Let $\hat{I}$ be an SMTI instance and let p be a person in $\hat{I}$ whose preference list contains a tie which includes persons q_1, q_2, $\cdots$, q_k. In this case, we write "$(\cdots q_1 \cdots q_2 \cdots q_k \cdots)$ in p's $\hat{I}$-list". Let I be an SMI instance that can be obtained by breaking all ties in $\hat{I}$, and suppose that the tie $(\cdots q_1 \cdots q_2 \cdots q_k \cdots)$ in p's $\hat{I}$-list is broken into $\cdots q_1 \succ \cdots \succ q_2 \succ \cdots \succ q_k \cdots$ in I. Then we write "$[\cdots q_1 \cdots q_2 \cdots q_k \cdots]$ in p's I-list." That is, "$[\cdots q_1 \cdots q_2 \cdots]$ in p's I-list" means that q_1 precedes q_2 in p's list in I but they are tied in p's list in $\hat{I}$.

Suppose that, in $\hat{I}$, a man m has a tie T of length ℓ consisting of women $w_1, w_2, \cdots, w_\ell$. Also, suppose that this tie T is broken into $[w_1\ w_2 \cdots w_\ell]$ in m's I-list. We say "shift tie T in I" to obtain a new SMI instance I' in which only the tie T is changed to $[w_2 \cdots w_\ell\ w_1]$ and other preference lists are the same as in I. If I' is the result of shifting all broken ties in men's lists in I, then we write "$I' = \mathit{Shift}_m(I)$". Similarly, if I' is the result of shifting all broken ties in women's lists in I, then we write "$I' = \mathit{Shift}_w(I)$". Let L be the maximum length of ties in $\hat{I}$. The full description of ShiftBrk is given in Fig. 3.

Algorithm ShiftBrk($\hat{I}$)

$I_{1,1}$:= an SMI instance obtained by breaking all ties of $\hat{I}$ in an arbitrary order;
for $i := 2$ **to** L
  $I_{i,1} := \mathit{Shift}_m(I_{i-1,1})$;
for $i := 1$ **to** L
  for $j := 2$ **to** L
    $I_{i,j} := \mathit{Shift}_w(I_{i,j-1})$;
for $i := 1$ **to** L
  for $j := 1$ **to** L
    $M_{i,j}$:= stable matching for $I_{i,j}$;
Output a largest matching among all $M_{i,j}$'s;

Figure 3: Algorithm ShiftBrk

Since stable matchings for SMI instances can be obtained in polynomial time using the Gale-Shapley algorithm, ShiftBrk runs in time polynomial in N. It is easy to see that all $M_{i,j}$ are stable for $\hat{I}$ (see [11] for example). Hence, ShiftBrk outputs a feasible solution.

4.1 Bidominating Pairs

Before analyzing the approximation ratio, we will define an important notion, a bidominating pair[1] , which plays an important role in our analysis. Let $\hat{I}$ be an SMTI instance and M_{opt} be a largest stable matching for $\hat{I}$. Let I be an SMI instance obtained by breaking all ties of $\hat{I}$ and M be a stable matching for I. A pair (m, w) is said to be a *bidominating pair for M* if they are matched together in M, both are

[1] In an earlier version of this article, this was denoted *annoying pair*.

matched to other people in M_{opt}, and both prefer each other (in I) to their partners in M_{opt}. That is, (a) $M(m) = w$, (b) m is matched in M_{opt} and $w \succ M_{opt}(m)$ in m's I-list, and (c) w is matched in M_{opt} and $m \succ M_{opt}(w)$ in w's I-list.

Lemma 4.1 *Let (m, w) be a bidominating pair for M. Then, one or both of the following holds: (i) $[\cdots w \cdots M_{opt}(m) \cdots]$ in m's I-list; (ii) $[\cdots m \cdots M_{opt}(w) \cdots]$ in w's I-list.*

Proof. If the strict preferences hold also in $\hat{I}$, i.e. $w \succ M_{opt}(m)$ in m's $\hat{I}$-list and $m \succ M_{opt}(w)$ in w's $\hat{I}$-list, then (m, w) blocks M_{opt} in $\hat{I}$. Thus, at least one of these preferences in I must have been caused by the breaking of ties in $\hat{I}$. ☐

Fig. 4 shows a simple example of a bidominating pair. (A dotted line means that the endpoints are matched in M_{opt} and a solid line means the same in M. In m_3's list, w_2 and w_3 are tied in $\hat{I}$ and this tie is broken into $[w_2\ w_3]$ in I.)

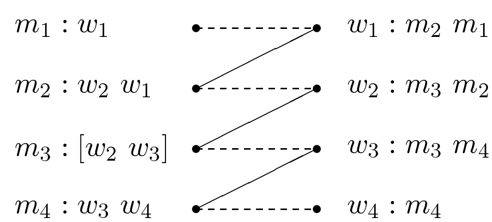

Figure 4: A bidominating pair (m_3, w_2) for M

Lemma 4.2 *$|M| \geq |M_{opt}| - k$, where k is the number of bidominating pairs for M.*

Proof. Consider a connected component of the bipartite graph $G_{M,M_{opt}}$ (see Sec. 2 for the definition), which is either a cycle or a path. A cycle contains equally many edges from M and M_{opt}, while a path can contain one more edge from one of the matchings. We show that each path in $G_{M,M_{opt}}$ contains at least one bidominating pair for M. This will imply the lemma.

Consider a path $m_1, w_1, m_2, w_2, \ldots, m_\ell, w_\ell$, where $w_s = M_{opt}(m_s)$ $(1 \leq s \leq \ell)$ and $m_{s+1} = M(w_s)$ $(1 \leq s \leq \ell - 1)$. (This path begins with a man and ends with a woman. Other cases can be proved in a similar manner.) Suppose that this path does not contain a bidominating pair for M. Since m_1 is single in M, $m_2 \succ m_1$ in w_1's I-list (otherwise, (m_1, w_1) blocks M). Then, consider the man m_2. Since we assume that (m_2, w_1) is not a bidominating pair, $w_2 \succ w_1$ in m_2's I-list. We can continue the same argument to show that $m_3 \succ m_2$ in w_2's I-list and $w_3 \succ w_2$ in m_3's I-list, and so on. Finally, we have that $w_\ell \succ w_{\ell-1}$ in m_ℓ's I-list. Since w_ℓ is single in M, (m_ℓ, w_ℓ) blocks M, a contradiction. Hence, every path must contain at least one bidominating pair. ☐

5 ShiftBrk for Instances Where Only Men Have Ties

In this section, we consider SMTI instances such that only men have ties each with length at most L. Note that we do not restrict the *number* of ties in the list; one man can have more than one tie, as long as each tie is of length at most L. We show that ShiftBrk achieves an approximation ratio of $2/(1 + L^{-2})$.

Let $\hat{I}$ be an SMTI instance. We fix a largest stable matching M_{opt} for $\hat{I}$ and denote $n = |M_{opt}|$. All preferences in this section are with respect to $\hat{I}$ unless otherwise stated. Since women do not have ties, ShiftBrk produces L instances $I_{1,1}, I_{2,1}, \ldots, I_{L,1}$. Let us denote them for simplicity as $I_1, I_2, \ldots, I_L$. Let $M_1, M_2, \ldots, M_L$ be corresponding stable matchings obtained in line 9 of ShiftBrk.

As shown in Lemma 4.2, if the number of bidominating pairs for a stable matching M is small, then M is relatively large. We will show that among L stable matchings, there is at least one stable matching that contains a small number of bidominating pairs. To do so, we observe the following property: Suppose that a man m is matched in M_{opt} and all of $M_1, M_2, \ldots, M_L$. If m does not contain $M_{opt}(m)$ in a tie, then m cannot be a part of a bidominating pair in any of L matchings. If m contains $M_{opt}(m)$ in a tie, then m cannot be a part of a bidominating pair in M_i, where in I_i, m's tie is broken so that $M_{opt}(m)$ comes the first place in the tie. In the proof, we estimate the lower bound on the number of such men in terms of the optimal size n.

Let V_{opt} and W_{opt} be the set of men and women, respectively, that are matched in M_{opt}. Let V_a be the subset of V_{opt} such that each man $m \in V_a$ has a partner in all of $M_1, \ldots, M_L$. Let $W_b = \{w | M_{opt}(w) \in V_{opt} \setminus V_a\}$. Note that, by definition, $W_b \subseteq W_{opt}$ and $|V_a| + |W_b| = n$. For each woman w, let $best(w)$ be the man that w prefers the most among $M_1(w), \ldots, M_L(w)$; if she is single in each of $M_1, \cdots, M_L$, then $best(w)$ is not defined.

Lemma 5.1 *Let w be in W_b. Then $best(w)$ exists and is in V_a, and is preferred by w over $M_{opt}(w)$. That is, $best(w) \in V_a$ and $best(w) \succ M_{opt}(w)$ in w's $\hat{I}$-list.*

Proof. By the definition of W_b, $M_{opt}(w) \in V_{opt} \setminus V_a$. By the definition of V_a, there is a matching M_i in which $M_{opt}(w)$ is single. Since M_i is a stable matching for $\hat{I}$, w has a partner in M_i and further, that partner $M_i(w)$ is preferred over $M_{opt}(w)$ (as otherwise, $(M_{opt}(w), w)$ blocks M_i). Since w has a partner in M_i, $best(w)$ is defined and differs from $M_{opt}(w)$. By the definition of $best(w)$, w prefers $best(w)$ over $M_{opt}(w)$. That implies that $best(w)$ is matched in M_{opt}, i.e. $best(w) \in V_{opt}$, as otherwise $(best(w), w)$ blocks M_{opt}. Finally, $best(w)$ must be matched in each $M_1, \ldots, M_L$, i.e. $best(w) \in V_a$, as otherwise $(best(w), w)$ blocks M_i in which $best(w)$ is single. □

Lemma 5.2 *Consider a man m and two women w_1, w_2, where $m = best(w_1) = best(w_2)$. Then w_1 and w_2 are tied in m's $\hat{I}$-list.*

Proof. Since $m = best(w_1) = best(w_2)$, there are matchings M_i and M_j such that $m = M_i(w_1) = M_j(w_2)$. First, suppose that $w_1 \succ w_2$ in m's list. Since $m = M_j(w_2)$, w_1 is not matched with m in M_j. By the definition of $best(w)$, w_1 is either single or matched with a man below m in her list, in the matching M_j. In either case, (m, w_1) blocks M_j in $\hat{I}$, a contradiction. By exchanging the role of w_1 and w_2, we can show that it is not the case that $w_2 \succ w_1$ in m's list. Hence, w_1 and w_2 must be tied in m's $\hat{I}$-list. □

By Lemma 5.2, each man can be $best(w)$ for at most L women w because the length of ties is at most L. Let us partition V_a into V_t and $\overline{V_t}$, where V_t is the set of men m such that m is $best(w)$ for exactly L women $w \in W_b$ and $\overline{V_t} = V_a \setminus V_t$.

Lemma 5.3 *There is a matching M_k for which the number of bidominating pairs is at most $|M_k| - (|V_t| + \frac{|\overline{V_t}|}{L})$.*

Proof. Consider a man $m \in V_t$. By definition, there are L women $w_1, \ldots, w_L$ such that $m = best(w_1) = \cdots = best(w_L)$, $M_i(w_i) = m$ for $1 \le i \le L$, and all these women are in W_b. By Lemma 5.2, all these women are tied in m's $\hat{I}$-list. By Lemma 5.1, each woman w_i prefers $best(w_i)(= m)$ to $M_{opt}(w_i)$ so that $m \neq M_{opt}(w_i)$ for any i. This means that none of these women can be $M_{opt}(m)$. For m to form a bidominating pair for M_i, $w_i(= M_i(m))$ and $M_{opt}(m)$ must be tied in m's list, due to Lemma 4.1 (i) (note that the case (ii) of Lemma 4.1 does not happen because women do not have ties). Hence, m cannot form a bidominating pair for any of M_1 through M_L.

Next, consider a man $m \in \overline{V_t}$. If $M_{opt}(m)$ is not in a tie in m's list, m cannot form a bidominating pair for any of M_1 through M_L, by the same argument as above. If m includes $M_{opt}(m)$ in a tie, there exists an instance I_i such that $M_{opt}(m)$ lies in first place in the broken tie of m's I_i-list. This means that m does not constitute a bidominating pair for M_i by Lemma 4.1.

Hence, there is a matching M_k for which at least $|V_t| + \frac{|\overline{V_t}|}{L}$ men, among those matched in M_k, do not form a bidominating pair. Hence, the number of bidominating pairs is at most $|M_k| - (|V_t| + \frac{|\overline{V_t}|}{L})$. □

Lemma 5.4 $|V_t| + \frac{|\overline{V_t}|}{L} \geq \frac{n}{L^2}$.

Proof. By the definition of V_t, a man in V_t is $best(w)$ for L different women in W_b, while a man in $\overline{V_t}$ is $best(w)$ for at most $L-1$ women in W_b. Recall that by Lemma 5.1, for each woman w in W_b, there is a man in V_a that is $best(w)$. Thus, W_b contains at most $|V_t|L + |\overline{V_t}|(L-1)$ women. Since $|V_a| + |W_b| = n$, we have that

$$n \leq |V_a| + |V_t|L + |\overline{V_t}|(L-1) = L|V_a| + |V_t|.$$

Now,

$$\begin{aligned}
|V_t| + \frac{|\overline{V_t}|}{L} &= |V_t| + \frac{|V_a| - |V_t|}{L} \\
&= \frac{1}{L}|V_a| + \frac{L-1}{L}|V_t| \\
&\geq \frac{1}{L}\frac{n - |V_t|}{L} + \frac{L-1}{L}|V_t| \\
&= \frac{n}{L^2} + \frac{L^2 - L - 1}{L^2}|V_t| \\
&\geq \frac{n}{L^2}.
\end{aligned}$$

The last inequality is due to the fact that $L^2 - L - 1 > 0$ since $L \geq 2$. □

Theorem 5.5 *The approximation ratio of ShiftBrk is at most* $2/(1+L^{-2})$ *for instances where only men have ties, and these ties are of length at most* L.

Proof. By Lemmas 5.3 and 5.4, there is a matching M_k for which the number of bidominating pairs is at most $|M_k| - n/L^2$. By Lemma 4.2, $|M_k| \geq n - \left(|M_k| - \frac{n}{L^2}\right)$, which implies that $|M_k| \geq \frac{L^2+1}{2L^2}n = \frac{1+L^{-2}}{2}n$. □

Remark. The same result holds for men's preference lists being arbitrary partial order. Suppose that each man m's list is a partial order with width at most L, i.e., such that the maximum number of mutually incomparable women for m is at most L. Then, we can partition its partial order into L chains by Dilworth's theorem [6]. In each "shift", we give the priority to one of L chains and the resulting totally ordered preference list is constructed so that it satisfies the following property: Each member (woman) of the chain with the priority lies top among all women mutually incomparable with her for m in the original partial order. It is not hard to see that the theorem holds for this case.

Lower Bounds for ShiftBrk We show that the upper bound of Theorem 5.5 is tight. We give an example for $L = 4$ to explain how it works. The following preference lists illustrate part of a worst-case example. There are $2L$ men and $2L$ women:

$$\begin{array}{llcll}
m_1^1: & (\ w_1^1\ \ w_1^2\ \ w_1^3\ \ w_1^4\ ) & \qquad & w_1^1: & m_1^1 \\
m_2^1: & (\ w_2^1\ \ w_2^2\ \ w_2^3\ \ w_2^4\ ) & & w_2^1: & m_2^1 \\
m_1^2: & (\ w_2^2\ \ w_1^2\ \ w_2^3\ \ w_2^4\ ) & & w_1^2: & m_1^1\ \ m_1^2 \\
m_2^2: & w_2^2 & & w_2^2: & m_2^1\ \ m_1^2\ \ m_2^2\ \ m_1^3\ \ m_1^4 \\
m_1^3: & (\ w_2^2\ \ w_2^3\ \ w_1^3\ \ w_2^4\ ) & & w_1^3: & m_1^1\ \ m_1^3 \\
m_2^3: & w_2^3 & & w_2^3: & m_2^1\ \ m_1^3\ \ m_2^3\ \ m_1^4\ \ m_1^2 \\
m_1^4: & (\ w_2^2\ \ w_2^3\ \ w_2^4\ \ w_1^4\ ) & & w_1^4: & m_1^1\ \ m_1^4 \\
m_2^4: & w_2^4 & & w_2^4: & m_2^1\ \ m_1^4\ \ m_2^4\ \ m_1^2\ \ m_1^3
\end{array}$$

The largest stable matching for this instance is of size $2L$ (m_j^i is matched with w_j^i for all i, j). Consider to apply ShiftBrk to the above instance. If the initial SMI instance I_1 is constructed by breaking ties in the same order as written above, then ShiftBrk produces L stable matchings $M_1, \ldots, M_L$, where $|M_1| = L + 1$ and $|M_2| = |M_3| = \cdots = |M_L| = L$.

To make the complete example, let $I^1, \ldots, I^L$ be L copies of the above instance and let I^{all} be the instance constructed by putting $I^1, \ldots, I^L$ together. Then, in the worst case initial tie-breaking, ShiftBrk produces L matchings each of which has size $(L+1)\cdot 1 + L\cdot(L-1) = L^2+1$, while a largest stable matching for I^{all} is of size $2L^2$. Hence, the approximation ratio of ShiftBrk for I^{all} is $2L^2/(L^2+1) = 2/(1+L^{-2})$, which proves the tightness of the analysis.

Finally, we briefly sketch how to make a sub-instance for general L. Prepare $2L$ men m_j^i and $2L$ women w_j^i ($1 \le i \le L$ and $1 \le j \le 2$). For $j = 1, 2$, m_j^1's list consists of one tie including women $w_j^1, w_j^2, \ldots, w_j^L$ in this order (this "order" is important only for considering worst case tie-breaking). For $2 \le i \le L$, m_2^i's list includes only one woman w_2^i. For $2 \le i \le L$, m_1^i's list consists of one tie of length L, $(w_2^2\ w_2^3\ \cdots\ w_2^{i-1}\ w_2^i\ w_1^i\ w_2^{i+1}\ \cdots\ w_2^L)$. Woman w_j^1 ($j = 1, 2$) includes the man m_j^1. For $2 \le i \le L$, woman w_1^i includes m_1^1 and m_1^i in this order. For $2 \le i \le L$, woman w_2^i's list is as follows: $m_2^1\ m_1^i\ m_2^i\ m_1^{i+1}\ m_1^{i+2}\ \cdots\ m_1^L\ m_1^2\ m_1^3\ \cdots\ m_1^{i-1}$.

6 ShiftBrk for Instances Where Both Men And Women Have Ties

In this section we show that when $L = 2$, the performance ratio of ShiftBrk is better than two even if we allow women's lists to include ties. ShiftBrk creates four SMI instances $I_{1,1}, I_{1,2}, I_{2,1}$ and $I_{2,2}$ from the SMTI input instance $\hat{I}$, since $L = 2$. Note that men have the same lists in $I_{i,1}$ and $I_{i,2}$ ($i \in \{1, 2\}$), while women have the same lists in $I_{1,j}$ and $I_{2,j}$ ($j \in \{1, 2\}$). Let $M_{i,j}$ ($i \in \{1, 2\}$, $j \in \{1, 2\}$) be a stable matching for $I_{i,j}$ obtained in line 9 of ShiftBrk. We fix an optimal solution M_{opt}, a largest stable matching for $\hat{I}$, and let $n = |M_{opt}|$ as before.

Let V and W be the sets of men and women in $\hat{I}$, respectively. For $i, j \in \{1, 2\}$, let $V_{i,j}$ ($W_{i,j}$, respectively) be the set of men (women, respectively) that are matched in $M_{i,j}$. Observe that $|V_{i,j}| = |W_{i,j}| = |M_{i,j}|$. Define $A_{i,j}$ to be the set of pairs $(m, w) \in M_{i,j}$ that are bidominating for $M_{i,j}$, and $B_{i,j}$ the set of pairs $(m, w) \in M_{i,j}$ that are matched but not bidominating in $M_{i,j}$, that is, $B_{i,j} = M_{i,j} \setminus A_{i,j}$. Observe that $|A_{i,j}| + |B_{i,j}| = |M_{i,j}|$.

The following lemma shows that the number of matched men for a given broken instance, but are not matched when the instance is shifted, is at most the number of non-bidominating pairs.

Lemma 6.1 *For $j = 1, 2$, $|V_{1,j} \setminus V_{2,j}| \leq |B_{1,j}|$.*

Proof. Let m_1 be in $V_{1,j} \setminus V_{2,j}$. Consider the bipartite graph $G_{M_{1,j},M_{2,j}}$ (see Sec. 2 for the definition). Since m_1 is matched in $M_{1,j}$ but single in $M_{2,j}$, there is a path in $G_{M_{1,j},M_{2,j}}$, starting from m_1. Assume that the path ends with a man (the other case can be discussed similarly and hence will be omitted). Let the path be $m_1, w_1, \ldots, m_\ell$, where $w_s = M_{1,j}(m_s)$ and $m_{s+1} = M_{2,j}(w_s)$ for $1 \leq s \leq \ell - 1$. We show that the path contains a pair $(m_i, w_i) \in B_{1,j}$.

Suppose that the path $m_1, w_1, \ldots, m_\ell$ does not contain a pair in $B_{1,j}$. $m_2 \succ m_1$ in w_1's $I_{2,j}$-list; otherwise, (m_1, w_1) blocks $M_{2,j}$. Then, since $I_{1,j} = \mathit{Shift}_m(I_{2,j})$, (a) $m_2 \succ m_1$ in w_1's $I_{1,j}$-list. By the assumption that $(m_1, w_1) \notin B_{1,j}$, (m_1, w_1) is a bidominating pair for $M_{1,j}$. By the definition of bidominating pairs, $m_1 \succ M_{opt}(w_1)$ in w_1's $I_{1,j}$-list. Using the above (a), we have that (b) $m_2 \succ m_1 \succ M_{opt}(w_1)$ in w_1's $I_{1,j}$-list.

Then we show the claim: For $1 \leq i \leq \ell - 2$, if $m_{i+1} \succ m_i \succ M_{opt}(w_i)$ in w_i's $I_{1,j}$-list, the following (x) and (y) hold.

(x) $w_{i+1} \succ w_i$ in m_{i+1}'s $I_{2,j}$-list.

(y) $m_{i+2} \succ m_{i+1} \succ M_{opt}(w_{i+1})$ in w_{i+1}'s $I_{1,j}$-list.

If the above claim holds, we can apply (x) and (y) repeatedly to (b), obtaining $m_\ell \succ m_{\ell-1} \succ M_{opt}(w_{\ell-1})$ in $w_{\ell-1}$'s $I_{1,j}$-list. Then, $(m_\ell, w_{\ell-1})$ blocks $M_{1,j}$, a contradiction.

Hence, in the following, we prove the above claim. Since $m_{i+1} \succ m_i$ in w_i's $I_{1,j}$-list by the assumption, (c) $w_{i+1} \succ w_i$ in m_{i+1}'s $I_{1,j}$-list (otherwise, (m_{i+1}, w_i) blocks $M_{1,j}$). Also, since we assume that $(m_{i+1}, w_{i+1}) \notin B_{1,j}$, (d) $w_{i+1} \succ M_{opt}(m_{i+1})$ in m_{i+1}'s $I_{1,j}$-list. Considering (c) and (d), we have following two cases: (A) $w_{i+1} \succ M_{opt}(m_{i+1}) \succ w_i$ in m_{i+1}'s $I_{1,j}$-list. (B) $w_{i+1} \succ w_i \succ M_{opt}(m_{i+1})$ in m_{i+1}'s $I_{1,j}$-list. (Note that $M_{opt}(m_{i+1}) \neq w_i$ because we assume that $m_{i+1} \succ m_i \succ M_{opt}(w_i)$ in w_i's $I_{1,j}$-list; hence $M_{opt}(w_i) \neq m_{i+1}$.)

Case (A): Since the length of ties is two, w_{i+1} and w_i are not tied in m_{i+1}'s list. Hence, the relative order of w_{i+1} and w_i is the same in $I_{1,j}$ and $I_{2,j}$. Thus (x) holds.

Case (B): Suppose that w_i and $M_{opt}(m_{i+1})$ are not tied in m_{i+1}'s $\hat{I}$-list. Then, since we assume that $w_i \succ M_{opt}(m_{i+1})$ in m_{i+1}'s $I_{1,j}$-list, this relation also holds for $\hat{I}$. Furthermore, we assume that $m_{i+1} \succ m_i \succ M_{opt}(w_i)$ in w_i's $I_{1,j}$-list. Hence, $m_{i+1} \succ M_{opt}(w_i)$ in w_i's $\hat{I}$-list. Then, (m_{i+1}, w_i) blocks M_{opt} in $\hat{I}$. Thus w_i and $M_{opt}(m_{i+1})$ are tied in m_{i+1}'s $\hat{I}$-list. Since $w_{i+1} \succ [w_i \ M_{opt}(m_{i+1})]$ in m_{i+1}'s $I_{1,j}$-list, $w_{i+1} \succ [M_{opt}(m_{i+1}) \ w_i]$ in m_{i+1}'s $I_{2,j}$-list. Thus (x) holds.

Since (x) holds, $m_{i+2} \succ m_{i+1}$ in w_{i+1}'s $I_{2,j}$-list (otherwise, (m_{i+1}, w_{i+1}) blocks $M_{2,j}$). This means that $m_{i+2} \succ m_{i+1}$ in w_{i+1}'s $I_{1,j}$-list because every woman's list is the same in $I_{1,j}$ and $I_{2,j}$. Since $(m_{i+1}, w_{i+1}) \notin B_{1,j}$, (m_{i+1}, w_{i+1}) is a bidominating pair for $M_{1,j}$, so that $m_{i+1} \succ M_{opt}(w_{i+1})$ in w_{i+1}'s $I_{1,j}$-list. Hence, $m_{i+2} \succ m_{i+1} \succ M_{opt}(w_{i+1})$ in w_{i+1}'s $I_{1,j}$-list. Thus (y) holds. □

By exchanging the role of men and women, we have the following lemma:

Lemma 6.2 $|W_{1,1} \setminus W_{1,2}| \leq |B_{1,1}|$.

Consider a matching $M_{i,j}$ ($i = 1, 2$ and $j = 1, 2$). We will define two subsets of men (women, respectively) participating in bidominating pairs for $M_{i,j}$, namely, $P^m_{i,j}$ and $Q^m_{i,j}$ ($P^w_{i,j}$ and $Q^w_{i,j}$, respectively). By Lemma 4.1, at least one of the following holds for a bidominating pair (m, w) for $M_{i,j}$: (i) $[w \ M_{opt}(m)]$ in m's $I_{i,j}$-list. (ii) $[m \ M_{opt}(w)]$ in w's $I_{i,j}$-list. If (i) holds, let $m \in P^m_{i,j}$ and $w \in P^w_{i,j}$. If (ii) holds, then let $m \in Q^m_{i,j}$ and $w \in Q^w_{i,j}$. Note that, $|P^m_{i,j}| = |P^w_{i,j}|$ and $|Q^m_{i,j}| = |Q^w_{i,j}|$. Also, notice that $P^m_{i,j}$ and $Q^m_{i,j}$ ($P^w_{i,j}$ and $Q^w_{i,j}$, respectively) are not necessarily disjoint.

In order to explain the following lemma, let us say that a woman w is *anxious* in a given matching if she is a part of a bidominating pair (m, w) and $[m\ M_{opt}(w)]$ in w's list. The next lemma says that the set of anxious women when the women's preference lists are broken in one way is disjoint from the set of anxious women when the preference lists are broken in the other way.

Lemma 6.3 $(Q^w_{1,1} \cup Q^w_{2,1}) \cap (Q^w_{1,2} \cup Q^w_{2,2}) = \emptyset$.

Proof. Suppose that $w \in (Q^w_{1,1} \cup Q^w_{2,1}) \cap (Q^w_{1,2} \cup Q^w_{2,2})$. Then, there are $i \in \{1,2\}$ and $j \in \{1,2\}$ such that $w \in Q^w_{i,1} \cap Q^w_{j,2}$. Since $w \in Q^w_{i,1}$, $[m\ M_{opt}(w)]$ in w's $I_{i,1}$-list for some man m. Since $w \in Q^w_{j,2}$, $[m'\ M_{opt}(w)]$ in w's $I_{j,2}$-list for some man m'. However, this is impossible because women's ties are broken in different ways in $I_{i,1}$ and $I_{j,2}$, a contradiction. □

The following lemma is the key to our argument that one of the matchings we find must be relatively large. By Lemma 4.2, it suffices to focus on the case when the number of bidominating pairs, $|A_{i,j}|$, is large, close to $\frac{n}{2}$. Then, $|B_{i,j}|$ is small, close to zero. In that case, the lemma shows that a certain subset of the matched women is large, close to $\frac{n}{2}$.

Lemma 6.4 *For* $j = 1, 2$, $|W_{1,j} \cap (Q^w_{1,j} \cup Q^w_{2,j})| \geq |A_{1,j}| - |V_{1,j} \setminus V_{2,j}| - |B_{2,j}|$.

Proof. Let $j \in \{1,2\}$. Define X_j to be the set of men that belong to bidominating pairs both for $M_{1,j}$ and for $M_{2,j}$:

$$X_j = \{m \mid (m, M_{1,j}(m)) \in A_{1,j} \cap (m, M_{2,j}(m)) \in A_{2,j}\}.$$

We first show that $|W_{1,j} \cap (Q^w_{1,j} \cup Q^w_{2,j})| \geq |X_j|$. For this purpose, we consider a man in $m \in X_j$, and show that there is a woman in $W_{1,j} \cap (Q^w_{1,j} \cup Q^w_{2,j})$ corresponding to m. We then show that a woman does not correspond to different men. When considering a man m in X_j, we consider two cases: $m \in X_j \cap P^m_{1,j}$ and $m \in X_j \setminus P^m_{1,j}$.

First, consider a man m_p in $X_j \cap P^m_{1,j}$. Since $m_p \in P^m_{1,j}$, $[M_{1,j}(m_p)\ M_{opt}(m_p)]$ in m_p's $I_{1,j}$-list. Since $I_{2,j} = \mathit{Shift}_m(I_{1,j})$, $[M_{opt}(m_p)\ M_{1,j}(m_p)]$ in m_p's $I_{2,j}$-list. Since $(m_p, M_{2,j}(m_p))$ is a bidominating pair for $M_{2,j}$ by the definition of X_j, $M_{2,j}(m_p) \succ M_{opt}(m_p)$ in m_p's $I_{2,j}$-list. It then follows that

$$M_{2,j}(m_p) \succ [M_{opt}(m_p)\ M_{1,j}(m_p)] \text{ in } m_p\text{'s } I_{2,j}\text{--list.} \quad \cdots \quad (*)$$

Thus, by Lemma 4.1, $[m_p\ M_{opt}(M_{2,j}(m_p))]$ in $M_{2,j}(m_p)$'s $I_{2,j}$-list. Hence, $M_{2,j}(m_p) \in Q^w_{2,j}$. Since $I_{1,j} = \mathit{Shift}_m(I_{2,j})$ and by $(*)$ above, $M_{2,j}(m_p) \succ [M_{1,j}(m_p)\ M_{opt}(m_p)]$ in m_p's $I_{1,j}$-list. It is easy to see that $M_{2,j}(m_p) \in W_{1,j}$ because otherwise, $(m_p, M_{2,j}(m_p))$ blocks $M_{1,j}$. Hence, $M_{2,j}(m_p) \in W_{1,j} \cap Q^w_{2,j}$.

Next, consider a man m_r in $X_j \setminus P^m_{1,j}$. By the definition of $W_{1,j}$, $M_{1,j}(m_r) \in W_{1,j}$, and since $m_r \notin P^m_{1,j}$, $M_{1,j}(m_r) \in Q^w_{1,j}$ by Lemma 4.1. Hence, $M_{1,j}(m_r) \in W_{1,j} \cap Q^w_{1,j}$.

Finally, we show that the above projection is an injection. To see this, it suffices to show that there is no woman w such that $w = M_{2,j}(m_p) = M_{1,j}(m_r)$ for different men $m_p \in X_j \cap P^m_{1,j}$ and $m_r \in X_j \setminus P^m_{1,j}$. Suppose such a woman w exists. Then, $w \in Q^w_{2,j} \cap Q^w_{1,j}$ by the above observation. Since $w \in Q^w_{2,j}$, $[m_p\ M_{opt}(w)]$ in w's $I_{2,j}$-list. Since $w \in Q^w_{1,j}$, $[m_r\ M_{opt}(w)]$ in w's $I_{1,j}$-list. Since $I_{1,j} = \mathit{Shift}_m(I_{2,j})$, women's lists are the same in $I_{1,j}$ and $I_{2,j}$, which implies that $m_p = m_r$. This is a contradiction.

Hence, we have that $|W_{1,j} \cap (Q^w_{1,j} \cup Q^w_{2,j})| \geq |X_j|$. Finally, we will show that $|X_j| \geq |A_{1,j}| - |V_{1,j} \setminus V_{2,j}| - |B_{2,j}|$. Let $A^m_{1,j}$ and $A^m_{2,j}$ be the sets of men participating $A_{1,j}$ and $A_{2,j}$, respectively, so that $|A^m_{1,j}| = |A_{1,j}|$ and $|A^m_{2,j}| = |A_{2,j}|$. Then, by the definition of X_j, $X_j = A^m_{1,j} \cap A^m_{2,j}$. It then follows that

$$\begin{aligned} |X_j| &= |A^m_{1,j}| + |A^m_{2,j}| - |A^m_{1,j} \cup A^m_{2,j}| \\ &= |A_{1,j}| + |A_{2,j}| - |A^m_{1,j} \cup A^m_{2,j}| \end{aligned}$$

$$
\begin{aligned}
&= |A_{1,j}| + |M_{2,j}| - |B_{2,j}| - |A^m_{1,j} \cup A^m_{2,j}| \\
&= |A_{1,j}| + |V_{2,j}| - |B_{2,j}| - |A^m_{1,j} \cup A^m_{2,j}| \\
&= |A_{1,j}| + |V_{1,j} \cup V_{2,j}| - |V_{1,j} \setminus V_{2,j}| - |B_{2,j}| - |A^m_{1,j} \cup A^m_{2,j}| \\
&\geq |A_{1,j}| - |V_{1,j} \setminus V_{2,j}| - |B_{2,j}|.
\end{aligned}
$$

The last inequality follows from the fact that $|V_{1,j} \cup V_{2,j}| \geq |A^m_{1,j} \cup A^m_{2,j}|$, which can be easily verified by definition. □

From the above lemmas, the following theorem holds.

Theorem 6.5 *The approximation ratio of ShiftBrk is at most* $13/7$ *for instances where the length of ties is two.*

Proof. We prove that $\max\{|M_{1,1}|, |M_{1,2}|, |M_{2,1}|, |M_{2,2}|\} \geq \frac{7}{13}n$. For this purpose, we prove the following statement: If all of $|M_{1,1}|$, $|M_{2,1}|$ and $|M_{2,2}|$ are smaller than $\frac{7}{13}n$, then $|M_{1,2}| \geq \frac{7}{13}n$.

By Lemma 6.3, $(Q^w_{1,1} \cup Q^w_{2,1}) \cap (Q^w_{1,2} \cup Q^w_{2,2}) = \emptyset$. Hence, $|W_{1,1} \cup W_{1,2}| \geq |W_{1,1} \cap (Q^w_{1,1} \cup Q^w_{2,1})| + |W_{1,2} \cap (Q^w_{1,2} \cup Q^w_{2,2})|$. So, $|W_{1,2}| = |W_{1,1} \cup W_{1,2}| - |W_{1,1} \setminus W_{1,2}| \geq |W_{1,1} \cap (Q^w_{1,1} \cup Q^w_{2,1})| + |W_{1,2} \cap (Q^w_{1,2} \cup Q^w_{2,2})| - |W_{1,1} \setminus W_{1,2}|$. By Lemma 6.4, $|W_{1,j} \cap (Q^w_{1,j} \cup Q^w_{2,j})| \geq |A_{1,j}| - |V_{1,j} \setminus V_{2,j}| - |B_{2,j}|$ for $j = 1, 2$. By Lemma 6.2, $|W_{1,1} \setminus W_{1,2}| \leq |B_{1,1}|$. By Lemma 6.1, $|V_{1,j} \setminus V_{2,j}| \leq |B_{1,j}|$. Hence,

$$
\begin{aligned}
|W_{1,2}| &\geq |W_{1,1} \cap (Q^w_{1,1} \cup Q^w_{2,1})| + |W_{1,2} \cap (Q^w_{1,2} \cup Q^w_{2,2})| - |W_{1,1} \setminus W_{1,2}| \\
&\geq |A_{1,1}| - |V_{1,1} \setminus V_{2,1}| - |B_{2,1}| + |A_{1,2}| - |V_{1,2} \setminus V_{2,2}| - |B_{2,2}| - |B_{1,1}| \\
&\geq |A_{1,1}| - |B_{1,1}| - |B_{2,1}| + |A_{1,2}| - |B_{1,2}| - |B_{2,2}| - |B_{1,1}|.
\end{aligned}
$$

Now, since $|M_{1,1}| < \frac{7}{13}n$, $|A_{1,1}| > \frac{6}{13}n$ by Lemma 4.2, and $|B_{1,1}| = |M_{1,1}| - |A_{1,1}| < \frac{1}{13}n$. For the same reason, $|B_{2,1}| < \frac{1}{13}n$ and $|B_{2,2}| < \frac{1}{13}n$. Hence,

$$
|W_{1,2}| \geq |A_{1,2}| - |B_{1,2}| + \frac{2}{13}n.
$$

Recall that $|W_{1,2}| = |M_{1,2}| = |A_{1,2}| + |B_{1,2}|$. Putting $|W_{1,2}| = |M_{1,2}|$ and $|B_{1,2}| = |M_{1,2}| - |A_{1,2}|$ to the above inequality, it follows that $|A_{1,2}| \leq |M_{1,2}| - \frac{1}{13}n$. By Lemma 4.2, we have that $|M_{1,2}| \geq n - |A_{1,2}|$. Hence, $|M_{1,2}| \geq \frac{7}{13}n$. □

7 Concluding Remarks

In this paper, we presented improved approximability and inapproximability results for the restricted instances of MAX SMTI. An obvious open problem is to close the gap between the upper and lower bounds on the approximation ratio in the general case. One direction would be to try to extend the result of Sec. 6 to arbitrary tie-lengths.

For inapproximability issues, Minimum Vertex Cover and Minimum Maximal matching have a property similar to MAX SMTI as mentioned in Sec. 1: 2-approximation is easy, $(2-\epsilon)$-approximation for a constant ϵ seems to be hard, but a tight lower bound on the approximation ratio is not yet proved. Finding closer relationships between Minimum Vertex Cover or Minimum Maximal Matching and MAX SMTI would be interesting.

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