Invited talk

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Title: Substitution decompositions and pattern classes of permutations

Abstract: Substitution (or, as they are sometimes known, modular or wreath) decompositions are a ubiquitous tool in many branches of combinatorics: graphs, tournaments, posets, etc. In this lecture I will lay down the foundations for the theory of substitution decompositions for pattern classes of permutations, and will then describe some initial work on this subject done jointly with Mike Atkinson and Rebecca Smith.

If $\sigma = s_1 \dots s_n$ is a permutation of length n, and if $\alpha_1, \dots, \alpha_n$ are arbitrary permutations, the *substitution* of $\alpha_1, \dots, \alpha_n$ into σ is the permutation $\tau = \sigma[\alpha_1, \dots, \alpha_n]$ consists of n consecutive subsequences isomorphic to $\alpha_1, \dots, \alpha_n$ respectively, whose relative order is the same as the relative order of the terms of σ. For example $231[21, 321, 12] = 4376512$. A permutation which does not admit a (non-trivial) such decomposition is said to be *simple*.

A set of permutations X is said to be a *pattern class* if it is closed under the *involvement*, or, equivalently, if it can be defined as the set of all permutations *avoiding* a certain set $B = B(X)$ of permutations (called the *basis* of X). The *substitution closure* of a pattern class X is the class

$$\mathcal{C}(X) = \{\sigma[\alpha_1, \dots, \alpha_n] \, : \, \sigma, \alpha_1, \dots, \alpha_n \in X, \ |\sigma| = n\}.$$

We say that X is *closed* if $X = \mathcal{C}(X)$. This happens if and only if the basis of X consists entirely of simple permutations.

The following questions naturally arise: *Given a pattern class X with basis B what is the basis for $\mathcal{C}(X)$? Under which conditions is it finite?* It is relatively easy to find a general answer to the first question: The basis of $\mathcal{C}(X)$ consists of all simple permutations minimal subject to not belonging to X. But this leaves the second question wide open. I will describe a complete solution to this question in the case where X is *principal*, i.e. its basis is a singleton. It turns out that even in this special case the demarcation line between finitely based and non-finitely based closures is remarkably involved. To give a taster: If X is the class with basis 1234 then $\mathcal{C}(X)$ is non-finitely based, while if Y has the basis 3412 then the basis of $\mathcal{C}(Y)$ is finite. I will

also discuss some obstacles and possible ways forward for resolving the full question.