ON SEMI-TRANSITIVE ORIENTATIONS AND GRAPHS REPRESENTABLE BY WORDS

Magnús M. Halldórsson[1]
School of Computer Science, Reykjavik University, Kringlan 1, 103 Reykjavik, Iceland
mmh@ru.is

Sergey Kitaev[2]
The Mathematics Institute, Reykjavik University, Kringlan 1, 103 Reykjavik, Iceland
sergey@ru.is

Artem Pyatkin[3]
Sobolev Institute of Mathematics, pr-t Koptyuga 4, 630090, Novosibirsk, Russia
artem@math.nsc.ru

Abstract

A graph $G = (V, E)$ is representable if there exists a word W over the alphabet V such that letters x and y alternate in W if and only if $(x, y) \in E$ for each $x \neq y$. If W is k-uniform (each letter of W occurs exactly k times in it) then G is called k-representable. The minimum k for which a representable graph G is k-representable is called its representation number.

In this paper we give a characterization of representable graphs in terms of orientations. Namely, we introduce a new type of graph orientations, the *semi-transitive orientations*, and show that a graph is representable if and only if it admits such an orientation. This allows us to prove a number of results about representable graphs, including that 3-colorable graphs are representable. It also shows that the recognition problem is in NP.

We obtain bounds on the representation number, showing that it is always at most n, while there exist graphs for which it is $n/2$. We also answer several open questions, in particular, on the representability of the Petersen graph and existence of triangle-free non-representable graphs.

Keywords: graphs, representation, words, orientations, complexity, circle graphs, comparability graphs.

1. Introduction

A directed graph (digraph) $G = (V, E)$ is *semi-transitive* if it is acyclic and for any directed path $v_1v_2...v_k$ either $v_1v_k \notin E$ or $v_iv_j \in E$ for all $1 \leq i < j \leq k$. Clearly, all transitive graphs

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are semi-transitive. This new notion proves to be a very helpful tool for solving open problems and deriving properties of graphs representable by words.

A graph $G = (V, E)$ is *representable* if there exists a word W over the alphabet V such that letters x and y alternate in W if and only if $(x, y) \in E$ for each $x \neq y$. It is k-representable if each letter appears exactly k times.

The notion of representable (directed) graphs was introduced in [6] to obtain asymptotic bounds on the free spectrum of the widely-studied Perkins semigroup which has played central role in semigroup theory since 1960, particularly as a source of examples and counterexamples.

Representable graphs appear also in periodic scheduling applications. Graham and Zang [3] study a counting problem relating to the cyclic movements of a robot arm. More generally, given a set of jobs to be performed periodically, certain pairs (a, b) must be done alternately, e.g. since the product of job a is used as a resource for job b. Any valid execution sequence corresponds to a word over the alphabet formed by the jobs. The representable graph given by the word must then contain the constraint pairs as a subgraph.

In [5] numerous properties of representable graphs are derived and several types of representable and non-representable graphs are pinpointed. Still, large gaps of knowledge of these graphs have remained, and the semi-transitive orientations introduced in this paper helps us to bridge them.

The main result of this paper is that the graph is representable if and only if it admits a semi-transitive orientation. This result allows us to make progress on the three most fundamental issues about representable graphs:

- Which types of graphs are representable and which ones are not?
- How large words can be needed to represent representable graphs?
- Are there alternative representations of these graphs that aid in reasoning about their properties?

We show that the class of representable graphs captures non-trivial graph properties. In particular, all 3-colorable graphs are representable, whereas various types of 4-chromatic graphs cannot all be represented. This resolves a conjecture of [5] regarding the Petersen graph, showing that it is representable. The result also properly captures all the previously known classes of representable graphs: outerplanar, prisms, and comparability graphs. On the negative side, we answer an open question of [5] by presenting a triangle-free non-representable graph.

Finally, we show that any representable graph is n-representable, again utilizing the semi-transitive orientability. This result implies that the problem of deciding whether a given graph is representable is contained in NP. Previously, no polynomial upper bound was known on the representation number, which is the smallest value k such that the given graph is k-representable. This bound on the representation number is tight up to a constant factor, as we construct graphs with representation number $n/2$. We also show that deciding if a representable graph is k-representable is NP-complete for $3 \leq k \leq n/2$, while the polynomically decidable class of circle graphs concides with the class of graphs with representation number at most 2.

The paper is organized as follows. In Section 2 we give definitions of objects of interest and review some of the known results. In Section 3 we give a characterization of representable graphs in terms of orientations and discuss some important corollaries of this fact. In Section 4 we examine the representation number, and show that it is always at most n but can be as much as $n/2$. We explore in Section 5 which classes of graphs are representable, showing in particular 3-colorable graphs to be representable, but numerous other properties to be orthogonal to representability. The construction for triangle-free non-representable graphs is also presented there. Finally, we conclude with a discussion of algorithmic complexity and some open problems in Section 6.

2. Definitions, notation, and known results

In this section we follow [5] to define the objects of interest.

Let W be a finite word. If W involves the letters $x_1, x_2, \ldots, x_n$ then we write $Var(W) = \{x_1, \ldots, x_n\}$. A word is *$k$-uniform* if each letter appears in it exactly k times. A 1-uniform word is also called a *permutation*. Denote by W_1W_2 the concatenation of words W_1 and W_2. We say that the letters x_i and x_j *alternate* in W if the word induced by these two letters contains neither x_ix_i nor x_jx_j as a factor. If a word W contains k copies of a letter x then we denote these k appearances of x by $x^1, x^2, \ldots, x^k$. We write $x_i^j < x_k^l$ if x_i^j stays in W before x_k^l, i. e., x_i^j is to the left of x_k^l in W.

We say that a word W *represents* the graph $G = (V, E)$ if there is a bijection $\phi : Var(W) \to V$ such that $(\phi(x_i), \phi(x_j)) \in E$ if and only if x_i and x_j alternate in W. It is convenient to identify the vertices of a representable graph and the corresponding letters of a word representing it. We call a graph G *representable* if there exists a word W that represents G. If G can be represented by a k-uniform word, then we say that G is *k-representable*. The *representation number* of a representable graph G is the minimum k such that G is k-representable.

We call a graph *permutationally representable* if it can be represented by a word of the form $P_1P_2 \ldots P_k$ where all P_i are permutations.

A digraph is *transitive* if the adjacency relation is transitive, i. e. for every vertices $x, y, z \in V$ the existence of the arcs $xy, yz \in E$ yields that $xz \in E$. A *comparability graph* is an undirected graph having an orientation of the edges that yields a transitive digraph.

The class of representable graphs is known to contain comparability graphs [6]; moreover, the comparability graphs are precisely the permutationally representable graphs. Further, outerplanar graphs are 2-representable, all prisms are 3-representable and so is the 3-subdivision of any graph [5].

The following properties of representable graphs are useful [5]. A graph G is representable if and only if it is k-representable for some k. If $W = AB$ is k-uniform word representing a graph G, then the word $W' = BA$ also k-represents G. If the bicomponents of G are representable, then G is also representable.

The following property is useful in proving the non-representability of graphs.

Theorem 1. *If G is representable, then for every $x \in V(G)$ the graph induced by the neighbors $N(x)$ of x is permutationally representable.*

To construct a non-representable graph we just take a graph that is not a comparability graph and add an all-adjacent vertex to it. The wheel W_5 is the smallest non-representable graph. Some non-representable graphs on 6 and 7 vertices are given in Fig. 1 (see [5]).

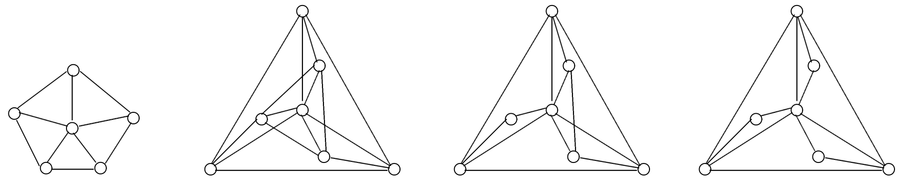

FIGURE 1. Small non-representable graphs

3. Characterization of Representable Graphs by Orientability

The word representation of representable graphs is simple and natural. Yet it does not lend to easy arguments for the characteristic of representable graphs. Non-representability is even harder to argue in terms of the many possible corresponding words. The main result of this section is a new characterization of representable graphs that leads easily to various results about representability.

We give a characterization in terms of orientability, which implies that representability corresponds to a property of a digraph obtained by directing the edges in certain way. Recall that a graph is permutationally representable if and only if it has a transitive orientation. We prove a similar fact on representable graphs, namely, that a graph is representable if and only if it has a so-called *semi-transitive orientation*. Our definition, in fact, generalizes that of a transitive orientation.

Other orientations have been defined in order to capture generalizations of comparability graphs. As transitive orientations form constraints on the orderings of induced P_3, these generalizations form constraints on the orderings of induced P_4. These include *perfectly orderable graphs* (and its subclasses) and *opposition graphs* [1]. Classes such as *chordal graphs* are defined in terms of vertex-orderings, and imply therefore indirectly acyclic orientations. None of these properties captures our definition below, nor does our characterization subsume any of them.

We turn to the characterization and start with definitions of certain directed graphs. A *semi-cycle* is the directed acyclic graph obtained by reversing the direction of one arc of a directed cycle. An acyclic digraph is a *shortcut* if it is induced by the vertices of a semi-cycle and contains a pair of non-adjacent vertices. Thus, a digraph on the vertex set $\{v_0, v_1, \ldots, v_t\}$, is a shortcut if it contains a directed path $v_0v_1 \ldots v_t$, the arc v_0v_t and it is missing an arc v_iv_j, $0 \leq i < j \leq t$ (in particular, $t \geq 3$).

Definition 2. A digraph is *semi-transitive* if it is acyclic and contains no shortcuts.

A graph is *semi-transitively orientable* if there exists an orientation of the edges that results in a semi-transitive graph.

Our main result in this paper is the following.

Theorem 3. *A graph is representable if and only if it is semi-transitively orientable.*

We first need some additional definitions and lemmas. A *topological order* (or *topsort*) of an acyclic digraph is a permutation of the vertices that obeys the arcs, i. e. for each arc uv, u precedes v in the permutation. For a node-labeled digraph, let the topsort also refer to the word obtained by visiting the nodes in that order. Let $D = (V, E)$ be a digraph. The *t-string* digraph

D^t of D is defined as follows. The vertices of D^t are v^i, for $v \in V$ and $i = 1, 2, \ldots, t$, and $v^i u^j$ is an arc in D^t if and only if either $i = j$ and $vu \in E$ or $i < j$ and $uv \in E$. Intuitively, the t-string digraph of D has t copies of D strung together. Given a word S, let G_S denote the graph represented by S. If S is a topsort of D^t then we also denote by G_S the graph represented by the word S' obtained from S by omitting the superindices of the vertices (i. e. the copies of the same vertex in S are considered as the same letters in S').

Given a digraph D, let G_D be the graph obtained by ignoring orientation.

We argue that the word representing a semi-transitive digraph comes from a special topological ordering of the t-string digraph D^t for some t. We first observe that any topological ordering of D^t preserves arcs.

Lemma 4. *Let D be a digraph with distinct node-labels. Let S be a topological ordering of a D^t. Then G_D is a subgraph of G_S.*

Proof. Consider an edge (u, v) in G_D, and suppose without loss of generality that it is directed as uv in D. Then, in D^t, there is a directed path $u^1v^1u^2v^2 \ldots u^tv^t$. Thus, occurrences of u and v in a topsort of D^t are alternating. Hence, $uv \in G_S$. □

To prove equivalence, we now give a method to produce a topological ordering that generates all non-arc. We say that a subgraph H *covers* a set A of non-arcs if each non-arc in A is also found in H. A word covers the non-arc if the digraph it represents covers them.

Lemma 5. *The non-arcs incident with a path in a semi-transitive digraph can be covered with a 2-uniform word.*

Proof. Let P be a path in a semi-transitive digraph D. We shall form a topsort S of the 2-string digraph D^2 and show that it covers all non-arcs having at least one endpoint on P.

We say that a node x of D^2 *depends on* node y if there is a directed path from y to x in D^2, i. e. y must appear before x in a topological ordering of D^2. We use the notation $y \rightsquigarrow x$ if x depends on y. A node is listed *earliest possible* if it is listed as soon as all nodes that it depends on have been listed. A node is listed *latest possible* if it is listed after all nodes that do not depend on it.

Let S be any topological ordering of D^2 where the first occurrences of nodes in P are as late as possible and the second occurrences are as early as possible. The ordering of other nodes is arbitrary, within these constraints.

We claim that this word S covers all non-arcs involving nodes in P. Consider a pair u, v, where $(u, v) \notin G_D$ and $u \in P$. Note that v may also belong to P, in which case we may assume that the path goes from u to v. Consider the listings of u^1, v^1, u^2, v^2, where the subscript refers to the occurrence number of the node. Observe that u may depend on v, or vice versa, but not both. There are three cases to consider.

Case (i): There is a path from u to v in D. We claim that u^2 does not depend on v^1. Suppose it does, i. e. $v^1 \rightsquigarrow u^2$. Then, there is an arc $x^1y^2 \in D^2$ such that $v^1 \rightsquigarrow x^1$ and $y^2 \rightsquigarrow u^2$. By the assumptions and the symmetry of the two copies of D in D^2, it follows that $y^1 \rightsquigarrow u^1 \rightsquigarrow v^1 \rightsquigarrow x^1$. By the definition of 2-string graphs, yx is an arc in D, so $(y^1, x^1) \in E(D^2)$. Then, by semi-transitivity, $u^1v^1 \in E(D^2)$, or $(u, v) \in E(G_D)$, which is a contradiction. It now follows that the nodes will occur as $u^1u^2v^1v^2$ in S, i. e. $uv \notin E(G_S)$.

Case (ii): There is a path from v to u in D. This is symmetric to case (i), with u replaced by v. Thus, the nodes will occur as $v^1v^2u^1u^2$ in S.

Case (iii): The nodes u and v are incomparable in D. In particular, v is not in P. Then, u^1 and v^1 do not depend on each other, nor do u^2 and v^2. If v^2 depends on u^1 then the nodes occur as $v^1u^1u^2v^2$ in S. Otherwise, their order is $v^1v^2u^1u^2$. □

We now return to the proof of Theorem 3, starting with the forward direction. Given a word-representant S, we direct an edge from x to y if the first occurence of x is before that of y in the word. Let us show that such orientation D of G_S is semi-transitive. Indeed, assume that $x_0x_t \in E(D)$ and there is a directed path $x_0x_1 \dots x_t$ in D. Then in the word S we have $x_0^i < x_1^i < \dots < x_t^i$ for every i. Since $x_0x_t \in E(D)$ we have $x_t^i < x_0^{i+1}$. But then for every $j < k$ and i there must be $x_j^i < x_k^i < x_j^{i+1}$, i. e. $x_ix_j \in E(D)$. So, D is semi-transitive.

For the other direction, denote by G the graph and by D its semi-transitive orientation. Let $P_1, P_2, \dots, P_\tau$ be the set of directed paths covering all vertices of D. For every $i = 1, 2, \dots, \tau$ denote by S_i the topsort of the digraph D^2 satisfying the conditions of Lemma 5 for the path P_i. Put $S = S_1S_2 \dots S_\tau$. Clearly, S is a 2τ-uniform word; it can be treated as a topsort of a 2τ-string $D^{2\tau}$. Then $G = G_S$. Indeed, by Lemma 4 we have $E(G) \subset E(G_S)$. On the other hand, if $uv \notin E(G)$ then $u \in P_i$ for some i, and thus by Lemma 5 the letters u and v are not alternating in the subword S_i. Therefore, $uv \notin E(S)$. Theorem 3 is proved. □

Theorem 3 makes clear the relationship to comparability graphs, which are those that have transitive orientations. Since transitive digraphs are also semi-transitive, this immediately implies that comparability graphs are representable.

The construction in Lemma 5 shows that all representable graphs can be represented "almost" permutationally. This is made more precise as follows.

Observation 6. *Let G be a representable graph. Then there is a word W representing G such that for any prefix P of W and any pair a, b of letters, the number of occurrences of a and b in P differ by at most two.*

4. The Representation Number of Graphs

We focus now on the following question: Given a representable graph, how large is its representation number? In [5], certain classes of graphs were proved to be 2- or 3-representable, and an example was given of a graph (the triangular prism) with the representation number of 3. On the other hand, no examples were known of graphs with representation numbers larger than 3, nor were there any non-trivial upper bounds known. We show here that the maximum representation number of representable graphs is linear in the number of vertices.

For the upper bound, we use the results of the preceding section. We have the following directly from the proof of Theorem 3.

Corollary 7. *A representative graph G is $2\tau(G)$-representable, where $\tau(G)$ is the minimum number of paths covering all nodes in some semi-transitive orientation of G.*

This immediately gives an upper bound of $2n$ on the representation number. We can improve this somewhat with an effective procedure.

Theorem 8. *Given a semi-transitive digraph D on n vertices, there is a polynomial time algorithm that generates an n-uniform word representing G_D. Thus, each representable graph is n-representable.*

Proof. The algorithm works as follows.

Step 0. Start with $A = \emptyset$ and $i = 1$.

Step i. If D contains a path P_i covering at least two vertices from $V \setminus A$ then let $A := A \cup V(P_i)$ and $i := i + 1$. Otherwise, let $B = V \setminus A$ and go to the Final Step.

Final Step. Let S_i be the topsort of the digraph D^2 satisfying the conditions of Lemma 5 for the path P_i and put $S' = S_1 S_2 \ldots S_t$ where t is the number of paths found at previous steps. If $|B| \leq 1$ then let $S = S'$. Otherwise, consider a topsort S_0 of D where the vertices of B are listed in a row (since the vertices of B do not depend on each other, such a topsort must exist) and in particular in the reverse order of their appearance in S_1. Let $S = S' S_0$.

Clearly, $G_D = G_S$ (the proof is the same as in Theorem 3). It is easy to verify that each letter appears in S at most n times. □

Theorem 8 implies that the graph representability is polynomially verifiable, answering an open question in [5]. Indeed, having a representable graph G, we may ask for a word representing it and verify this fact in time bounded by the polynomial in n.

Corollary 9. *The recognition problem for representable graphs is in NP.*

We now show that there are graphs with representation number of $n/2$, matching the upper bound within a factor of 2.

The *cocktail party graph* $H_{k,k}$ is the graph obtained from the complete bipartite graph $K_{k,k}$ by removing a perfect matching. Denote by G_k the graph obtained from a cocktail party graph $H_{k,k}$ by adding an all-adjacent vertex.

Theorem 10. *The graph G_k has representation number $k = \lfloor n/2 \rfloor$.*

The proof is based on three statements.

Lemma 11. *Let H be a graph and G be the graph obtained from H by adding an all-adjacent vertex. Then G is k-representable if and only if H is permutationally k-representable.*

Proof. Let 0 be the letter corresponding to the all-adjacent vertex. Then every other letter of the word W representing G must appear exactly once between two consecutive zeroes. We may assume also that W starts with 0. Then the word $W \setminus \{0\}$, formed by deleting all occurrences of 0 from W, is a permutational k-representation of H. Conversely, if W' is a word permutationally k-representing H, then we insert 0 in front of each permutation to get a (permutational) k-representation of G. □

Recall that the *order dimension* of a poset is the minimum number of linear orders such that their intersection induces this poset.

Lemma 12. *A comparability graph is permutationally k-representable if and only if the poset induced by this graph has dimension at most k.*

Proof. Let H be a comparability graph and W be a word permutationally k-representing it. Each permutation in W can be considered as a linear order where $a < b$ if a meets before b in the permutation (and vice versa). We want to show that the comparability graph of the poset induced by the intersection of these linear orders coincides with H.

Two vertices a and b are adjacent in H if and only if their letters alternate in the word. So, they must be in the same order in each permutation, i. e. either $a < b$ in every linear order or $b < a$ in every linear order. But this means that a and b are comparable in the poset induced by the intersection of the linear orders, i. e. a and b are adjacent in its comparability graph. □

The next statement most probably is known but we give its proof here for the sake of completeness.

Lemma 13. *The poset P over $2k$ elements $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ such that $a_i < b_j$ for every $i \neq j$ and all other elements are not comparable has dimension k.*

Proof. Assume that this poset is the intersection of t linear orders. Since a_i and b_i are not comparable for each i, their must be a linear order where $b_i < a_i$. If we have in some linear order both $b_i < a_i$ and $b_j < a_j$ for $i \neq j$, then either $a_i < a_j$ or $a_j < a_i$ in it. In the first case we have that $b_i < a_j$, in the second that $b_j < a_i$. But each of these inequalities contradicts the definition of the poset. Therefore, $t \geq k$.

In order to show that $t = k$ we can consider a linear order $a_1 < a_2 < \ldots < a_{k-1} < b_k < a_k < b_{k-1} < \ldots < b_2 < b_1$ together with all linear orders obtained from this order by the simultaneous exchange of a_k and b_k with a_m and b_m respectively $(m = 1, 2, \ldots, k-1)$. It can be verified that the intersection of these k linear orders coincides with our poset. □

Now we can prove Theorem 10. Since the cocktail party graph $H_{k,k}$ is a comparability graph of the poset P, we deduce from Lemmas 13 and 12 that $H_{k,k}$ is permutationally k-representable but not permutationally $(k-1)$-representable. Then by Lemma 11 we have that G_k is k-representable but not $(k-1)$-representable. Theorem 10 is proved. □

The above arguments help us also in deciding the complexity of determinining the representation number. From Lemmas 11 and 12, we see that it is as hard as determining the dimension k of a poset. Yannakakis [9] showed that the latter is NP-hard, for any $3 \leq k \leq \lceil n/2 \rceil$. We therefore obtain the following.

Proposition 14. *Deciding whether a given graph is k-representable, for any given $3 \leq k \leq \lceil n/2 \rceil$, is NP-complete.*

It was further shown by Hegde and Jain [4] that it is NP-hard to approximate the dimension of a poset within almost a square root factor. We therefore obtain the same hardness for the representation number.

Proposition 15. *Approximating the representation number within $n^{1/2-\epsilon}$-factor is NP-hard, for any $\epsilon > 0$.*

In contrast with these hardness results, the case $k = 2$ turns out to be easier and admits a succinct characterization. The following fact essentially appears in [2]. Recall that a graph is called *a circle graph* if we can arrange its vertices as chords on a circle in such a way that two nodes in the graph are adjacent if and only if the corresponding chords overlap.

Observation 16. *A graph is 2-representable if and only if it is a circle graph.*

Indeed, given a circle graph G, consider the ends of the chords on a circle as a letters and read the obtained word in a clockwise order starting from an arbitrary point. It is easy to see that two chords intersect if and only if the corresponding letters alternate in the word. For the opposite direction, place $2n$ nodes at a circle in the same order as they meet in the word and connect the same letters by chords.

Since outerplanar graphs are 2-representable, it follows from Theorems 16 that outerplanar graphs are circle graphs. Theorem 16 can also be useful as a tool in proving that a graph is not a circle graph. For example, non-representable graphs (for instance, all odd wheels W_{2t+1} for $t \geq 2$) are not circle graphs.

5. Characteristics of Representable Graphs

When faced with a new graph class, the most basic questions involve the kind of properties it satisfies: which known classes are properly contained (and which not), which graphs are otherwise contained (and which not), what operations preserve representability (or non-representability), and which properties hold for these graphs.

Previously, it was known that the class of representable graphs includes comparability graphs, outerplanar graphs, subdivision graphs, and prisms. The purpose of this section is to clarify this situation significantly, include resolving some conjectures. We start with exploring the impact of colorability on representability.

Chromatic number and representability.

Theorem 17. *3-colorable graphs are semi-transitive, and thus representable.*

Proof. Given a 3-coloring of a graph, direct its edges from the first color class through the second to the third class. It is easy to see that we obtain a semi-transitive digraph. □

This implies a number of earlier results on representability, including that of outerplanar graphs, subdivision graphs, and prisms The theorem also shows that 2-degenerate graphs graphs of maximum degree 3 (via Brooks theorem), and triangle-free planar graphs (via Grötzch's theorem) are all representable.

This result does not extend to higher chromatic numbers. The examples in Fig. 1 show that 4-colorable graphs can be non-representable. We can, however, obtain a results in terms of the *girth* of the graph, which is the length of its shortest cycle.

Proposition 18. *Let G be a graph whose girth is greater than its chromatic number. Then, G is representable.*

Proof. Suppose the graph is colored with $\chi(G)$ natural numbers. Orient the edges of the graph from small to large colors. There is no directed path with more than $\chi(G) - 1$ arcs, but since G contains no cycle of $\chi(G)$ or fewer edges, there can be no shortcut. Hence, the digraph is semi-transitive. □

Theorem 17 also implies that the Petersen graph is representable, turning down a conjecture in [5]. We can show that the graph is actually 3-representable. We give here two of its 3-representations, related to the numbering in Fig. 2, that were found in [7]:

- 1, 3, 8, 7, 2, 9, 6, 10, 7, 4, 9, 3, 5, 4, 1, 2, 8, 3, 10, 7, 6, 8, 5, 10, 1, 9, 4, 5, 6, 2
- 1, 3, 4, 10, 5, 8, 6, 7, 9, 10, 2, 7, 3, 4, 1, 2, 8, 3, 5, 10, 6, 8, 1, 9, 7, 2, 6, 4, 9, 5

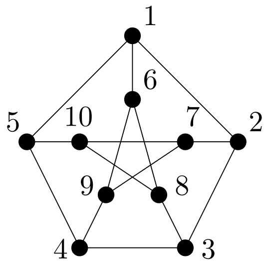

FIGURE 2. Petersen's graph

The following argument shows that Petersen's graph is *not* 2-representable. Suppose that the graph is 2-representable and W is a word 2-representing it. Let x be a letter in W such that there are minimum number of letters between the two appearances of x. Clearly, there are exactly three different letters between them. By symmetry, we can assume that $x = 1$ we can assume that W starts with 1. So, letters 2,5, and 6 are between the two 1's and because of symmetry, the fact that Petersen's graph is edge-transitive (that is, each of its edges can be made "internal"), and taking into account that nodes 2, 5, and 6 are pairwise not adjacent, we can assume that $W = 12561W_1 6W_2 5W_3 2W_4$ where W_i's are some factors for $i = 1, 2, 3, 4$. To alternate with 6 but not to alternate with 5, we must have $8 \in W_1$ and $8 \in W_2$. Also, to alternate with 2 but not to alternate with 5, we must have $3 \in W_3$ and $3 \in W_4$. But then 8833 is a subsequence in W and thus 8 and 3 are not adjacent in the graph, a contradiction.

We explore now further graph properties that are orthogonal to representability.

Non-representable graphs. One of the open problems posed in [5] was the following.

Problem 1. *Are there any non-representable graphs that do not satisfy the conditions of Theorem 1? In particular, are there triangle-free non-representable graphs?*

We give here a positive answer. The smallest found counterexample to the converse of Theorem 1 is given by the graph in Fig. 3 called co-(T_2) in [8]. It is easy to check that the induced neighborhood of any node of the graph co-(T_2) is a comparability graph.

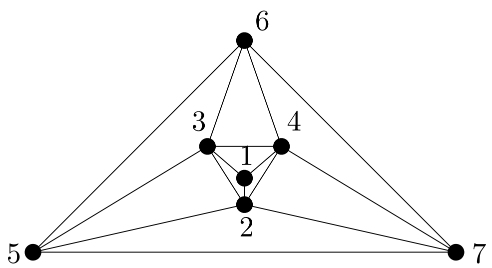

FIGURE 3. Co-(T_2) graph

Theorem 19. *The graph co-(T_2) is non-representable.*

Proof. Assume that the graph in Fig. 3 is k-representable for some k and W is a word-representant for it. The vertices 1,2,3,4 form a clique; so, their appearances $1^i, 2^i, 3^i, 4^i$ in W must be in the same order for each $i = 1, 2, \ldots, k$. By symmetry we may assume that the order is 1234. Now let I_1, $I_2, \ldots, I_k$ be the set of all $[2^i, 4^i]$-intervals in W. Two cases are possible.

1. There is an interval I_j such that 7 belongs to it. Then since 2,4,7 form a clique, 7 must be inside each of the intervals I_1, $I_2, \ldots, I_k$. But then 7 is adjacent to 1, a contradiction.
2. 7 does not belong to any of the intervals I_1, $I_2, \ldots, I_k$. Again, since 7 is adjacent to 2 and 4, each pair of consecutive intervals I_j, I_{j+1} must be separated by a single 7. But then 7 is adjacent to 3, a contradiction.

□

Remark 20. Note that the existence of edges between the vertices 5,6,7 was not used in the proof of Theorem 19. So, Theorem 19 actually gives us four counter-examples to the converse of Theorem 1.

The next theorem shows us how to construct an infinite series of triangle-free non-representable graphs.

Theorem 21. *There exist triangle-free non-representable graphs.*

Proof. Let H be a 4-chromatic graph with girth at least 10 (such graphs exist by Erdös theorem). For every path P of length 3 in H add to H the edge e_P connecting its ends. Denote the obtained graph by G. Let us show that G is a triangle-free non-representable graph.

If G contains a triangle on the vertices u, v, w then H contains three paths P_{uv}, P_{uw}, and P_{vw} of lengths 1 or 3 connecting these vertices. Let T be a graph spanned by these three paths. Since T has at most 9 edges and the girth of H is at least 10, T is a tree. Clearly, it cannot be a path. So, it is a subdivision of $K_{1,3}$ with the leafs u, v, w. But then at least one of the paths P_{uv}, P_{uw}, P_{vw} must have an even length, a contradiction.

So, G is triangle-free. Assume that G has a semi-transitive orientation. Then it induces a semi-transitive orientation on H. Since H is 4-chromatic, each of its acyclic orientation must contain a directed path P of length at least 3. But then the orientation of the edge e_P in G produces either a 4-cycle or a shortcut, contradicting the semi-transitivity. So, G is a triangle-free non-representable graph. □

What about other non-representable graphs? Or, rather, which important classes of graphs are not contained in the class of representable graphs? We establish the following classes to be not necessarily representable:

- chordal (see the rightmost graph in Fig. 1), and thus perfect,
- line (second graph in Fig. 1),
- co-trees (the graph co-T_2 earlier), and thus co-bipartite and co-comparability,
- 2-outerplanar (the first and last graphs in Fig. 1), and thus planar
- split (the last graph in Fig. 1),
- 3-trees (the last graph in Fig. 1), and thus partial 3-trees.

On the other hand, the 4-clique, K_4, is representable and it belongs to all these classes.

The effect of graph operations. One may want to explore which operations on graphs preserve representability (or non-representability). We pinpoint one such operation, and list others that are orthogonal.

(1) The following operation on a representable graph yields a representable graph: Replace any node with a comparability graph, connecting all the new nodes to the neighbors of the original node. I. e., replacing a node with a *module* that is a comparability graph.
(2) A generalization of a previous result about biconnected components is false. Namely, if we identify cliques of size more than 1 from two representable graphs we don't necessarily get a representable graph. Indeed, consider the rightmost graph in Fig. 1 without a node of degree 2 connected to the endpoints of edge e (denote this graph by G), and identify e with an edge in a triangle T resulting in obtaining the rightmost graph in Fig. 1. Both G and T are representable, but gluing them through an edge (a clique of size 2) results in a non-representable graph.
(3) The complement to a non-representable graph can be permutationally representable: see, for example, the second graph in Fig. 1.
(4) Not much can be said in general about the operation of taking the line graph. For example, the second non-representable graph in Fig. 1 is obtained from $K_{2,3}$ together with an edge between the nodes of degree 3, which is representable. On the other hand, there are many easy examples when representable graphs go to representable graphs by taking line graph operation.

6. Concluding Remarks and Open Questions

It is natural to ask about optimization problems on representable graphs. Theorem 17 implies that many classical optimization problems are NP-hard on representable graphs:

Observation 22. *The optimization problems Independent Set, Dominating Set, Graph Coloring, Clique Partition, Clique Covering are NP-hard on representative graphs.*

Note that it may be relevant whether the representation of the graph as a semi-transitive digraph is given; solvability under these conditions is open.

However, some problems remain polynomially solvable:

Observation 23. *The Clique problem is polynomially solvable on representation graphs.*

Indeed, we can simply use the fact that the neighborhood of any node is a comparability graph. The clique problem is easily solvable on comparability graphs. Thus, it suffices to search for the largest clique within all induced neighborhoods.

We conclude with several open questions about representable graphs:

(1) Is it NP-hard to decide whether a graph is representable?
(2) What is the maximum representation number of a graph? We know that it lies between $n/2$ and n.
(3) Is there a forbidden subgraph characterization of representative graphs? This problem seems to be hard since even for 2-representable (i. e. circle) graphs such a characterization is unknown.
(4) Are the graphs of maximum degree 4 representable?

(5) Is there an algorithm that forms an $f(k)$-representation of a k-representable graph, for some function f? Namely, can the representation number be approximated as a function of itself? The same question holds also for the partial order dimension.

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