Constrained stabilization via $(k, \lambda)$-contractive sets with an application to Buck converters

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Paper:
Outline

- Setting: nonautonomous homogeneous dynamics
- Finite-time control Lyapunov functions
- Motivating case-study
- Periodic control laws
- Buck converter application
- Conclusions
Setting

Homogeneous dynamics of order one

\[ x^+ = \Phi(x), \ x \in \mathbb{R}^n, \ \Phi(\alpha x) = \alpha \Phi(x) \] for all \( \alpha \in \mathbb{R}_+ \).

\[ x^+ = \Phi(x, u), \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^n, \ \Phi(\alpha x, u) = \alpha \Phi(x, \frac{1}{\alpha}), \ \Phi(x, \alpha u) = \alpha \Phi(\frac{1}{\alpha} x, u) \] for all \( \alpha \in \mathbb{R}_+ \).

State constraints

\[ x_t \in X \subset \mathbb{R}^m, \ X \text{ is a proper } C\text{-set}, \ \forall t \in \mathbb{N} \]

Input Constraints

\[ u_t \in U \subset \mathbb{R}^n, \ U \text{ is a proper } C\text{-set}, \ \forall t \in \mathbb{N} \]

\( S \subset \mathbb{R}^n \): compact, convex and contains the origin

\( S \): proper \( C\)-set if contains the origin in interior
Controlled $\lambda$-contractive sets

controlled $\lambda$–contractive sets

$\lambda \in [0, 1)$, $\exists g : \mathbb{U} \rightarrow \mathbb{X}$

$x \in S$ implies $\Phi(x, g(x)) \in \lambda S$
Controlled \( (k, \lambda) \)-contractive sets

\[
\lambda \in [0,1), \exists \ g : \mathbb{U} \to \mathbb{X}, \quad x \in S \implies \Phi(x, g(x)) \in \lambda S
\]

controlled \( \lambda \)-contractive sets

\[
\lambda \in [0,1), \ k \in \mathbb{N}, \ \exists \ g : \mathbb{U} \to \mathbb{X}, \quad x \in S \implies \Phi^i(x, g(x)) \in \mathbb{X}, \ i \in \mathbb{N}_{[1, k-1]}, \quad \Phi^k(x, g(x)) \in \lambda S
\]

controlled \( (k, \lambda) \)-contractive sets.
Control Lyapunov functions

$V : \mathbb{X} \to \mathbb{R}_+, \mathcal{S}:$ controlled invariant

$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|), \ \forall \xi \in \mathcal{S}$

$V(\Phi(\xi, g(\xi))) \leq \lambda V(\xi), \ \forall \xi \in \mathcal{S}$

Control Lyapunov function
Finite Time Control Lyapunov functions

\[ V : \mathbb{X} \to \mathbb{R}_+, \ S: \text{controlled invariant} \]
\[ \alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|), \ \forall \xi \in S \]
\[ V(\Phi(\xi, g(\xi))) \leq \lambda V(\xi), \ \forall \xi \in S \]

Finite Time Control Lyapunov function

\[ V : \mathbb{X} \to \mathbb{R}_+, \ S: (k, 1)\text{-controlled invariant} \]
\[ \alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|), \ \forall \xi \in \mathbb{X} \]
\[ V(\Phi^k(\xi, g(\xi))) \leq \lambda V(\xi), \ \forall \xi \in S \]

Control Lyapunov function

\[ \Phi^k(\xi, g(\xi)) := \Phi(\Phi^{k-1}(\xi, g(\xi)), g(\Phi^{k-1}(\xi, g(\xi))))), \text{ for any } k \geq 1 \]
\[ \Phi^0(\xi, g(\xi)) = 0 \]
Set-induced Lyapunov functions

- $x_{t+1} = A_t x_t + B_t u_t$
- $S \subset \mathbb{R}^n$ is a controlled $\lambda$-contractive set
- $V(x) := \text{gauge}(S, x)$ is a control Lyapunov function
Set-induced Lyapunov functions

- \( x_{t+1} = A_t x_t + B_t u_t \)
- \( S \subseteq \mathbb{R}^n \) is a controlled \( \lambda \)-contractive set
- \( V(x) := \text{gauge}(S, x) \) is a control Lyapunov function

- \( x_{t+1} = \Phi(x_t, u_t) \), homogeneous w.r.t. both arguments
- \( S \subseteq \mathbb{R}^n \) is a controlled \( (k, \lambda) \)-contractive set
- \( V(x) := \text{gauge}(S, x) : \) finite-time control Lyapunov function
Set-induced Lyapunov functions

- \( x_{t+1} = A_t x_t + B_t u_t \)
- \( S \subset \mathbb{R}^n \) is a controlled \( \lambda \)-contractive set
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Finite time control Lyapunov functions \( \iff \) controlled \((k, \lambda)\)-contractive sets.

- \( x_{t+1} = \Phi(x_t, u_t) \), homogeneous w.r.t. both arguments
- \( S \subset \mathbb{R}^n \) is a controlled \((k, \lambda)\)-contractive set
- \( V(x) := \text{gauge}(S, x) \) : finite-time control Lyapunov function
Motivating case study

Constrained stabilization of a Buck converter

Averaged discrete-time model:
\[ x^+ = Ax + Bu + f, \quad f = 0, \]
\[ x := (v_C \quad i_L)^\top \]

Set-point ≠ origin:
\[ x_s := (10 \quad 1)^\top \]

\[
A = \begin{pmatrix} 0.9456 & 0.4388 \\ -0.0439 & 0.9719 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2019 \\ 0.8978 \end{pmatrix}, \quad \rho(A) = 0.9687
\]

Hard constraints on:
- the duty-cycle ratios \( u \in [0, 1] \)
- the voltage of the output capacitor \( v_C \in [0, 22.5] \)
- the current through the filter inductor \( i_L \in [0, 3] \)
Motivating case study

Maximal controlled $\lambda$-contractive set:
- does not contain the initial condition $(0, 0)$ for any $\lambda < 1$
- previous construction results which included $(0, 0)$ make use of relaxed constraints (voltage through capacitor negative)

Maximal controlled invariant set:
- is controlled $(k, \lambda)$-contractive
- contains the significant initial condition $(0, 0)$
Problem formulation

Newly introduced concepts:

- controlled \((k, \lambda)\)-contractive sets
- finite-time control Lyapunov functions

Finite time control Lyapunov functions \(\Leftrightarrow\) controlled \((k, \lambda)\)-contractive sets.

**Aim:** exploit new concepts for constrained stabilization of linear systems.
Problem formulation

Newly introduced concepts:

- \textit{controlled} \((k, \lambda)\)-contractive sets
- \textit{finite-time} control Lyapunov functions

Finite time control Lyapunov functions ⇔ controlled \((k, \lambda)\)-contractive sets.

\textbf{Aim:} exploit new concepts for constrained stabilization of linear systems.

Compute a \(k \in \mathbb{N}\) such that a set \(S\) is controlled \((k, \lambda)\)-contractive and a corresponding periodic stabilizing state-feedback control law

- \(x_{t+1} = \Phi(x_t, u_t)\), homogeneous w.r.t. both arguments
- \(S \subset \mathbb{R}^n\) is a controlled \((k, \lambda)\)-contractive set
- \(V(x) := \text{gauge}(S, x)\), is a finite-time control Lyapunov function
Problem formulation

\[ x_{t+1} = \Phi(x_t, u_t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \]
\[ \Phi(\alpha x_t, u_t) = \alpha \Phi(x_t, \frac{1}{\alpha} u_t), \quad \Phi(x_t, \alpha u_t) = \alpha \Phi(\frac{1}{\alpha} x_t, u_t) \quad \text{for all } \alpha \in \mathbb{R}_+. \]

Hard constraints: \( x \in X, \ u \in U, \ X \in \mathbb{R}^n, \ U \in \mathbb{R}^m \) proper \( C \)-polytopic set.

Synthesis of stabilizing and admisible control laws:

- controlled \( (k, \lambda) \)-contractive proper \( C \)-set \( S \subseteq X \)
- find sequence \( \{g_i(\cdot)\}_{i \in \mathbb{N}_{0:k-1}}, \ g_i : X \rightarrow U \) such that for all \( x_0 \in S \):

\[
\begin{align*}
  x_{i+1} &= \Phi(x_i, g_i(x_i)), \quad i \in \mathbb{N}_{0:k-1} \\
  x_i &\in X, \quad i \in \mathbb{N}_{0:k-1} \\
  g_i(x_i) &\in U, \quad i \in \mathbb{N}_{0:k-1} \\
  x_k &\in \lambda S
\end{align*}
\]
Synthesis algorithms for linear systems

\[ x_{t+1} = Ax_t + Bu_t, \forall t \in \mathbb{N} \]

\[ S := \{ H_0 x \leq 1_p \} = \text{convh}(\{v_0^j\}_{j \in \mathbb{N}_{1,q}}) \]

\[ V_0 := [v_0^1, v_0^2, \ldots, v_0^q] \in \mathbb{R}^{n \times q} \]

\[ \mathbf{X} := \{ H_x x \leq 1_{p_x} \}, \mathbf{U} := \{ H_u u \leq 1_{p_u} \}. \]

**Prototype problem: LP**

- for any \( S \subset \mathbf{X} \), compute a \( k \in \mathbb{N} \), such that \( S \) is controlled \((k, \lambda)\)-contractive

- compute control actions for the vertices of \( S \) such that they enter \( \lambda S \) in \( k \) steps without violating the constraints

**Enables construction of:**

- two periodic state feedback control laws:
  - vertex interpolation
  - conewise linear
Outputs of the feasibility LP:

- $(k, \lambda)$-contractiveness check
- sequence of vertices: $V_i = [v_{i1}^1, v_{i2}^2, \ldots, v_{iL}^q], \forall i \in \mathbb{N}_{[1,k-1]}, V_k = V_0$
- sequence of corresponding inputs: $U_i = [u_{i1}^1, u_{i2}^2, \ldots, u_{iL}^q], \forall i \in \mathbb{N}_{[0,k-1]}$
Periodic vertex-interpolation control law

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Control law: \(\pi(x_t) := U_i \mu_i(x_t), \text{ if } t = kN + i, N \in \mathbb{N}\)

Where: \(\mu \in \mathbb{R}_{+}^q:\n\begin{align*}
x_t &= V_0 \mu \\
V_i \mu &= (AV_{i-1} + BU_{i-1}) \mu \\
1_T^T \mu &\leq 1, i \in \mathbb{N}_{[1,k-1]}\end{align*}
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\end{align*}
\]

Optimized parametrization:
- \(\mu_i(x_t)\) now uniquely defined:
- solve an optimization problem online at every \(k\) instants
Periodic vertex-interpolation control law

Outputs of the feasibility LP:
- \((k, \lambda)\)-contractiveness check
- sequence of vertices: \(V_i = [v_i^1, v_i^2, \ldots, v_i^q] \), \(\forall i \in \mathbb{N}_{[1,k-1]}\), \(V_k = V_0\)
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Optimized parametrization:
- \(\mu_i(x_t)\) now uniquely defined:
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Optimal state–feedback control law:
\(\pi(x_t) := U_i \mu_i^*,\) if \(t = kN + i\), \(N \in \mathbb{N}\),
\(\mu_i^*\)-optimal solution
The state–feedback control law is:

$$\pi(x_t) = g_i(x_t), \text{ if } t = kN + i, \; N \in \mathbb{N}$$

$$g_i(x_t) = U_i^s (V_i^s)^{-1} x_{t+i}, \text{ if } x_{t+i} \in D_i^s, \; s \in \mathbb{N}_{[1,p_i]}$$

ex: $$V_1^1 = [[V_1]:1][V_1]:2$$
Application to the Buck converter

Averaged discrete-time model:
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Set-point \( \neq \) origin:
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Periodic vertex-interpolation solution

Results:

- $k = 4, \lambda = 0.99$
- worst case computational time: 0.08sec
Periodic conewise solution

Results:

- $k = 3$, $\lambda = 0.99$
- worst case computational time: $60\mu\text{sec}$
Conclusions

- the relaxed notions of controlled contractive sets and finite-time control Lyapunov functions have been introduced

- two novel synthesis methods for constrained stabilization of linear dynamics were constructed exploiting the results

- effectiveness demonstrated for the Buck converter