Nonlinear Perron-Frobenius Theory and Lyapunov functions for monotone systems

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Positive matrices

A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ is called positive (nonnegative) if $a_{ij} > 0$ ($a_{ij} \geq 0$) for all i, j. For matrices $A \in \mathbb{R}^{n \times n}$,

$$\rho(A) := \max\left\{|\lambda| : Ax = \lambda x \text{ for some } \lambda \in \mathbb{C} \text{ and } x \in \mathbb{C}^n \setminus \{0\}\right\}$$

is called the spectral radius of A.

Theorem (Perron 1907)

Let A be a positive $n \times n$ matrix. Then

(i) $\rho(A) > 0$ is an algebraically simple eigenvalue of A

(ii) the corresponding eigenvector $v \in \mathbb{R}^n$ is unique and positive

(iii) any nonnegative eigenvector is a multiple of v

(iv) each eigenvalue $\lambda \neq \rho(A)$ satisfies $|\lambda| < \rho(A)$

Monotone systems

- Let A be a nonnegative $n \times n$ matrix. Then $0 \leq x \leq y$ implies $Ax \leq Ay$, when $\leq$ is the componentwise partial ordering.
- Assume $\rho(A) < 1$. Then the origin is globally asymptotically stable for

  $$x^+ = Ax.$$

  Consider this system on $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$.
- Let $\widetilde{A} := A + \epsilon E$ for some $\epsilon > 0$ such that $\rho(\widetilde{A}) < 1$ and denote by $0 \ll \sigma \in \mathbb{R}^n$ the associated Perron vector.
- We have

  $$A\sigma \ll A\sigma + \epsilon E\sigma = \widetilde{A}\sigma = \rho(\widetilde{A})\sigma \ll \sigma.$$

A simple Lyapunov function

- Observe that for any $x \in \mathbb{R}^n_+$ there is a unique smallest scalar $r \geq 0$ such that $x \leq r\sigma$.
- This r is given by $V(x) \coloneqq \max_{i=1\dots n} \frac{x_i}{\sigma_i}$.
- V is order-preserving, radially unbounded, and positive definite.
- For $x \in \mathbb{R}^n_+$, $x \neq 0$, V satisfies

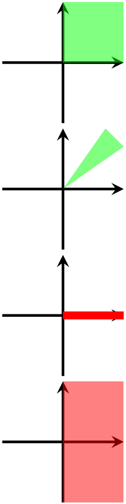
$$
\begin{aligned}
V(Ax) \leq V\big(AV(x)\sigma\big) \leq V(V(x)\rho(\widetilde{A})\sigma) \\
= V(x)\rho(\widetilde{A})V(\sigma) < V(x).
\end{aligned}
$$

- So V is a 'global' Lyapunov function for $x^+ = Ax$ on the set $\mathbb{R}^n_+$.

Outline

- Monotone (=order-preserving) systems defined on cones
- Local in addition to global asymptotic stability
- Lyapunov functions inspired by Perron vectors

Space and cone



- Let X a locally compact real Hilbert space (i.e., $\mathbb{R}^n$)
- Let $K \subset X$ a closed, pointed, and salient cone with nonempty interior $\operatorname{int} K$, i.e.,
  $$K + K \subset K, \quad rK = K \quad \forall r \geq 0, \quad K \cap (-K) = \{0\}.$$
  Denote by $\mathbb{1}$ a distinguished element of $\operatorname{int} K$.

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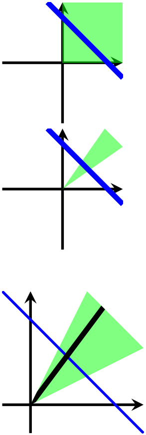

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  Denote by $\mathbb{1}$ a distinguished element of $\operatorname{int} K$.
- Let $H_r \coloneqq \{x \in X \colon \langle x, \mathbb{1} \rangle = r\}$ a hyperplane for all $r > 0$, such that

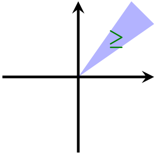
$$C_r \coloneqq H_r \cap K$$

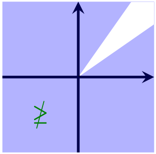
  is compact (it will also be convex).
- For $r > 0$ we define the projection $P_r \colon K \setminus \{0\} \to C_r$ by

$$P_r x \coloneqq \arg\min_{y \in C_r} \min_{\alpha \in \mathbb{R}} \|\alpha x - y\|.$$

Closed partial ordering

- We write $x \leq y$ iff $y - x \in K$;
- we write $x < y$ iff $x \leq y$ and $x \neq y$;
- we write $x \ll y$ iff $y - x \in \operatorname{int} K$.





Also important is the notation $x \ngeq y$. It means that $x \geq y$ does not hold. This is not the same as $x < y$ or $x \ll y$.

Dynamical system

$\mathbb{T}$ denotes either $\mathbb{R}_+$ or $\mathbb{Z}_+$. A forward complete dynamical system is a continuous map $\phi\colon \mathbb{T} \times X \to X$ satisfying

$$\phi(0,x) = x \qquad \forall x \in X \text{ and}$$
$$\phi(t,\phi(s,x)) = \phi(t+s,x) \qquad \forall t,s,\ \forall x \in X.$$

The system is monotone if

$$x \leq y$$

implies

$$\phi(t,x) \leq \phi(t,y)$$

for all $t > 0$.

Stability notions

A point $x^* \in X$ is an equilibrium if $\phi(t, x^*) \equiv x^*$. We consider only systems with a unique equilibrium, which without loss of generality is the origin.

An equilibrium point x^* is (globally) attractive if for all x in a neighborhood of x^* (for all $x \in X$), $\lim_{t\to\infty} \phi(t, x) = x^*$.

An equilibrium point x^* is stable, if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x - x^*\| < \delta$ implies $\|\phi(t, x) - x^*\| < \epsilon$ for all $t > 0$.

trivial Lemma

A monotone system with unique equilibrium at the origin leaves K and $-K$ invariant.

Monotone systems have monotone Lyapunov functions

Converse Lyapunov results based on Sontag's Lemma on $\mathcal{KL}$-functions define

$$V(x) := \sup_{t \geq 0} \alpha(|\phi(t,x)|)e^t$$

where α is continuous, positive definite, strictly increasing, unbounded, locally Lipschitz, cf. yesterday's talk by Chris.

When restricted to K, V is monotone!

Lemma

If a monotone system evolving on a cone is (globally) asymptotically stable, then it admits a monotone Lyapunov function.

Non-ordering conditions as definition

We say that ϕ with equilibrium at the origin satisfies the non-ordering conditions globally, if

- for all $x > 0$ all $t > 0$, we have $\boxed{\phi(t,x) \ngeq x}$;
- for all $x < 0$ all $t > 0$, we have $\boxed{\phi(t,x) \nleq x}$.

Non-ordering conditions as definition

We say that ϕ with equilibrium at the origin satisfies the non-ordering conditions locally, if

- for all $x > 0$ in a neighborhood of the origin and all $t > 0$, we have $\boxed{\phi(t,x) \ngeq x}$;
- for all $x < 0$ in a neighborhood of the origin and all $t > 0$, we have $\boxed{\phi(t,x) \nleq x}$.

Attractivity $\implies$ non-ordering conditions

Lemma

Let the origin be attractive. Then

(i) for all $x > 0$ in the region of attraction and all $t > 0$, we have $\boxed{\phi(t,x) \not\geq x}$;

(ii) for all $x < 0$ in the region of attraction and all $t > 0$, we have $\boxed{\phi(t,x) \not\leq x}$.

Proof. Assume that there were $\overline{x} > 0$, $t > 0$, s.t. $\phi(t,\overline{x}) \geq \overline{x} > 0$.

Applying $\phi(t,\cdot)$ repeatedly yields $\phi((k+1)t,\overline{x}) \geq \overline{x} > 0$.

Letting $k \to \infty$ we obtain a contradiction to the attractivity of 0.

Hence such $\overline{x}$ cannot exist and necessarily $\phi(t,x) \not\geq x$ for all $x > 0$ and all $t > 0$, proving the lemma (other case follows by symmetric argument). □

A fixed point result

Let the origin be an equilibrium for ϕ and let $Tx \coloneqq \phi(1,x)$. Then $T\colon K \to K$ with $T0 = 0$ is a monotone map. For $r > 0$ assume that $Tx \neq 0$ for all $x \in C_r$ and define $T_r\colon C_r \to C_r$ by

$$T_r x \coloneqq (P_r \circ T)(x) = P_r(Tx).$$

Lemma on fixed points

Assume (for now) that $Tx \neq 0$ for all $x \in C_r$. Then the map T_r has a fixed point in C_r.

Proof. By construction T_r is a map from a compact convex set into itself, so the result is an application of Brouwer's or Schauder's fixed point theorem. □

Thoughts about the fixed point

The fixed point does not have to be unique. Think of $Tx := qx$ with $q \in (0,1)$. Then T_r is just the identity map on C_r.

Every fixed point x_r^* of T_r must satisfy

$$Tx_r^* = \lambda x_r^*$$

for some $\lambda \in (0,\infty)$.

$\lambda \geq 1$ implies $Tx_r^* \geq x_r^*$, which is not compatible with $Tx \ngeq x$ for all $x \in C_r$. So necessarily $\lambda \in (0,1)$ if we assume the non-ordering condition and hence

$$Tx_r^* < x_r^*.$$

The sequence $0 \leq T^{k+1}x_r^* \leq T^k x_r^* \leq \ldots \leq Tx_r^* < x_r^*$ is bounded, ordered, hence convergent, and it can only converge to the origin if the non-ordering conditions hold.

A converse result

Lemma

Let the origin be an equilibrium of ϕ and let ϕ satisfy the non-ordering conditions.

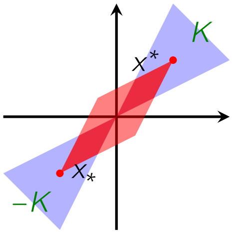
Assume there exist points $x_* \ll 0 \ll x^*$ satisfying $Tx_* \gg x_*$ and $Tx^* \ll x^*$.

Then the origin is asymptotically stable (w.r.t. ϕ) and the order intervals

$$[x_*, 0] \qquad [0, x^*] \qquad [x_*, x^*]$$

are contained in the region of attraction.

Proof



Let us consider $x \in [0, x^*]$. Monotonicity implies $0 \leq \phi(k+1, x) \leq T^{k+1}x \leq T^k x \leq T^k x^* \longrightarrow 0$ as $k \to \infty$.

The case $x \in [x_*, 0]$ is shown with symmetric arguments.

If $x \not\gtrsim 0$ and $x \not\lesssim 0$ but $x \in [x_*, x^*]$ we can *wedge* it from two sides with similar arguments. This shows attractivity.

The fact that $T([x_*, x^*])$ is bounded implies stability. □

A ‘fixed’ point with strict descent

So far, the non-ordering conditions alone for $r > 0$ only give us $x^* \in C_r$ such that $Tx^* < x^*$. We want $Tx^* \ll x^*$.
As $Tx \ngeq x$ for all $x \in C_r$, we can find an $\epsilon = \epsilon(r) > 0$ such that

$$\widetilde{T}x := Tx + \epsilon 1 \ngeq x \quad \text{for all } x \in C_r.$$

By application of the fixed point lemma we find $\widetilde{x}_r^* \in C_r$ such that

$$T\widetilde{x}_r^* \ll \widetilde{T}\widetilde{x}_r^* < \widetilde{x}_r^*.$$

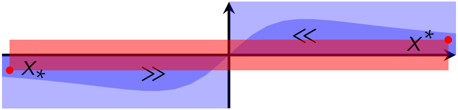
A local converse result

In summarizing the construction on the previous slides we obtain:

Theorem

If the origin is an equilibrium for ϕ and if ϕ satisfies the non-ordering conditions (locally is enough) then the origin is locally asymptotically stable.

However, we do not obtain a global result: For the standard partial order on $\mathbb{R}^2$, $\mathbb{T} = \mathbb{Z}$, any $\lambda \in (0,1)$, and $\phi(1,x) := \begin{pmatrix} \lambda x_1 + x_1^2 x_2 + x_2 \\ \lambda x_2 \end{pmatrix}$ one can show that the origin is not globally attractive [R-2010 *Positivity*] (but the non-ordering conditions are satisfied).

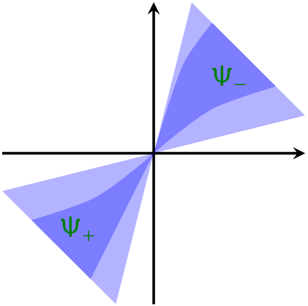


Ingredients for a global result

Clearly, we have to assume more.
Define the sets

$$\Psi_- := \{x \in K \colon Tx \leq x\} \subset K$$
$$\Psi_+ := \{x \in -K \colon Tx \geq x\} \subset -K.$$

and write again $\Psi_\pm$ to refer to either of the two.



We say that a set Y is positively (negatively) unbounded (±-unbounded for short) if for all $x \in X$ there is a $y \in Y$ such that $y \geq x$ ($y \leq x$).

A global result

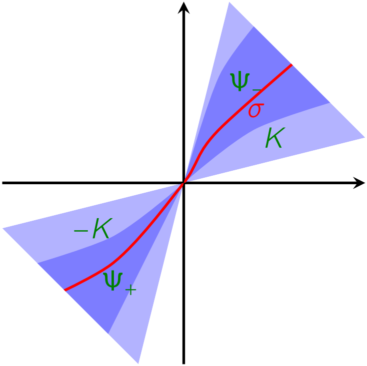
Proposition

If the origin is an equilibrium for ϕ, if ϕ satisfies the non-ordering conditions on X, and if the sets $\Psi_\pm$ are, respectively, $\pm$-unbounded, then the origin is globally asymptotically stable.

Proof. For any $x > 0$ we can find a $y \geq x$ with $y \in \Psi_-$, so that $0 \leq x \leq y$ and $Ty \leq y$. Combined this yields $0 \leq T^k x \leq T^k y \to 0$. The remainder is similar to the proof of the local result. □

See [R-2010 *Positivity*] for some classes of systems and conditions guaranteeing $\pm$-unboundedness of $\Psi_\pm$. Homogeneity also does the trick.

Canonical Lyapunov functions (discrete-time case)



Let T have an equilibrium at the origin. Assume we can find a parameterized path $\sigma\colon \mathbb{R} \to K \cup (-K)$ such that

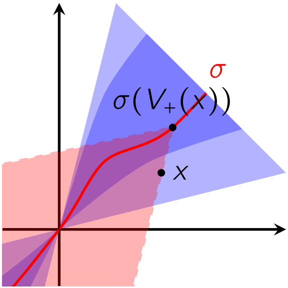
- σ is continuous, $\sigma(0) = 0$;
- $r < s$ implies $\sigma(r) \ll \sigma(s)$;
- $T\sigma(r) \ll \sigma(r)$ for all $r > 0$ and $T\sigma(r) \gg \sigma(r)$ for all $r < 0$;
- the image of σ is $\pm$-unbounded.

A Lyapunov function on the positive cone

Think of $x \geq 0$, but define for all $x \in X$,

$$V_+(x) := \min\{r \geq 0 \colon \sigma(r) \geq x\}$$
$$= \max \sigma_i^{-1}(x_i)$$

when $K = \mathbb{R}^n_+$



- V_+ is strictly increasing: $x \ll y$ implies $V_+(x) < V_+(y)$;
- We can find $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(\|x\|) \leq V_+(x) \leq \alpha_2(\|x\|)$ for $x \geq 0$;
- $$x \leq \sigma(V_+(x)) \quad \Big| \; T(\cdot)$$

$$Tx \leq T\sigma(V_+(x)) \ll \sigma(V_+(x)) \quad \Big| \; V_+(\cdot)$$

$$V_+(Tx) < V_+\big(\sigma(V_+(x))\big) = V_+(x)$$

The resulting global Lyapunov function

Define $V_-(x) \coloneqq \min\{r \geq 0 \colon \sigma(-r) \leq x\}$ for all $x \in X$ but consider $x \leq 0$.

Now combine V_+ and V_- to obtain

$$V(x) \coloneqq \max\{V_+(x), V_-(x)\}.$$

Theorem (discrete-time version)

- Let ϕ be given by $\phi(1, x) = Tx$ and assume $T0 = 0$.
- Assume the existence of σ introduced two slides ago.
- Then V is a strict, global Lyapunov function for ϕ.

The ODE version is conceptually very similar.

Existence of σ I

Nonlinear Perron-Frobenius Theory[1] provides conditions for existence of a Perron vector for, e.g., homogeneous maps on general cones K, that is, maps $T : K \to K$ such that

$$T(\lambda x) = \lambda T(x)$$

for all $x \in K$, and all scalar $\lambda > 0$.

[1] Highly recommended reading:
NPFT by Bas Lemmens and Roger Nussbaum, CUP 2012

Existence of σ II

Proposition

Let ϕ be given by $\phi(1,x) = Tx$ and assume $T0 = 0$ and that T satisfies the non-ordering conditions (globally).
The path σ as introduced three slides ago exists if $\Psi_\pm$ is $\pm$-unbounded.

(Sketch:) Origin is GAS by previous proposition, so there exists some Lyapunov function (ask Chris for details).

Robustness argument gives a $\widetilde{T} \gg T$ so that origin is GAS w.r.t. $x^+ = \widetilde{T}x$, and $\Psi_\pm(\widetilde{T})$ is $\pm$-unbounded.

Fixed point argument yields two $\pm$-unbounded solutions $\overline{\phi}$ and $\underline{\phi}$ that evolve in Ψ_-, resp., Ψ_+ for all times. These can be reparameterized to become σ.

The robustness guarantees strict descent of T along σ.

Unboundedness of $\Psi_\pm$

Lemma

Let ϕ be given by $\phi(1,x) = Tx$ and assume $T0 = 0$ and that T satisfies the non-ordering conditions (globally).
The sets $\Psi_\pm$ are $\pm$-unbounded if there exists an $\alpha \in \mathcal{K}_\infty$ s.t.

$$T(x) \geq \alpha(\|x\|)\mathbb{1} \text{ for all } x \in K$$

and

$$T(x) \leq -\alpha(\|x\|)\mathbb{1} \text{ for all } x \in -K$$

(Sketch:) Fixed point theorem on a set C_r + the yellow condition to guarantee that the fixed point can be chosen arbitrarily large.

Not a necessary condition, e.g., $T \equiv 0$.

Conclusions and outlook

Conclusion

- Asymptotic stability $\Longleftrightarrow$ local non-ordering conditions
- GAS does not imply the existence of a Lyapunov function of the presented type
- Level sets of V_+ are order intervals

Directions for future work

- Other types of Lyapunov functions, e.g. if $\nu^T A \ll \nu^T$ and $\nu \gg 0$, then $W(x) = \nu^T x$ is a Lyapunov function for $x^+ = Ax$ on $\mathbb{R}^n_+$.
- How does this extend to general monotone mappings T?