Dynamical Systems, Vol. 20, No. 3, September 2005, 281–299

Taylor & Francis
Taylor & Francis Group

A constructive converse Lyapunov theorem on asymptotic stability for nonlinear autonomous ordinary differential equations

S. F. HAFSTEIN*

University Duisburg-Essen, Germany

(*Received 7 June 2004; in final form 21 April 2005*)

An ordinary differential equation's (ODE) equilibrium is asymptotically stable, if and only if the ODE possesses a Lyapunov function, that is, an energy-like function decreasing along any trajectory of the ODE and with exactly one local minimum. Theorems regarding the 'only if' part are called converse theorems. Recently, the author presented a linear programming problem, of which every feasible solution parameterizes a Lyapunov function for the nonlinear autonomous ODE in question. In 2004 the author proved the first general constructive converse theorem by showing that if the equilibrium of the ODE is exponentially stable, then the linear programming problem possesses a feasible solution. In this paper we prove a constructive converse theorem on asymptotic stability for nonlinear autonomous ODEs and so improve the 2004 results. The only restriction on the ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is that $\mathbf{f}$ is a class $\mathcal{C}^2$ function. Note, that these results imply that the algorithm presented by the author in 2002 is capable of constructing a Lyapunov function for all nonlinear systems, of which the equilibrium is asymptotically stable.

1. Introduction

The Lyapunov stability theory is the most useful general theory for studying the stability of the equilibria of ordinary differential equations (ODEs). It is covered in practically all textbooks on dynamical systems, on control theory and in many on ODEs. It was introduced by Alexandr M. Lyapunov in 1892 and includes two methods: Lyapunov's indirect method and Lyapunov's direct method. An English translation of his work can be found in [1].

Lyapunov's direct method is a mathematical extension of the fundamental physical observation that an energy dissipative system must eventually settle down to an equilibrium point. It states that if there is an energy-like function V for a system, that is strictly decreasing along every trajectory of the system, then the trajectories are asymptotically attracted to an equilibrium. The function V is then said to be a Lyapunov function for the system (an exact mathematical definition follows below). The region (basin, domain) of attraction of a dynamical system's equilibrium

*Former name Sigurður Freyr Marinósson. Email: hafstein@traffic.uni-duisburg.de

Dynamical Systems
ISSN 1468–9367 print/ISSN 1468–9375 online © 2005 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/14689360500164873

is the set of those initial values that are attracted to the equilibrium by the dynamics of the system. A Lyapunov function provides through its preimages a lower bound on the region of attraction. This bound is non-conservative in the sense that it extends to the boundary of the domain of the Lyapunov function.

The original Lyapunov theory did not secure the existence of non-local Lyapunov functions for nonlinear systems with asymptotically stable equilibrium points. The first results on this subject are due to K. Perdeskii in 1933 [2]. The general case was resolved somewhat later, mainly by Massera [3, 4] and Malkin [5].

Theorems, which secure the existence of a Lyapunov or a Lyapunov-like function for a system possessing an equilibrium, stable in some sense, are called converse theorems in the theory of dynamical systems. The first constructive converse theorem was presented in 2004 by the author [6]. Former converse theorems were proved by constructing by a finite or a transfinite procedure a Lyapunov(-like) function using the trajectories of the respective ODE. Hence, these earlier converse theorems are pure existence theorems. However, one of them was used in the proof of the constructive converse theorem on exponential stability in [6] and we will use another one here to prove a constructive converse theorem on asymptotic stability.

There are several possibilities to formulate Lyapunov's direct method. In this work we follow [7] and only consider autonomous systems, where the dynamics of the system are modelled by an ODE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{f} \in [\mathcal{C}^2(\mathcal{U})]^n$ is a function from a domain $\mathcal{U} \subset \mathbb{R}^n$ into $\mathbb{R}^n$, of which every component f_i is two times continuously differentiable, and such that $\mathbf{0} \in \mathcal{U}$ and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. We denote by ϕ *the 'solution'* of (1), that is, $\dot{\phi}(t, \xi) = \mathbf{f}(\phi(t, \xi))$ and $\phi(0, \xi) = \xi$ for all $\xi \in \mathcal{U}$ and all (possible) t. In this case the direct method of Lyapunov states (proved in this form in chapter 1 in [7]):

Proposition 1: *Consider the ODE* (1) *and assume there is a domain* $\mathcal{M}$ *in* $\mathbb{R}^n$, $\mathbf{0} \in \mathcal{M} \subset \mathcal{U}$, *and a locally Lipschitz and positive definite function* $V : \mathcal{M} \longrightarrow \mathbb{R}$, *that is,* $V(\mathbf{0}) = 0$ *and* $V(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \mathcal{M} \setminus \{\mathbf{0}\}$, *such that*

$$D_t^+ V(\phi(t, \xi)) := \limsup_{s \to 0+} \frac{V(\phi(t, \xi) + s\mathbf{f}(\phi(t, \xi))) - V(\phi(t, \xi))}{s} < 0$$

for all $\phi(t, \xi) \in \mathcal{M}$. *Then every compact and connected component of every preimage* $V^{-1}([0, c])$, $c > 0$, *that contains the origin is a subset of the region of attraction*

$$\left\{ \xi \in \mathcal{U} \;\middle|\; \limsup_{t \to +\infty} \phi(t, \xi) = \mathbf{0} \right\}$$

of the equilibrium at the origin.

Proposition 1 is particularly useful when $V \in \mathcal{C}^1(\mathcal{M})$ and $\mathbf{f} \in [\mathcal{C}^1(\mathcal{U})]^n$. Then

$$\limsup_{s \to 0+} \frac{V(\phi(t, \xi) + s\mathbf{f}(\phi(t, \xi))) - V(\phi(t, \xi))}{s} = [\nabla V](\phi(t, \xi)) \cdot \mathbf{f}(\phi(t, \xi))$$

by the chain rule and the right-hand side of this equation can be checked for negativity without knowing the solution ϕ. The function V in Proposition 1 is called a Lyapunov function for the ODE (1). For every $\xi \neq \mathbf{0}$ in the domain of the Lyapunov function, the function $t \mapsto V(\phi(t, \xi))$ is strictly decreasing on its domain. This implies that every solution of (1) either leaves the boundary of the domain of the Lyapunov function or is asymptotically attracted to the origin. The latter is necessarily the case if the initial value ξ is in a connected compact component of a set of the form $V^{-1}([0, c])$, $c > 0$, that contains the origin, for else there would be a contradiction to $t \mapsto V(\phi(t, \xi))$ being decreasing.

The origin is said to be an asymptotically stable equilibrium of (1), if and only if:

(i) for every $\varepsilon > 0$ there is a $\delta > 0$, such that $\|\xi\|_2 < \delta$ implies $\|\phi(t, \xi)\|_2 < \varepsilon$ for all $t \geq 0$,
(ii) and the set $\{\xi \in \mathcal{U} \mid \limsup_{t \to +\infty} \phi(t, \xi) = \mathbf{0}\}$ is a neighbourhood of the origin.

Hence, the origin is an asymptotically stable equilibrium of (1) if it possesses a Lyapunov function. If, additionally, there exist real numbers $m \geq 1$ and $\alpha > 0$ and a neighbourhood $\mathcal{M}$ of the origin, such that $\|\phi(t, \xi)\|_2 \leq me^{-\alpha t}\|\xi\|_2$ for all $\xi \in \mathcal{M}$ and all $t \geq 0$, then the origin is said to be an exponentially stable equilibrium of (1).

We denote by $\mathcal{K}$ the set of all continuous and strictly monotonically increasing functions $\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ vanishing at the origin. If the closure of $\mathcal{M}$ is compact in $\mathbb{R}^n$, then $V : \mathcal{M} \longrightarrow \mathbb{R}$ is a Lyapunov function for (1), if and only if for an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$, there are functions $\alpha, \beta, \omega \in \mathcal{K}$ such that

$$\alpha(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \beta(\|\mathbf{x}\|)$$

and

$$D_t^+ V(\phi(t, \xi)) \leq -\omega(\|\phi(t, \xi)\|)$$

for all $\mathbf{x}, \phi(t, \xi) \in \mathcal{M}$. A function $\psi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ is said to be convex if

$$\lambda\psi(x) + (1 - \lambda)\psi(y) \geq \psi(\lambda x + (1 - \lambda)y)$$

for all $\lambda \in [0, 1]$ and all $x, y \in \mathbb{R}_{\geq 0}$. Clearly, without loss of generality, we can assume that α and ω are convex functions.

Further, note that if the closure of $\mathcal{M}$ is a compact set, then the concept 'exponentially stable' for an asymptotically stable equilibrium is a purely local property. The origin is an exponentially stable equilibrium, if and only if all real parts of the eigenvalues of the Jacobian $\nabla\mathbf{f}(\mathbf{0})$ are strictly negative, that is, if the matrix $\nabla\mathbf{f}(\mathbf{0})$ is Hurwitz. If all real parts of the eigenvalues of $\nabla\mathbf{f}(\mathbf{0})$ are negative and some are equal to zero, then the origin is not exponentially stable but might be asymptotically stable, and if some real parts are larger than zero, then the origin is an unstable equilibrium point.

For our proof of the constructive converse theorem presented in this work we will use a well-known non-constructive converse theorem on asymptotic stability.

Theorem 1: *Assume the origin is an asymptotically stable equilibrium of the ODE* (1) *and let $\mathcal{M} \subset \mathcal{U}$ be a domain containing the origin, of which the closure $\overline{\mathcal{M}}$ is a compact* *subset of the equilibrium's region of attraction. Then, for every norm* $\|\cdot\|$ *on* $\mathbb{R}^n$, *there are functions* $\alpha, \beta, \omega \in \mathcal{K}$ *and a function* $W \in \mathcal{C}^2(\overline{\mathcal{M}})$, *such that*

$$\alpha(\|\mathbf{x}\|) \leq W(\mathbf{x}) \leq \beta(\|\mathrm{x}\|)$$

and

$$\nabla W(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq -\omega(\|\mathbf{x}\|)$$

for all $\mathbf{x} \in \mathcal{M}$.

Proof: Follows, for example, from Theorem 24 in section 5.7 in [8]. □

The Lyapunov theory is covered in numerous textbooks on dynamical systems, for example, [2, 8–10] to name a few.

The structure of the rest of this paper is as follows: in section 2 we give a short description of linear programming problems. In section 3 we introduce a vector space of continuous piecewise affine functions. In section 4 we state the linear programming problem $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$, of which every feasible solution parameterizes a continuous piecewise affine Lyapunov function. In section 5 we give an algorithm that systematically applies the linear programming problem $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$ in a search for a parameterized Lyapunov function for the ODE in question. Then we prove that if the origin is an asymptotically stable equilibrium point, the algorithm finds a Lyapunov function for the ODE in a finite number of steps. Further, we give an example of its use. Finally, in section 6, we give some conclusions and ideas for future research.

2. Linear programming problems

A linear programming problem is a set of linear constraints, under which a linear function is to be minimized. There are several equivalent forms for linear programming problems, one of them being

$$\begin{aligned} \text{minimize} \quad & g(\mathbf{x}) := \mathbf{c}^T\mathbf{x}, \\ \text{given} \quad & C\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{2}$$

where $r, s > 0$ are integers, $C \in \mathbb{R}^{s \times r}$ is a matrix, $\mathbf{b} \in \mathbb{R}^s$ and $\mathbf{c} \in \mathbb{R}^r$ are vectors, and $\mathbf{x} \leq \mathbf{y}$ denotes $x_i \leq y_i$ for all i. The function $\mathbf{x} \longmapsto \mathbf{c}^T\mathbf{x}$ is called the objective of the linear programming problem and the conditions $C\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ together are called the constraints. A feasible solution of the linear programming problem is a vector $\mathbf{x}' \in \mathbb{R}^r$ that satisfies the constraints, that is, $\mathbf{x}' \geq \mathbf{0}$ and $C\mathbf{x}' \leq \mathbf{b}$. There are numerous algorithms known for solving linear programming problems, the most commonly used being the simplex method [11] or interior-point algorithms [12], for example, the primal-dual logarithmic barrier method. Both need a feasible starting solution for initialization. A feasible solution to (2) can be found by introducing slack variables $\mathbf{y} \in \mathbb{R}^s$ and solving the linear programming problem:

$$\begin{aligned} \text{minimize} \quad & g\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) := \sum_{i=1}^{s} y_i, \\ \text{given} \quad & \begin{bmatrix} C & -I_s \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \mathbf{b}, \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \geq \mathbf{0}, \end{aligned} \tag{3}$$

which has the feasible solution $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} = (|b_1|, |b_2|, \dots, |b_s|)^T$. If this linear programming problem has the solution $g([\mathbf{x}'\ \mathbf{y}']^T) = 0$, then $\mathbf{x}'$ is a feasible solution to (2). If the minimum of g is larger than zero, then (2) possesses no feasible solution.

3. CPWA Lyapunov functions

In order to construct a Lyapunov function from a feasible solution to a linear programming problem, one needs a class of continuous functions that can be parameterized. The class of the continuous piecewise affine (often called piecewise linear) functions is an obvious candidate. In this section we introduce continuous piecewise affine (CPWA) functions $\mathbb{R}^n \longrightarrow \mathbb{R}$. The advantage of this function space is that it is a finite dimensional vector space over $\mathbb{R}$ in a canonical way.

Let $N > 0$ be an integer and $\mathbf{y} := (y_0, y_1, \dots, y_N)^T \in \mathbb{R}^{N+1}$ a vector such that $0 = y_0 < y_1 < \dots < y_N$. Let $P : [0, N] \longrightarrow [0, y_N]$ be the unique continuous function, of which the restriction on every interval $[i, i+1]$, $i = 0, 1, \dots, N-1$, is affine, and such that $P(i) = y_i$ for all $i = 0, 1, \dots, N$. Define the function $\mathbf{PS} : [-N, N]^n \longrightarrow [-y_N, y_N]^n$ through

$$\mathbf{PS}(\mathbf{x}) := \sum_{i=1}^{n} \operatorname{sign}(x_i) P(|x_i|) \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the ith unit vector. Denote by Sym_n the set of permutations of $\{1, 2, \dots, n\}$ and define for every $\sigma \in \mathrm{Sym}_n$ the simplex

$$S_\sigma := \{\mathbf{y} \in \mathbb{R}^n \mid 0 \le y_{\sigma(1)} \le y_{\sigma(2)} \le \dots \le y_{\sigma(n)} \le 1\}.$$

Denote by $\mathfrak{P}(\{1, 2, \dots, n\})$ the power-set of $\{1, 2, \dots, n\}$ and define the function $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for every $\mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$ through

$$\mathbf{R}^{\mathcal{J}}(\mathbf{x}) := \sum_{i=1}^{n} (-1)^{\chi_{\mathcal{J}}(i)} x_i \mathbf{e}_i,$$

where $\chi_{\mathcal{J}} : \{1, 2, \dots, n\} \longrightarrow \{0, 1\}$ is the characteristic function of the set $\mathcal{J}$. A continuous function $G : [-y_N, y_N]^n \longrightarrow \mathbb{R}$ is defined to be an element of $\mathrm{CPWA}[\mathbf{PS}, [-N, N]^n]$, if and only if its restriction $G|_{\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+S_\sigma))}$ to the set $\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + S_\sigma))$ is affine for every $\mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$, every $\sigma \in \mathrm{Sym}_n$ and every $\mathbf{z} \in \{0, 1, \dots, N-1\}^n$. It is proved in chapter 4 in [7] that the mapping

$$\mathrm{CPWA}[\mathbf{PS}, [-N, N]^n] \longrightarrow \mathbb{R}^{(2N+1)^n}, \quad G \longmapsto (a_\mathbf{z})_{\mathbf{z} \in \{-N, -N+1, \dots, N\}^n},$$

where $a_\mathbf{z} = G(\mathbf{PS}(\mathbf{z}))$ for all $\mathbf{z} \in \{-N, -N+1, \dots, N\}^n$ is a vector space isomorphism. This means that we can uniquely define a function in CPWA $[\mathbf{PS}, [-N, N]^n]$ by assigning it values on the grid $\{-y_N, -y_{N-1}, \dots, y_0, y_1, \dots, y_N\}^n$. In the next section we state a linear programming problem, of which every feasible solution parameterizes a CPWA Lyapunov function.

4. The linear programming problem LP($\mathbf{f}, d, \mathbf{y}, \|\cdot\|$)

The linear programming problem **LP**($\mathbf{f}, d, \mathbf{y}, \|\cdot\|$), defined below, is not the first effort to construct Lyapunov functions by linear programming. In [13] there is an earlier, simpler effort, to do the same. However, it includes an a posteriori analysis of the quality of the Lyapunov function, which renders this method inapplicable for a constructive converse theorem. For a detailed discussion of the differences we refer to [7, 14].

In chapter 5 in [7] it is proved that every feasible solution of the following linear programming problem parameterizes a CPWA Lyapunov function for (1).

Linear programming problem LP($\mathbf{f}, d, \mathbf{y}, \|\cdot\|$): *Consider the system* (1). *Let* $N>0$ *be an integer and let* $0 = y_0 < y_1 < \cdots < y_N$ *be real numbers, such that* $[-y_N, y_N]^n \subset \mathcal{U}$. *Let* $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be defined through the constants* $y_0, y_1, \ldots, y_N$ *as in the last section and let d be an integer,* $0 \leq d < N$. *Finally, let* $\|\cdot\|$ *be an arbitrary norm on* $\mathbb{R}^n$. *Then the linear programming problem is constructed in the following way*:

(i) *Define the sets*

$$\mathcal{X}^{\|\cdot\|} := \{\|\mathbf{x}\| | \mathbf{x} \in \{y_0, y_1, \ldots, y_N\}^n\}$$

and

$$\mathcal{G} := \{-y_N, -y_{N-1}, \ldots, y_0, y_1, \ldots, y_N\}^n \setminus \{-y_{d-1}, -y_{d-2}, \ldots, y_0, y_1, \ldots, y_{d-1}\}^n.$$

(ii) *Define for every* $\sigma \in \mathrm{Sym}_n$ *and every* $i = 1, 2, \ldots, n+1$, *the vector*

$$\mathbf{x}_i^\sigma := \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)}.$$

(iii) *Define the set*

$$\mathcal{Z} := [\{0, 1, \ldots, N-1\}^n \setminus \{0, 1, \ldots, d-1\}^n] \times \mathfrak{P}(\{1, 2, \ldots, n\}).$$

(iv) *For every* $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$ *define for every* $\sigma \in \mathrm{Sym}_n$ *and every* $i = 1, 2, \ldots, n+1$, *the vector*

$$\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} := \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + \mathbf{x}_{\mathrm{i}}^{\sigma})).$$

(v) *Define the set*

$$\mathcal{Y} := \left\{ \left\{ \mathbf{y}_{\sigma,k}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,k+1}^{(\mathbf{z},\mathcal{J})} \right\} \middle| \sigma \in \mathrm{Sym}_n, (\mathbf{z}, \mathcal{J}) \in \mathcal{Z} \text{ and } k \in \{1, 2, \ldots, n\} \right\}.$$

The set $\mathcal{Y}$ *is the set of neighbouring grid-points in the grid* $\mathcal{G}$.

(vi) *For every* $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$ *and every* $r, s = 1, 2, \ldots, n$ *let* $B_{rs}^{(\mathbf{z},\mathcal{J})}$ *be a real constant, such that*

$$B_{rs}^{(\mathbf{z},\mathcal{J})} \geq \max_{i=1,2,\ldots,n} \sup_{\mathbf{x} \in \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+(0,1)^n))} \left| \frac{\partial^2 f_i}{\partial x_r \partial x_s}(\mathbf{x}) \right|.$$

(vii) *For every* $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, *every* $k, i = 1, 2, \ldots, n$ *and every* $\sigma \in \text{Sym}_n$, *define*

$$A_{\sigma,k,i}^{(\mathbf{z},\mathcal{J})} := \left| \mathbf{e}_k \cdot \left(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})} \right) \right|.$$

(viii) *Define the constant*

$$x_{\min} := \min\{\|\mathbf{x}\| \;\; \mathbf{x} \in \mathcal{G} \text{ and } \|\mathbf{x}\|_\infty = y_N\}.$$

(ix) *Let* $\varepsilon > 0$ *and* $\delta > 0$ *be arbitrary constants.*

The variables of the linear programming problem are:

$$\begin{array}{ll} \Psi[x], & \text{for all } x \in \mathcal{X}^{\|\cdot\|}, \\ \Gamma[x], & \text{for all } x \in \mathcal{X}^{\|\cdot\|}, \\ V[\mathbf{x}], & \text{for all } \mathbf{x} \in \mathcal{G}, \\ C[\{\mathbf{x},\mathbf{y}\}], & \text{for all } \{\mathbf{x},\mathbf{y}\} \in \mathcal{Y}. \end{array}$$

The linear constraints of the linear programming problem are:

(LC1) *Let* $x_1, x_2, \ldots, x_K$ *be the elements of* $\mathcal{X}^{\|\cdot\|}$ *in an increasing order. Then*

$$\begin{aligned} \Psi[x_1] = \Gamma[x_1] &= 0, \\ \varepsilon x_2 &\le \Psi[x_2], \\ \varepsilon x_2 &\le \Gamma[x_2], \end{aligned}$$

and for every $i = 2, 3, \ldots, K-1$:

$$\frac{\Psi[x_i] - \Psi[x_{i-1}]}{x_i - x_{i-1}} \le \frac{\Psi[x_{i+1}] - \Psi[x_i]}{x_{i+1} - x_i}$$

and

$$\frac{\Gamma[x_i] - \Gamma[x_{i-1}]}{x_i - x_{i-1}} \le \frac{\Gamma[x_{i+1}] - \Gamma[x_i]}{x_{i+1} - x_i}.$$

(LC2) *For every* $\mathbf{x} \in \mathcal{G}$:

$$\Psi[\|\mathbf{x}\|] \le V[\mathbf{x}].$$

If $d = 0$, *then*

$$V[\mathbf{0}] = 0.$$

If $d \ge 1$, *then for every* $\mathbf{x} \in \mathcal{G}$, *such that* $\|\mathbf{x}\|_\infty = y_d$:

$$V[\mathbf{x}] \le \Psi[x_{\min}] - \delta.$$

(LC3) *For every* $\{\mathbf{x}, \mathbf{y}\} \in \mathcal{Y}$:

$$-C[\{\mathbf{x}, \mathbf{y}\}] \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \leq V[\mathbf{x}] - V[\mathbf{y}] \leq C[\{\mathbf{x}, \mathbf{y}\}] \cdot \|\mathbf{x} - \mathbf{y}\|_\infty.$$

(LC4) *For every* $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, *every* $\sigma \in \mathrm{Sym}_n$ *and every* $i = 1, 2, \ldots, n+1$:

$$\begin{aligned}-\Gamma[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|] \geq &\sum_{j=1}^{n} \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\ &+ \frac{1}{2} \sum_{r,s=1}^{n} B_{rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} (A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})}) \sum_{j=1}^{n} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}].\end{aligned}$$

Note that the values of the constants $\varepsilon > 0$ and $\delta > 0$ do not affect whether there is a feasible solution to the linear programming problem or not. If there is a feasible solution for $\varepsilon := \varepsilon' > 0$ and $\delta := \delta' > 0$, then there is a feasible solution for all $\varepsilon := \varepsilon^* > 0$ and $\delta := \delta^* > 0$. Just multiply all variables of a feasible solution with $\max\{\varepsilon^*/\varepsilon', \delta^*/\delta'\}$. The objective of the linear programming problem is not needed. It can, however, be used to optimize the Lyapunov function in some way.

Assume that the linear programming problem $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$ has a feasible solution. Then we can define the functions $\psi, \gamma : [0, +\infty[\longrightarrow \mathbb{R}$ by using the values of the variables $\Psi[x], \Gamma[x]$ and the function $V^{\mathrm{Lya}} : [-y_N, y_N]^n \longrightarrow \mathbb{R}$ by using the values of the variables $V[\mathbf{x}]$ in the following way.

Let $x_1, x_2, \ldots, x_K$ be the elements of $\mathcal{X}^{\|\cdot\|}$ in an increasing order. We define the piecewise affine functions

$$\psi(y) := \Psi[x_i] + \frac{\Psi[x_{i+1}] - \Psi[x_i]}{x_{i+1} - x_i}(y - x_i)$$

and

$$\gamma(y) := \Gamma[x_i] + \frac{\Gamma[x_{i+1}] - \Gamma[x_i]}{x_{i+1} - x_i}(y - x_i),$$

for all $y \in [x_i, x_{i+1}]$ and all $i = 1, 2, \ldots, K-1$. The values of ψ and γ on $]x_K, +\infty[$ do not really matter, but to have everything properly defined, we set

$$\psi(y) := \Psi[x_{K-1}] + \frac{\Psi[x_K] - \Psi[x_{K-1}]}{x_K - x_{K-1}}(y - x_{K-1})$$

and

$$\gamma(y) := \Gamma[x_{K-1}] + \frac{\Gamma[x_K] - \Gamma[x_{K-1}]}{x_K - x_{K-1}}(y - x_{K-1})$$

for all $y > x_K$. Clearly, the functions ψ and γ are continuous. The function $V^{\mathrm{Lya}} \in \mathrm{CPWA}[\mathbf{PS}, [-N, N]^n]$ is defined by assigning

$$V^{\mathrm{Lya}}(\mathbf{x}) := V[\mathbf{x}]$$

for all $\mathbf{x} \in \mathcal{G}$. In chapter 5 in [7] it is proved that ψ and γ are convex and strictly increasing and that

$$\psi(\|\mathbf{x}\|) \leq V^{\mathrm{Lya}}(\mathbf{x})$$

for all $\mathbf{x} \in [-y_N, y_N]^n \backslash (-y_d, y_d)^n$, and

$$\limsup_{s \to 0+} \frac{V^{\mathrm{Lya}}(\phi(t,\xi) + s\mathbf{f}(\phi(t,\xi))) - V^{\mathrm{Lya}}(\phi(t,\xi))}{s} \leq -\gamma(\|\phi(t,\xi)\|),$$

for all $\phi(t,\xi) \in (-y_N, y_N)^n \backslash (-y_d, y_d)^n$. This implies that if $d = 0$, then $V^{\mathrm{Lya}} : [-y_N, y_N]^n \longrightarrow \mathbb{R}$ is a Lyapunov function for (1). Further, it is proved for $d > 0$, that for every $c > 0$, such that the connected component of

$$\left\{\mathbf{x} \in (-y_N, y_N)^n \backslash [-y_d, y_d]^n \mid V^{\mathrm{Lya}}(\mathbf{x}) \leq c\right\} \cup [-y_d, y_d]^n$$

containing the origin is compact, there is a $t' \geq 0$ for every ξ in this component such that $\phi(t', \xi) \in [-y_d, y_d]^n$. It is not difficult to see that for every $t \geq t'$ we have

$$\phi(t,\xi) \in \left\{\mathbf{x} \in \mathbb{R}^n \mid V^{\mathrm{Lya}}(\mathbf{x}) \leq \max_{\|\mathbf{y}\|_\infty = y_d} V^{\mathrm{Lya}}(\mathbf{y})\right\} \cup [-y_d, y_d]^n.$$

Hence, the function $V^{\mathrm{Lya}} : [-y_N, y_N]^n \backslash (-y_d, y_d)^n \longrightarrow \mathbb{R}$ is essentially a Lyapunov function for the ODE (1).

5. The constructive converse theorem

In this section we prove a constructive converse theorem on asymptotic stability for (1). We will prove that if the origin is an asymptotically stable equilibrium point of the ODE (1) and $a > 0$ a real number such that $[-a, a]^n$ is contained in its region of attraction, then, for an arbitrary small neighbourhood $\mathcal{N} \subset \mathbb{R}^n$ of the origin, we can use the linear programming problem from the last section to parameterize a CPWA Lyapunov function

$$V^{\mathrm{Lya}} : [-a, a]^n \setminus \mathcal{N} \longrightarrow \mathbb{R}$$

for the system. Note, that it is not possible to prove such a theorem for $\mathcal{N} = \emptyset$. The reason is, that for a CPWA Lyapunov function $V^{\mathrm{Lya}} : [-a, a]^n \longrightarrow \mathbb{R}$, there exist constants $b, c, d > 0$, such that

$$b\|\mathbf{x}\| \leq V^{\mathrm{Lya}}(\mathbf{x}) \leq c\|\mathbf{x}\| \quad \text{for all } \mathbf{x} \in [-a, a]^n$$

and

$$D_t^+ V^{\mathrm{Lya}}(\phi(t,\xi)) \leq -d\|\phi(t,\xi)\| \quad \text{for all } \phi(t,\xi) \in (-a, a)^n.$$

These inequalities imply that

$$D_t^+ V^{\mathrm{Lya}}(\phi(t,\xi)) \le -\frac{d}{c} V^{\mathrm{Lya}}(\phi(t,\xi)),$$

which in turn implies

$$D_t^+\big[V^{\mathrm{Lya}}(\phi(t,\xi))e^{(d/c)t}\big] = \big[D_t^+ V^{\mathrm{Lya}}(\phi(t,\xi))\big]e^{(d/c)t} + \frac{d}{c} V^{\mathrm{Lya}}(\phi(t,\xi))e^{(d/c)t} \le 0,$$

that is

$$\|\phi(t,\xi)\| \le \frac{c}{b} e^{-(d/c)t}\|\xi\|,$$

so the origin must be an exponentially stable equilibrium point.

We prove our constructive converse theorem by showing that the following systematic scan of the parameters d and $\mathbf{y}$ of the linear programming problem $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$ will, in a finite number of steps, deliver a CPWA Lyapunov function for (1).

Algorithm 1: *Consider the system* (1) *and let* $a > 0$ *be a constant such that* $[-a,a]^n \subset \mathcal{U}$ *and let* $\mathcal{N} \subset \mathcal{U}$ *be an arbitrary neighbourhood of the origin. Set* $D := 0$ *and let* m *be the smallest positive integer, such that* $(-a2^{-m}, a2^{-m})^n \subset \mathcal{N}$. *Then the algorithm is as follows:*

(i) *Set* $\mathbf{y} := a2^{-m}(0,1,2,\ldots,2^m)^T$.
(ii) *If* $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$ *possesses a feasible solution for some* $d = 2^0, 2^1, \ldots, 2^D$, *then go to step* (iii). *If* $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$ *does not possess a feasible solution for any* $d = 2^0, 2^1, \ldots, 2^D$, *then set* $m := m+1$, $D := D+1$, *and go back to step* (i).
(iii) *Use the feasible solution to parameterize a* CPWA *Lyapunov function for the system.*

We come to the main contribution of this work, a constructive converse theorem on asymptotic stability.

Theorem 2 (Constructive converse theorem on asymptotic stability): *Algorithm 1 terminates in a finite number of steps whenever the origin is an asymptotically stable equilibrium point of the system* (1) *and* $[-a,a]^n$ *is a subset of its region of attraction.*

Proof: We split the proof into two parts. In part I we prove that there are positive integers m and d, such that the linear programming problem $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$, where $\mathbf{y} := a2^{-m}(0,1,2,\ldots,2^m)^T$, possesses a feasible solution. We do this by assigning appropriate values to the constants ε, δ and $B_{rs}^{(\mathbf{z},\mathcal{J})}$ and the variables $\Psi[x]$, $\Gamma[x]$ $V[\mathbf{x}]$, and $C[\{\mathbf{x},\mathbf{y}\}]$ of the linear programming problem and then we show that the linear constraints **(LC1)**, **(LC2)**, **(LC3)** and **(LC4)** are fulfilled when the variables and constants have these values. Then, in part II, we use the results from part I to prove that Algorithm 1 terminates in a finite number of steps.

Part I: By Theorem 1 there are class $\mathcal{K}$ functions α, β and ω, and a class $\mathcal{C}^2$ function $W : [-a,a]^n \longrightarrow \mathbb{R}$, such that

$$\alpha(\|\mathbf{x}\|) \le W(\mathbf{x}) \le \beta(\|\mathbf{x}\|)$$

and

$$\nabla W(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq -\omega(\|\mathbf{x}\|)$$

for all $\mathbf{x} \in (-a, a)^n$. Further, without loss of generality, we can assume that α and ω are convex functions. With

$$x^*_{\min} := \min_{\|\mathbf{x}\|_\infty = a} \|\mathbf{x}\|$$

we set

$$\delta := \frac{\alpha(x^*_{\min})}{2}$$

and denote by m^* the smallest positive integer, such that

$$\left[-\frac{a}{2^{m^*}}, \frac{a}{2^{m^*}}\right]^n \subset \{\mathbf{x} \in \mathbb{R}^n | \beta(\|\mathbf{x}\|) \leq \delta\} \cap \mathcal{N}.$$

Set

$$\begin{aligned}
x^* &:= \min_{\|\mathbf{x}\|_\infty = a2^{-m^*}} \|\mathbf{x}\|, \\
\omega^* &:= \frac{1}{2}\omega(x^*), \\
\varepsilon &:= \min\{\omega^*, \alpha(x_2)/x_2\}, \\
C &:= \max_{\substack{i=1,2,\ldots,n \\ \mathbf{x} \in [-a,a]^n}} \left|\frac{\partial W}{\partial x_i}(\mathbf{x})\right|,
\end{aligned}$$

and determine a constant B such that

$$B \geq \max_{\substack{i,k,l=1,2,\ldots,n \\ \mathbf{x} \in [-a,a]^n}} \left|\frac{\partial^2 f_i}{\partial x_k \partial x_l}(\mathbf{x})\right|.$$

Assign

$$\begin{aligned}
A^* &:= \sup_{\substack{\mathbf{x} \in [-a,a]^n \\ \mathbf{x} \neq \mathbf{0}}} \frac{\|\mathbf{f}(\mathbf{x})\|_2}{\|\mathbf{x}\|}, \\
B^* &:= n \cdot \max_{\substack{k,l=1,2,\ldots,n \\ \mathbf{x} \in [-a,a]^n}} \left|\frac{\partial^2 W}{\partial x_k \partial x_l}(\mathbf{x})\right|, \\
C^* &:= n^3 BC,
\end{aligned}$$

and denote by $m \geq m^*$ the smallest positive integer, such that

$$\frac{a}{2^m} \leq \frac{\sqrt{(x^* A^* B^*)^2 + 4x^*\omega^* C^*} - x^* A^* B^*}{2C^*}$$

and set

$$d := 2^{m-m^*}.$$

With $\mathbf{y} := a2^{-m}(0, 1, \ldots, 2^m)^T$ we assign the following values to the variables and the remaining constants of the linear programming problem $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$:

$$B_{rs}^{(\mathbf{z},\mathcal{J})} := B, \quad \text{for all } (\mathbf{z},\mathcal{J}) \in \mathcal{Z} \quad \text{and all } r, s = 1, 2, \ldots, n,$$

$$\Psi[x] := \alpha(x), \quad \text{for all } x \in \mathcal{X}^{\|\cdot\|},$$

$$\Gamma[x] := \omega^* x, \quad \text{for all } x \in \mathcal{X}^{\|\cdot\|},$$

$$V[\mathbf{x}] := W(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathcal{G},$$

$$C[\{\mathbf{x},\mathbf{y}\}] := C, \quad \text{for all } \{\mathbf{x},\mathbf{y}\} \in \mathcal{Y}.$$

We now consequently show that the linear constraints of the linear programming problem $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$ are satisfied by these values.

(LC1): The constraints LC1 are trivially fulfilled.

(LC2): Clearly,

$$\Psi[\|\mathbf{x}\|] = \alpha(\|\mathbf{x}\|) \le W(\mathbf{x}) = V[\mathbf{x}]$$

for all $\mathbf{x} \in \mathcal{G}$ and for every $\mathbf{x} \in \mathcal{G}$ such that $\|\mathbf{x}\|_\infty = y_d$, we have

$$V[\mathbf{x}] \le \beta(\|\mathbf{x}\|) \le \delta = \alpha(x^*_{\min}) - \delta \le \alpha(x_{\min}) - \delta = \Psi[x_{\min}] - \delta.$$

(LC3): Follows directly by the Mean-value theorem.

(LC4): Let $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$ and $\sigma \in \mathrm{Sym}_n$ be arbitrary. We have to show that

$$\begin{aligned} -\Gamma[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|] \ge \sum_{j=1}^{n} &\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\ &+ \frac{1}{2}\sum_{r,s=1}^{n} B_{rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} (A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})}) \sum_{j=1}^{n} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]. \end{aligned} \tag{4}$$

With the values we have assigned to the variables and the constants of the linear programming problem, the inequality (4) holds true if

$$-\omega^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| \ge \sum_{j=1}^{n} \frac{W[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - W[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + h^2 C^*$$

with $h := a2^{-m}$. Now, by Theorem 1, the Mean-value theorem, and because

$\omega(x) \geq 2\omega^* x$ for all $x \geq x^*$,

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{W[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - W[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + h^2 C^* \\
&= \sum_{j=1}^{n} \left(\frac{W[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - W[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} - \frac{\partial W}{\partial \xi_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \right) f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\
&\quad + \nabla W(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \cdot \mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + h^2 C^* \\
&\leq \left\| \sum_{j=1}^{n} \left(\frac{W[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - W[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} - \frac{\partial W}{\partial \xi_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \right) \mathbf{e}_j \right\|_2 \| f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \|_2 \\
&\quad - \omega(\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|) + h^2 C^* \\
&\leq B^* h A^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| - 2\omega^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| + h^2 C^*.
\end{aligned}$$

Hence, if

$$-\omega^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| \geq h A^* B^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| - 2\omega^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| + h^2 C^*,$$

the inequality (4) follows. But, this last inequality follows from

$$h := \frac{a}{2^m} \leq \frac{\sqrt{(x^* A^* B^*)^2 + 4x^* \omega^* C^*} - x^* A^* B^*}{2C^*},$$

which implies

$$0 \geq h A^* B^* - \omega^* + h^2 \frac{C^*}{x^*},$$

and that

$$h A^* B^* - \omega^* + h^2 \frac{C^*}{x^*} \geq h A^* B^* - \omega^* + h^2 \frac{C^*}{\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|}.$$

Part II: Now, consider Algorithm 1. It will start with $D = 0$ and $m = m_0$, where m_0 is the smallest integer such that

$$\left(-\frac{a}{2^{m_0}}, \frac{a}{2^{m_0}} \right)^n \subset \mathcal{N}.$$

Then, in the worst case, the algorithm will fail to find a feasible solution to the linear programming problems until m is so large that $m \geq m^*$ and

$$\frac{a}{2^m} \leq \frac{\sqrt{(x^* A^* B^*)^2 + 4x^* \omega^* C^*} - x^* A^* B^*}{2C^*}.$$

We showed in part I of the proof that the linear programming problem $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$, where $\mathbf{y} := a2^{-m}(0, 1, 2, \ldots, 2^m)^T$ and $d = 2^{m-m^*}$, possesses a feasible solution. The only fact remaining to be shown is that $2^{m-m^*} \in \{2^0, 2^1, \ldots, 2^D\}$. But this follows from $D = m - m_0$ and $m_0 \leq m^*$ and we have completed the proof. □

As an example of the use of Theorem 2 we consider the system (1) with

$$\mathbf{f}(x, y) := \begin{pmatrix} x^3(y-1) \\ -\dfrac{x^4}{(1+x^2)^2} - \dfrac{y}{1+y^2} \end{pmatrix}. \tag{5}$$

This system is taken from Example 65 in section 5.3 in [8]. The Jacobian of $\mathbf{f}$ at the origin has the eigenvalues 0 and -1. Hence, the origin is not an exponentially stable equilibrium point. We initialize Algorithm 1 with

$$a := \frac{8}{15} \quad \text{and} \quad \mathcal{N} := \left(-\frac{2}{15}, \frac{2}{15}\right)^2.$$

Further, with

$$x_{\mathbf{z}} := \mathbf{e}_1 \cdot \mathbf{PS}(\mathbf{z} + \mathbf{e}_1) \quad \text{and} \quad y_{\mathbf{z}} := \mathbf{e}_2 \cdot \mathbf{PS}(\mathbf{z} + \mathbf{e}_2),$$

we set

$$B_{11}^{(\mathbf{z},\mathcal{J})} := 6x_{\mathbf{z}}(1 + y_{\mathbf{z}}),$$

$$B_{12}^{(\mathbf{z},\mathcal{J})} := 3x_{\mathbf{z}}^2,$$

$$B_{22}^{(\mathbf{z},\mathcal{J})} := \begin{cases} \dfrac{6y_{\mathbf{z}}}{(1+y_{\mathbf{z}}^2)^2} - \dfrac{8y_{\mathbf{z}}^3}{(1+y_{\mathbf{z}}^2)^3}, & \text{if } y_{\mathbf{z}} \leq \sqrt{2} - 1, \\ 1.46, & \text{else,} \end{cases}$$

for all $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$ in the linear programming problems. This is more effective than using one constant B larger than all $B_{rs}^{(\mathbf{z},\mathcal{J})}$ for all $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$ and all $r, s = 1, 2, \ldots, n$, as done to shorten the proof of Theorem 2. Algorithm 1 succeeds in finding a feasible solution to the linear programming problem with $m = 4$ and $D = 2$. The corresponding CPWA Lyapunov function is drawn in figure 1. We used this Lyapunov function as a starting point for parameterizing a CPWA Lyapunov function with a larger domain and succeeded with

$$\mathbf{y} := (0, 0.033, 0.067, 0.1, 0.133, 0.18, 0.25, 0.3, 0.38, 0.45, 0.55, 0.7, 0.85, 0.93, 1)^T.$$

It is drawn in figure 2. In figure 3 the sets discussed at the end of section 4 are drawn for this particular Lyapunov function. Every solution to the ODE with an initial value ξ in the largest set will reach the square $[-0.133, 0.133]^2$ in a finite time t' and will stay in the smaller set containing the square for all $t \geq t'$.

The stability of switched systems has been under focus recently, see, for example, [15–17]. Therefore, the second example we present is a switched system under

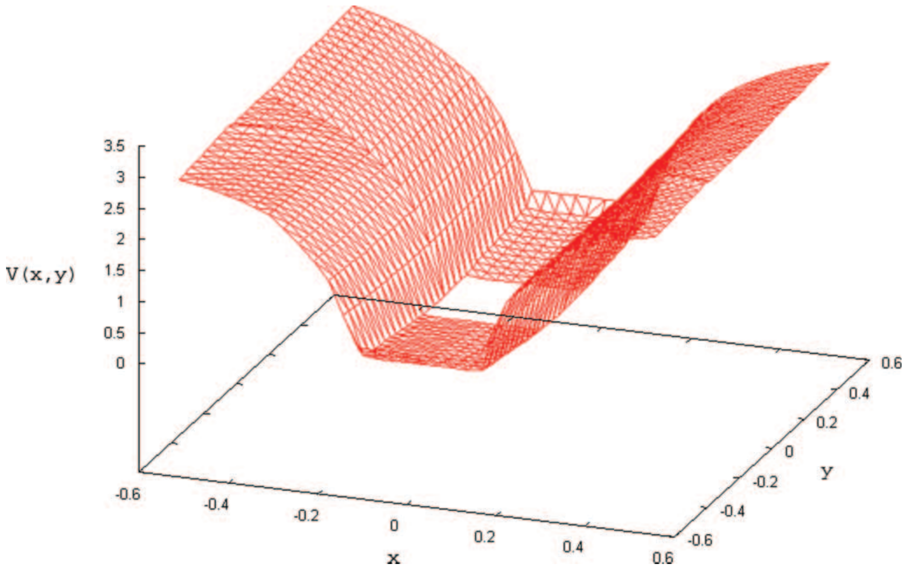

Figure 1. A CPWA Lyapunov function for (5) generated by Algorithm 1.

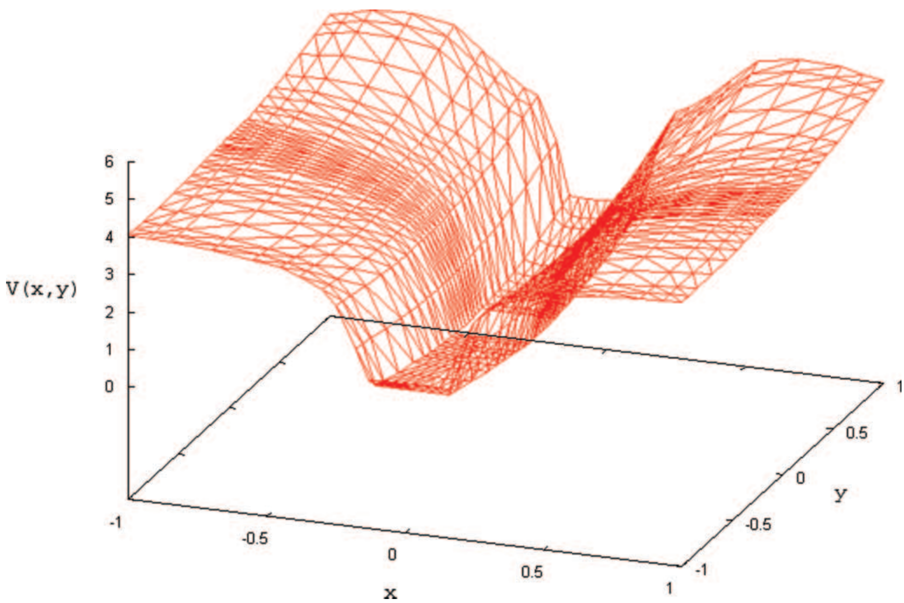

Figure 2. A CPWA Lyapunov function for (5) with a larger domain.

arbitrary switching. A switched system under arbitrary switching is a non-empty set $\mathcal{P}$ equipped with the discrete metric $d(p, q) := 1$ if $p \neq q$ and a collection of systems

$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}), \quad p \in \mathcal{P}. \tag{6}$$

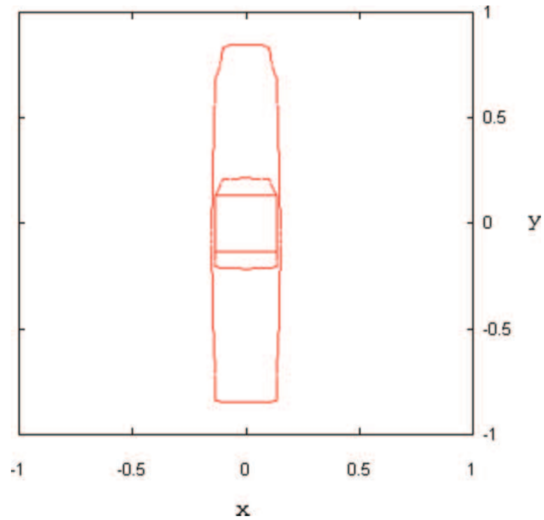

Figure 3. The sets discussed at the end of section 4 for the Lyapunov function generated for (5).

For every right-continuous function $\sigma : \mathbb{R}_{\geq 0} \longrightarrow \mathcal{P}$, such that the discontinuity-points of σ form a discrete set in $\mathbb{R}_{\geq}0$, the solution $t \mapsto \phi_\sigma(t, \xi)$ of the switched system $\dot{\mathbf{x}} = \mathbf{f}_\sigma(\mathbf{x})$ is defined by gluing together the solution-trajectories of the corresponding systems, using $\dot{\mathbf{x}} = \mathbf{f}_{\sigma(0)}(\mathbf{x})$ for t between 0 and the first discontinuity-point t_1 of σ, $\dot{\mathbf{x}} = \mathbf{f}_{\sigma(t_1)}(\mathbf{x})$ between t_1 and the second largest discontinuity-point t_2 of σ, and so on. The origin is said to be an asymptotically stable equilibrium of the switched system (6) under arbitrary switching, if and only if there exist continuous functions $\kappa, \ell : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$, such that $\kappa(0) = 0$, κ is strictly monotonically increasing, ℓ is strictly monotonically decreasing, and $\lim_{x \to +\infty} \ell(x) = 0$, and, for all ξ in some neighbourhood of the origin, all $t \geq 0$, and all $\sigma : \mathbb{R}_{\geq 0} \longrightarrow \mathcal{P}$ as described above, we have

$$\|\phi_\sigma(t, \xi)\| \leq \kappa(\|\xi\|)\ell(t).$$

It is not difficult to show, that if the systems (6) possess a common Lyapunov function, that is, a function that is a Lyapunov function for all of the systems individually, then the equilibrium at the origin is an asymptotically stable equilibrium of the switched system.

Consider the switched system $\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x})$, $p \in \{1, 2, 3\}$, with

$$\mathbf{f}_1(x, y) := \begin{pmatrix} -y \\ x - y(1 - x^2 + 0.1x^4) \end{pmatrix},$$

$$\mathbf{f}_2(x, y) := \begin{pmatrix} -y + x(x^2 + y^2 - 1) \\ x + y(x^2 + y^2 - 1) \end{pmatrix}$$

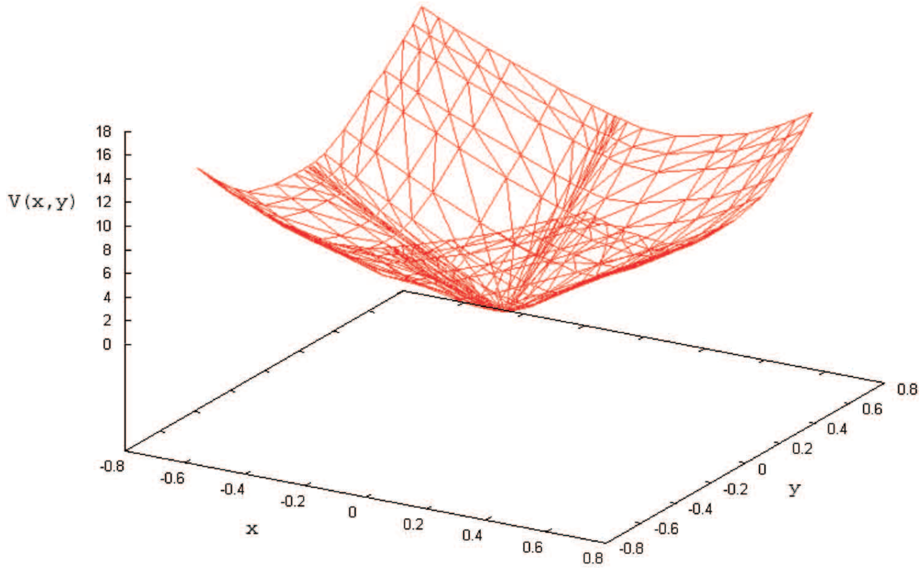

Figure 4. A common CPWA Lyapunov function for the systems $\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}), \quad p \in \{1,2,3\}$.

and

$$\mathbf{f}_3(x,y) := \begin{pmatrix} -1.5y \\ \frac{x}{1.5} + y\left(\left(\frac{x}{1.5}\right)^2 + y^2 - 1\right). \end{pmatrix}$$

A closer look at the linear programming problem in section 4 reveals that a feasible solution to an adapted linear programming problem, which incorporates **(LC1)**, **(LC2)** and **(LC3)** once and **(LC4)** for each of the functions $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{f}_3$, parameterizes a common Lyapunov function for the systems $\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x})$, $p \in \{1,2,3\}$.

We succeeded in parameterizing a Lyapunov function $V : [-0.648, 0.648]^2 \backslash]-0.01, 0.01[^2 \longrightarrow \mathbb{R}_{\geq 0}$ for the switched system. This Lyapunov function is plotted in figure 4.

In figure 5 the region of attraction secured by the Lyapunov function on figure 4 is plotted. Every solution starting in the region will reach the square at the origin in a finite time, regardless of the switching.

6. Conclusions

A constructive converse theorem on asymptotic stability is proved for class $\mathcal{C}^2$ autonomous ODEs. The Lyapunov function from Theorem 1, which is a non-constructive converse theorem, is used to assign values to the variables of the linear programming problem $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$ introduced in [7, 14] and defined in section 4 here. We prove that the linear constraints of $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$ are satisfied

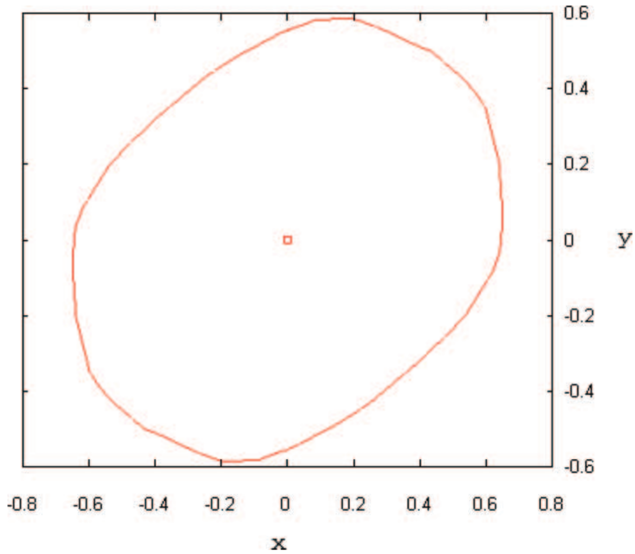

Figure 5. The region of attraction secured by the Lyapunov function in figure 4 for the switched system.

by these values. It follows that Algorithm 1 can be used to generate a Lyapunov function, which can be used to estimate the basin of attraction of the corresponding equilibrium point.

It is the belief of the author, that this general method to numerically generate Lyapunov functions for (nonlinear) ODEs might lead to advantages in the stability theory of ODEs, the stability theory of continuous dynamical systems, and control theory. However, there are a few open problems regarding the numerics that should be addressed first. The numerical experience in using this method is limited to several two-dimensional systems [6, 7, 14]. Higher dimensional systems are certainly of interest, inclusive of a reasonable method to visualize and extract information from the Lyapunov function generated. Sometimes, especially when the grid $\mathcal{G}$ in $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$ is regular like in Algorithm 1, numerical instability of the simplex method implementation (Gnu Linear Programming Kit 3.2.2 by Andrew Makhorin) used in the search for a feasible solution is an issue. It is not clear whether this is a fundamental drawback of the linear programming problem $\mathbf{LP}(\mathbf{f}, d, \mathbf{y}, \|\cdot\|)$ or an artefact of the simplex algorithm or its implementation in the liner solver used.

Software, written in the C++ programming language, to generate arbitrary dimensional CPWA Lyapunov functions is available for free on the Internet at the URL http://www.traffic.uni-duisburg.de/~hafstein. It was used for the examples presented in this work. The interested user is encouraged to download the software and apply it to some other ODEs.

References

[1] Lyapunov, A., 1992, The general problem of the stability of motion. *International Journal of Control*, **55**, 531–773.
[2] Hahn, W., 1967, *Stability of Motion* (New York and Berlin: Springer).

[3] Massera, J., 1949, On Liapunoff's conditions of stability. *Annals of Mathematics*, **50**, 705–721.
[4] Massera, J., 1956, Contributions to stability theory. *Annals of Mathamatics*, **64**, 182–206.
[5] Malkin, I., 1977, On a question of reversability of Liapunov's theorem on asymptotic stability. In J. Aggarwal and M. Vidyasagar (Eds) *Nonlinear Systems: Stability Analysis*, pp. 161–170 (Stroudsburg: Dowden, Hutchinson & Ross).
[6] Hafstein, S., 2004, A constructive converse Lyapunov theorem on exponential stability. *Discrete and Continuous Dynamical Systems — Series A*, **10**(3), 657–678.
[7] Marinosson, S., 2002, Stability analysis of nonlinear systems with linear programming: a Lyapunov functions based approach. PhD thesis, Gerhard-Mercator-University, Duisburg, Germany. Available online at: http://www.traffic.uni-duisburg.de/~hafstein
[8] Vidyasagar, M., 1993, *Nonlinear System Analysis* (Englewood Cliffs, NJ: Prentice Hall).
[9] Khalil, H., 1992, *Nonlinear Systems* (New York: Macmillan).
[10] Sastry, S., 1999, *Nonlinear Systems: Analysis, Stability, and Control* (New York and Berlin: Springer).
[11] Schrijver, A., 1998, *Theory of Linear and Integer Programming* (New York: John Wiley).
[12] Roos, C., Terlaky, T. and Vial, J., 1997, *Theory and Algorithms for Linear Optimization* (New York: John Wiley).
[13] Julian, P., 1999, A high level canonical piecewise linear representation: theory and applications. PhD thesis, Universidad Nacional del Sur, Bahia Blanca, Argentina.
[14] Marinosson, S., 2002, Lyapunov function construction for ordinary differential equations with linear programming. *Dynamical Systems: An International Journal*, **17**, 137–150.
[15] Dayawansa, W. and Martin, C., 1999, A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Transactions on Automatic Control*, **44**(4), 751–760.
[16] Liberzon, D. and Morse, A., 1999, Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, **19**(5), 59–70.
[17] Vu, L. and Liberzon, D., 2005, Common Lyapunov functions for families of commuting nonlinear systems. *Systems & Control Letters*, **54**(5), 405–416.