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A CONSTRUCTIVE CONVERSE LYAPUNOV THEOREM ON EXPONENTIAL STABILITY

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Abstract. Closed physical systems eventually come to rest, the reason being that due to friction of some kind they continuously lose energy. The mathematical extension of this principle is the concept of a Lyapunov function. A Lyapunov function for a dynamical system, of which the dynamics are modelled by an ordinary differential equation (ODE), is a function that is decreasing along any trajectory of the system and with exactly one local minimum. This implies that the system must eventually come to rest at this minimum. Although it has been known for over 50 years that the asymptotic stability of an ODE's equilibrium is equivalent to the existence of a Lyapunov function for the ODE, there has been no constructive method for non-local Lyapunov functions, except in special cases. Recently, a novel method to construct Lyapunov functions for ODEs via linear programming was presented [5], [6], which includes an algorithmic description of how to derive a linear program for a continuous autonomous ODE, such that a Lyapunov function can be constructed from any feasible solution of this linear program. We will show how to choose the free parameters of this linear program, dependent on the ODE in question, so that it will have a feasible solution if the equilibrium at the origin is exponentially stable. This leads to the first constructive converse Lyapunov theorem in the theory of dynamical systems/ODEs.

1. Introduction.

The Lyapunov theory of dynamical systems is the most useful general theory for studying the stability of nonlinear systems. It is covered in practically all textbooks on dynamical systems, on control theory, and in many on ordinary differential equation. It was introduced by Alexandr M. Lyapunov in 1892 and includes two methods, Lyapunov's indirect method and Lyapunov's direct method. An English translation of his work can be found in [4]

Lyapunov's direct method is a mathematical extension of the fundamental physical observation, that an energy dissipative system must eventually settle down to an equilibrium point. It states that if there is an energy-like function V for a system that is strictly decreasing along its trajectories, then the trajectories are asymptotically attracted to an equilibrium. The function V is then said to be a Lyapunov function for the system. The region (basin, domain) of attraction of a dynamical systems' equilibrium is the set of those initial values that are attracted to the equilibrium by the dynamics of the system. A Lyapunov function provides through its preimages a lower bound of the region of attraction. This bound is non-conservative in the sense that it extends to the boundary of the domain of the Lyapunov function.

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There are several possibilities to formulate Lyapunov's direct method. In this work we follow [5] and only consider autonomous systems, where the dynamics of the system are modelled by an ODE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{f} \in [\mathcal{C}^2(\mathcal{U})]^n$ is a function from a domain $\mathcal{U} \subset \mathbb{R}^n$ into $\mathbb{R}^n$, of which every component f_i is two-times continuously differentiable, and such that $\mathbf{0} \in \mathcal{U}$ and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. We denote by $\boldsymbol{\phi}$ *the solution* of (1.1), i.e., $\dot{\boldsymbol{\phi}}(t,\boldsymbol{\xi}) = \mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))$ and $\boldsymbol{\phi}(0,\boldsymbol{\xi}) = \boldsymbol{\xi}$ for all $\boldsymbol{\xi} \in \mathcal{U}$ and all (possible) t. In this case the direct method of Lyapunov states (proved in Chapter 1 in [5]):

Proposition 1.1. *Consider the ODE (1.1) and assume there is a domain $\mathcal{M}$ in $\mathbb{R}^n$, $\mathbf{0} \in \mathcal{M} \subset \mathcal{U}$, and a locally Lipschitz and positive definite function $V : \mathcal{M} \longrightarrow \mathbb{R}$, i.e. $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{M} \setminus \{\mathbf{0}\}$, such that*

$$\limsup_{s\to 0+} \frac{V(\boldsymbol{\phi}(t,\boldsymbol{\xi}) + s\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))) - V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))}{s} < 0$$

for all $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in \mathcal{M}$. Then every compact and connected component of every preimage $V^{-1}([0,c])$, $c > 0$, that contains the origin is a subset of the region of attraction

$$\{\boldsymbol{\xi} \in \mathcal{U} \big| \limsup_{t\to +\infty} \boldsymbol{\phi}(t,\boldsymbol{\xi}) = \mathbf{0}\}$$

of the equilibrium at the origin.

The function V in Proposition 1.1 is called a Lyapunov function for the ODE (1.1). For every $\boldsymbol{\xi} \neq \mathbf{0}$ in the domain of the Lyapunov function, the function $t \mapsto V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))$ is strictly decreasing on its domain. This implies, that every solution of (1.1) does either leave the boundary of the domain of the Lyapunov function or it is asymptotically attracted to the origin. The latter is necessarily the case if the initial value $\boldsymbol{\xi}$ is in a connected compact component of a set of the form $V^{-1}([0,c])$, $c > 0$, that contains the origin, for else there would be a contradiction to $t \mapsto V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))$ being decreasing.

Proposition 1.1 is particularly useful when $V \in \mathcal{C}^1(\mathcal{M})$ and $\mathbf{f} \in [\mathcal{C}^1(\mathcal{U})]^n$. Then

$$\limsup_{s\to 0+} \frac{V(\boldsymbol{\phi}(t,\boldsymbol{\xi}) + s\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))) - V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))}{s} = [\nabla V](\boldsymbol{\phi}(t,\boldsymbol{\xi})) \cdot \mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))$$

by the chain rule and the right-hand side of this equation can be checked for negativity without knowing the solution $\boldsymbol{\phi}$.

Although the direct method of Lyapunov is a powerful tool for stability analysis, its main drawback has been the lack of a general constructive method to generate non-local Lyapunov functions for nonlinear ODEs. A local Lyapunov function can be constructed if the Jacobian $A \in \mathbb{R}^{n\times n}$ of $\mathbf{f}$ at the origin is Hurwitz, i.e., if all its eigenvalues have strictly negative real parts. Then, by the indirect method of Lyapunov, $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$ is a Lyapunov function for the system, where $P \in \mathbb{R}^{n\times n}$ is the unique positive definite solution to the matrix equation †

$$PA + A^T P = -I_n.$$

However, the domain of this Lyapunov function depends on the approximation error $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}) - A\mathbf{x}$ and, except when $\mathbf{f}$ is a linear function, almost certainly does not give a good estimate of the region of attraction.

† A^T and $\mathbf{x}^T$ denote the transposes of the matrix A and the vector $\mathbf{x}$ respectively and I_n denotes the $n \times n$-identity matrix.

Let us make this point more clear. The function $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$ is a global Lyapunov function for the linearized ODE $\dot{\mathbf{x}} = A\mathbf{x}$. This follows by $V(\mathbf{0}) = 0$, $V(\mathbf{0}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, and

$$\begin{aligned}\frac{d}{dt}V(\boldsymbol{\phi}_{lin}) &= \boldsymbol{\phi}_{lin}^T P\dot{\boldsymbol{\phi}}_{lin} + \dot{\boldsymbol{\phi}}_{lin}^T P\boldsymbol{\phi}_{lin}\\ &= \boldsymbol{\phi}_{lin}^T PA\boldsymbol{\phi}_{lin} + \boldsymbol{\phi}_{lin}^T A^T P\boldsymbol{\phi}_{lin}\\ &= -\|\boldsymbol{\phi}_{lin}\|_2^2 \ \ (< 0 \ \text{ for all } \boldsymbol{\phi}_{lin} \in \mathbb{R}^n \setminus \{\mathbf{0}\}),\end{aligned}$$

where $\boldsymbol{\phi}_{lin}$ is the solution of the linear ODE. From this it deduces that

$$\begin{aligned}\frac{d}{dt}V(\boldsymbol{\phi}) &= -\|\boldsymbol{\phi}\|_2^2 + \boldsymbol{\phi}^T P(\mathbf{f}(\boldsymbol{\phi}) - A\boldsymbol{\phi}) + (\mathbf{f}(\boldsymbol{\phi}) - A\boldsymbol{\phi})^T P\boldsymbol{\phi}\\ &\le -\|\boldsymbol{\phi}\|_2^2 + 2\|\boldsymbol{\phi}\|_2\|P\|_2\|\mathbf{f}(\boldsymbol{\phi}) - A\boldsymbol{\phi}\|_2\end{aligned}$$

and because $\mathbf{f}$ is differentiable at the origin there is a neighborhood of the origin, such that

$$\|\mathbf{f}(\mathbf{x}) - A\mathbf{x}\|_2 < \frac{\|\mathbf{x}\|_2}{2\|P\|_2}, \tag{1.2}$$

for all $\mathbf{x}$ in this neighborhood. By these calculations, V is a Lyapunov function for the ODE (1.1) too. However, its domain is not only restricted by the equilibrium's region of attraction, but also to the neighborhood in which (1.2) is satisfied, that can be very small in comparison to the equilibrium's region of attraction.

The original Lyapunov theory did not secure the existence of non-local Lyapunov functions for nonlinear systems with asymptotically stable equilibrium points. The first results on this subject are due to K. P. Perdeskii in 1933 [2]. The general case was resolved somewhat later. Theorems, which secure the existence of a Lyapunov or a Lyapunov-like function for a system possessing an equilibrium, stable in some sense, are called converse theorems in the theory of dynamical systems. Most of the converse theorems are proved by actually constructing by a finite or a transfinite procedure a Lyapunov(-like) function. Unfortunately, the trajectories of the respective systems are used by the construction methods. Hence, the converse theorems have up-to-date been pure existence theorems.

In this work we will prove a constructive converse theorem on exponential stability. The origin is said to be an exponentially stable equilibrium of (1.1), if and only if there is a neighborhood $\mathcal{N}$ of the origin and constants $\alpha > 0$ and $m \ge 1$, such that $\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|_2 \le me^{-\alpha t}\|\boldsymbol{\xi}\|_2$ for all $\boldsymbol{\xi} \in \mathcal{N}$ and all $t \ge 0$. The concept of an exponentially stable equilibrium point is mathematically more restrictive than the concept of an asymptotically stable equilibrium, where it is only demanded that there exists a neighborhood of the origin, such that all trajectories starting in this neighborhood are attracted to the equilibrium by the dynamics of the system. Although asymptotically stable equilibrium points that are not exponentially stable are an interesting mathematical phenomena (bifurcations), most equilibrium points are either exponentially stable or not asymptotically stable. If the real parts of all eigenvalues of the Jacobian of $\mathbf{f}$ at the origin are strictly negative, then the origin is exponentially stable, if one is strictly positive then it is unstable, and if all are negative and some are equal to zero, then the origin might be asymptotically stable but is not exponentially stable. In the last case the stability is usually not robust to perturbations and is therefore not desirable in engineering applications. A well known non-constructive converse theorem on exponential stability states:

Proposition 1.2. *Assume there is an open neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin and constants $\alpha > 0$ and $m \geq 1$, such that the solution $\boldsymbol{\phi}$ of (1.1) satisfies $\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|_2 \leq m e^{-\alpha t}\|\boldsymbol{\xi}\|_2$ for all $\boldsymbol{\xi} \in \mathcal{N}$ and all $t \geq 0$. Suppose the set $\{\mathbf{y} \in \mathbb{R}^n \big| \|\mathbf{y}\|_2 \leq m \sup_{\mathbf{z} \in \mathcal{N}} \|\mathbf{z}\|_2\}$ is a compact subset of $\mathcal{U}$ and let L be a Lipschitz constant for $\mathbf{f}$ on this compact set. Let T be a constant satisfying*

$$T > \frac{1}{\alpha} \ln(m).$$

Then the function $W : \mathcal{N} \longrightarrow \mathbb{R}$,

$$W(\boldsymbol{\xi}) := \int_0^T \|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|_2^2 d\tau$$

for all $\boldsymbol{\xi} \in \mathcal{N}$, satisfies the inequalities

$$\frac{1 - e^{-2LT}}{2L}\|\boldsymbol{\xi}\|_2^2 \leq W(\boldsymbol{\xi}) \leq m^2 \frac{1 - e^{-2\alpha T}}{2\alpha}\|\boldsymbol{\xi}\|_2^2$$

and

$$\nabla W(\boldsymbol{\xi}) \cdot \mathbf{f}(\boldsymbol{\xi}) \leq -(1 - m^2 e^{-2\alpha T})\|\boldsymbol{\xi}\|_2^2$$

for all $\boldsymbol{\xi} \in \mathcal{N}$, and is therefore a Lyapunov function for (1.1).

This converse theorem is useful because it gives an explicit formula for a Lyapunov function for the system. However, it is non-constructive because this formula involves the solution of the ODE, which, in general, is not known. We will use this Lyapunov function formula to prove that the linear program in the next section has a feasible solution.

The Lyapunov theory is covered in numerous textbooks on dynamical systems, e.g., [3], [7], [2], [9]. In [5] Proposition 1.1 (Theorem 1.16) and Proposition 1.2 (Theorem 1.18) are proved in the form stated here.

2. **Lyapunov functions with linear programming.** A linear programming problem is a set of linear constraints, under which a linear function is to be minimized. There are several equivalent forms for a linear programming problem, one of them being

$$\begin{aligned} \text{minimize} \quad & g(\mathbf{x}) := \mathbf{c}^T\mathbf{x}, \\ \text{given} \quad & C\mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{2.3}$$

where $r, s > 0$ are integers, $A \in \mathbb{R}^{s \times r}$ is a matrix, $\mathbf{b} \in \mathbb{R}^s$ and $\mathbf{c} \in \mathbb{R}^r$ are vectors, and $\mathbf{x} \leq \mathbf{y}$ denotes $x_i \leq y_i$ for all i. The function $\mathbf{x} \mapsto \mathbf{c}^T\mathbf{x}$ is called the objective of the linear program and the conditions $C\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ together are called the constraints. A feasible solution of the linear program is a vector $\mathbf{y} \in \mathbb{R}^s$ that satisfies the constraints, i.e., $\mathbf{y} \geq \mathbf{0}$ and $C\mathbf{y} \leq \mathbf{b}$. There are numerous algorithms known for solving linear programming problems, the most commonly used being the simplex method [8] or interior point algorithms [10], e.g., Karmarkar's algorithm. Both need a starting feasible solution for initialization. A feasible solution to (2.3) can be found by introducing slack variables $\mathbf{y} \in \mathbb{R}^s$ and solving the linear program:

$$\begin{aligned} \text{minimize} \quad & g(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}) := \sum_{i=1}^{s} y_i, \\ \text{given} \quad & \begin{bmatrix} C & -I_s \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \mathbf{b}, \ \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \geq \mathbf{0}, \end{aligned} \tag{2.4}$$

which has the feasible solution $\mathbf{x} = \mathbf{0}$ and $\mathbf{y}^T = (|b_1|, |b_2|, \dots, |b_s|)$. If this linear program has the solution $g([\mathbf{x}'\ \mathbf{y}']^T) = 0$, then $\mathbf{x}'$ is a feasible solution to (2.3), if the minimum of g is strictly larger than zero, then (2.3) has no feasible solution.

In order to construct a Lyapunov function with linear programming, one needs a class of continuous functions that are easily parameterized. The class of the continuous piecewise affine ‡ functions is an obvious candidate. In this section we first introduce continuous piecewise affine (CPWA) functions $\mathbb{R}^n \longrightarrow \mathbb{R}$. The advantage of this function space is, that it is isomorphic to a vector space where the vectors are finite tuples of real numbers. Then we state a linear program, where the components of the matrix C and the vector $\mathbf{b}$ above are calculated using the function $\mathbf{f}$ from (1.1), that has the property, that every feasible solution to it parameterizes a CPWA Lyapunov function for (1.1).

CPWA **Lyapunov functions.** Let $N > 0$ be an integer and $0 = y_0 < y_1 < \dots < y_N$ be real numbers. Let $P : [0, N] \longrightarrow [0, y_N]$ be the unique continuous function, of which the restriction on every interval $[i, i+1]$, $i = 0, 1, \dots, N-1$, is affine, and such that $P(i) = y_i$ for all $i = 0, 1, \dots, N$. Define the function $\mathbf{PS} : [-N, N]^n \longrightarrow [-y_N, y_N]^n$ through

$$\mathbf{PS}(\mathbf{x}) := \sum_{i=1}^{n} \operatorname{sign}(x_i) P(|x_i|) \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the i-th unit vector. Denote by Sym_n the set of the permutations of $\{1, 2, \dots, n\}$ and define for every $\sigma \in \mathrm{Sym}_n$ the set

$$S_\sigma := \{\mathbf{y} \in \mathbb{R}^n \big|\ 0 \le y_{\sigma(1)} \le y_{\sigma(2)} \le \dots \le y_{\sigma(n)} \le 1\}.$$

Denote by $\mathfrak{P}(\{1, 2, \dots, n\})$ the power-set of $\{1, 2, \dots, n\}$ and define the function $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for every $\mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$ through

$$\mathbf{R}^{\mathcal{J}}(\mathbf{x}) := \sum_{i=1}^{n} (-1)^{\chi_{\mathcal{J}}(i)} x_i \mathbf{e}_i,$$

where $\chi_{\mathcal{J}} : \{1, 2, \dots, n\} \longrightarrow \{0, 1\}$ is the characteristic function of the set $\mathcal{J}$. A continuous function $G : [-y_N, y_N]^n \longrightarrow \mathbb{R}$ is an element of $\mathrm{CPWA}[\mathbf{PS}, [-N, N]^n]$, if and only if its restriction $G|_{\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+S_\sigma))}$ to the set $\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + S_\sigma))$ is affine for every $\mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$ and every $\mathbf{z} \in \{0, 1, \dots, N-1\}^n$. It is proved in Chapter 4 in [5] that the mapping

$$\mathrm{CPWA}[\mathbf{PS}, [-N, N]^n] \longrightarrow \mathbb{R}^{(2N+1)^n}, \quad G \mapsto (a_\mathbf{z})_{\mathbf{z} \in \{-N, -N+1, \dots, N\}^n},$$

where $a_\mathbf{z} = G(\mathbf{PS}(\mathbf{z}))$ for all $\mathbf{z} \in \{-N, -N+1, \dots, N\}^n$, is a vector space isomorphism. This means, that we can uniquely define a function in $\mathrm{CPWA}[\mathbf{PS}, [-N, N]^n]$ by assigning it values on the grid $\{-y_N, -y_{N-1}, \dots, y_0, y_1, \dots, y_N\}^n$.

In Chapter 5 in [5] it is proved, that every feasible solution of the following linear program § parameterizes a CPWA Lyapunov function for (1.1).

‡The popular term for piecewise affine is piecewise linear. In higher mathematics the term linear is reserved for affine mappings that vanish at the origin, so we use the term affine to avoid confusion.

§Actually, the linear program in [5] is more general than the linear program presented here.

The linear program. Consider the system (1.1). Let $N > 0$ be an integer and let $0 = y_0 < y_1 < \ldots < y_N$ be real numbers, such that $[-y_N, y_N]^n \subset \mathcal{U}$. Let $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be defined through the constants $y_0, y_1, \ldots, y_N$ as above and let d be an integer, $0 \leq d < N$. The linear program is constructed in the following way:

i) *Define the sets*

$$\mathcal{X}^{\|\cdot\|_2} := \{\|\mathbf{x}\|_2 \big| \ \mathbf{x} \in \{y_0, y_1, \ldots, y_N\}^n\}$$

and

$$\mathcal{G} := \{-y_N, -y_{N-1}, \ldots, y_0, y_1, \ldots, y_N\}^n \setminus \{-y_{d-1}, -y_{d-2}, \ldots, y_0, y_1, \ldots, y_{d-1}\}^n.$$

ii) *Define for every $\sigma \in \mathrm{Sym}_n$ and every $i = 1, 2, \ldots, n+1$, the vector*

$$\mathbf{x}_i^\sigma := \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)}.$$

iii) *Define the set*

$$\mathcal{Z} := [\{0, 1, \ldots, N-1\}^n \setminus \{0, 1, \ldots, d-1\}^n] \times \mathfrak{P}(\{1, 2, \ldots, n\}).$$

iv) *For every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$ define for every $\sigma \in \mathrm{Sym}_n$ and every $i = 1, 2, \ldots, n+1$, the vector*

$$\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} := \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + \mathbf{x}_i^\sigma)).$$

v) *Define the set*

$$\mathcal{Y} := \{\{\mathbf{y}_{\sigma,k}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,k+1}^{(\mathbf{z},\mathcal{J})}\} \big| (\mathbf{z}, \mathcal{J}) \in \mathcal{Z} \text{ and } k \in \{1, 2, \ldots, n\}\}.$$

The set $\mathcal{Y}$ is the set of neighboring grid points in the grid $\mathcal{G}$.

vi) *For every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$ and every $r, s = 1, 2, \ldots, n$ let $B_{rs}^{(\mathbf{z},\mathcal{J})}$ be a real constant, such that*

$$B_{rs}^{(\mathbf{z},\mathcal{J})} \geq \max_{i=1,2,\ldots,n} \sup_{\mathbf{x} \in \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} +]0,1[^n))} \left| \frac{\partial^2 f_i}{\partial x_r \partial x_s}(\mathbf{x}) \right|.$$

vii) *For every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, every $k, i = 1, 2, \ldots, n$, and every $\sigma \in \mathrm{Sym}_n$, define*

$$A_{\sigma,k,i}^{(\mathbf{z},\mathcal{J})} := |\mathbf{e}_k \cdot (\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})|.$$

viii) *Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary constants.*

The variables of the linear program are:

$$\Psi[x], \quad \text{for all } x \in \mathcal{X}^{\|\cdot\|_2},$$
$$\Gamma[x], \quad \text{for all } x \in \mathcal{X}^{\|\cdot\|_2},$$
$$V[\mathbf{x}], \quad \text{for all } \mathbf{x} \in \mathcal{G},$$
$$C[\{\mathbf{x}, \mathbf{y}\}], \quad \text{for all } \{\mathbf{x}, \mathbf{y}\} \in \mathcal{Y}.$$

The linear constraints of the linear program are:

LC1) *Let $x_1, x_2, \ldots, x_K$ be the elements of $\mathcal{X}^{\|\cdot\|_2}$ in an increasing order. Then*

$$\Psi[x_1] = \Gamma[x_1] = 0,$$
$$\varepsilon x_2 \leq \Psi[x_2],$$
$$\varepsilon x_2 \leq \Gamma[x_2],$$

and for every $i = 2, 3, \ldots, K-1$:

$$\frac{\Psi[x_i] - \Psi[x_{i-1}]}{x_i - x_{i-1}} \leq \frac{\Psi[x_{i+1}] - \Psi[x_i]}{x_{i+1} - x_i}$$

and
$$\frac{\Gamma[x_i]-\Gamma[x_{i-1}]}{x_i-x_{i-1}} \le \frac{\Gamma[x_{i+1}]-\Gamma[x_i]}{x_{i+1}-x_i}.$$

LC2) *For every* $\mathbf{x}\in\mathcal{G}$*:*
$$\Psi[\|\mathbf{x}\|_2] \le V[\mathbf{x}].$$
If $d=0$*, then*
$$V[\mathbf{0}]=0.$$
If $d\ge 1$*, then for every* $\mathbf{x}\in\mathcal{G}\cap\{-y_d,-y_{d-1},\dots,y_0,y_1,\dots,y_d\}^n$*:*
$$V[\mathbf{x}] \le \Psi[y_N]-\delta.$$

LC3) *For every* $\{\mathbf{x},\mathbf{y}\}\in\mathcal{Y}$*:*
$$-C[\{\mathbf{x},\mathbf{y}\}]\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \le V[\mathbf{x}]-V[\mathbf{y}] \le C[\{\mathbf{x},\mathbf{y}\}]\cdot\|\mathbf{x}-\mathbf{y}\|_\infty.$$

LC4) *For every* $(\mathbf{z},\mathcal{J})\in\mathcal{Z}$*, every* $\sigma\in\mathrm{Sym}_n$*, and every* $i=1,2,\dots,n+1$*:*
$$-\Gamma[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2] \ge \sum_{j=1}^{n}\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}]-V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}-\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})$$
$$+\frac{1}{2}\sum_{r,s=1}^{n}B_{rs}^{(\mathbf{z},\mathcal{J})}A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})}(A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})}+A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})})\sum_{j=1}^{n}C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})},\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}].$$

Note that the values of the constants $\varepsilon>0$ and $\delta>0$ do not affect whether there is a feasible solution of the linear program or not. If there is a feasible solution for $\varepsilon:=\varepsilon'>0$ and $\delta:=\delta'>0$, then there is a feasible solution for all $\varepsilon:=\varepsilon^*>0$ and $\delta:=\delta^*>0$. Just multiply all variables with $\max\{\varepsilon^*/\varepsilon',\delta^*/\delta'\}$. Those familiar with linear programming might wonder why there is no objective defined. The reason is, that the objective of the linear program is not needed. It can, however, be used to optimize the Lyapunov function in some way.

Assume that the linear program above has a feasible solution. Then we can define the functions $\psi,\gamma:[0,+\infty[\longrightarrow\mathbb{R}$ by using the values of the variables $\Psi[x],\Gamma[x]$ and the function $V^{Lya}:[-y_N,y_N]^n\longrightarrow\mathbb{R}$ by using the values of the variables $V[\mathbf{x}]$ in the following way:

Let $x_1,x_2,\dots,x_K$ be the elements of $\mathcal{X}^{\|\cdot\|_2}$ in an increasing order. We define the piecewise affine functions
$$\psi(y):=\Psi[x_i]+\frac{\Psi[x_{i+1}]-\Psi[x_i]}{x_{i+1}-x_i}(y-x_i)$$
and
$$\gamma(y):=\Gamma[x_i]+\frac{\Gamma[x_{i+1}]-\Gamma[x_i]}{x_{i+1}-x_i}(y-x_i),$$
for all $y\in[x_i,x_{i+1}]$ and all $i=1,2,\dots,K-1$. The values of ψ and γ on $]x_K,+\infty[$ do not really matter, but to have everything properly defined, we set
$$\psi(y):=\Psi[x_{K-1}]+\frac{\Psi[x_K]-\Psi[x_{K-1}]}{x_K-x_{K-1}}(y-x_{K-1})$$
and
$$\gamma(y):=\Gamma[x_{K-1}]+\frac{\Gamma[x_K]-\Gamma[x_{K-1}]}{x_K-x_{K-1}}(y-x_{K-1})$$
for all $y>x_K$. Clearly the functions ψ and γ are continuous. The function $V^{Lya}\in\mathrm{CPWA}[\mathbf{PS},[-N,N]^n]$ is defined by assigning
$$V^{Lya}(\mathbf{x}):=V[\mathbf{x}]$$

for all $\mathbf{x} \in \mathcal{G}$. In Chapter 5 in [5] it is proved, that ψ and γ are convex and strictly increasing and that

$$\psi(\|\mathbf{x}\|) \leq V^{Lya}(\mathbf{x})$$

for all $\mathbf{x} \in [-y_N, y_N]^n \setminus]-y_d, y_d[^n$, and

$$\limsup_{s \to 0+} \frac{V(\boldsymbol{\phi}(t,\boldsymbol{\xi}) + s\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))) - V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))}{s} \leq -\gamma(\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2),$$

for all $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in]-y_N, y_N[^n \setminus [-y_d, y_d]^n$. This implies that if $d = 0$, then V^{Lya} is a Lyapunov function for (1.1). Further, it is proved for $d > 0$, that for every $c > 0$, such that the connected component of

$$\{\mathbf{x} \in]-y_N, y_N[^n \setminus [-y_d, y_d]^n \big| V^{Lya}(\mathbf{x}) \leq c\} \cup [-y_d, y_d]^n$$

containing the origin is compact, there is a $t_{\boldsymbol{\xi}} \geq 0$ for every $\boldsymbol{\xi}$ in this component such that $\boldsymbol{\phi}(t_{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in [-y_d, y_d]^n$.

3. **The constructive converse theorem.** In this section we prove the main results of this work, a constructive converse theorem on exponential stability for (1.1). We do this by using the Lyapunov function from Proposition 1.2 to assign values to the variables in the linear program, and then we prove that the linear constraints of the linear program are satisfied with these values. Let us discuss this central point of this work in detail.

We want to prove that the linear program presented in the last section always succeeds in parameterizing a Lyapunov function on a domain $[-a, a]^n$, $a > 0$, for an ODE of the form (1.1) if:

- $\mathbf{f}$ is a class $\mathcal{C}^2$ function.
- There are constants $m \geq 1$ and $\alpha > 0$, such that the inequality $\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2 \leq me^{-\alpha t}\|\boldsymbol{\xi}\|_2$ is satisfied for all $\boldsymbol{\xi} \in [-a, a]^n$ and all $t \geq 0$.

If we do this, then we have proved a constructive converse theorem on exponential stability. To prove that the linear program always succeeds in parameterizing a Lyapunov function, we show that it has at least one feasible solution. This is sufficient, because there are algorithms, e.g., the simplex method, that find a feasible solution if the set of feasible solutions is not empty. From the elementary theory of ODEs, e.g., the theorem of Picard-Lindelöf, we know that the system (1.1) possesses a unique solution $\boldsymbol{\phi}$ and from Proposition 1.2 we know that the function

$$W(\boldsymbol{\xi}) := \int_0^T \|\boldsymbol{\phi}(\tau,\boldsymbol{\xi})\|_2^2 d\tau$$

is a Lyapunov function for the system on the domain $[-a, a]^n$. Because this formula for W involves the solution of the ODE (1.1) and its algebraic form, in general, is not known, Proposition 1.2 is not constructive. However, if we can use W and $\mathbf{f}$ to assign values to the variables of the linear program in the last section, e.g., $V[\mathbf{x}] := W(\mathbf{x})$, and then show, that the constraints of the linear program are satisfied when the variables have these values, then we have proved that its set of feasible solutions is not empty. Note, that we do not know the numeric values we assign to the variables of the linear program. We only know their formulas, which involve the (unknown) solution $\boldsymbol{\phi}$ of the ODE (1.1).

First, we state a well known theorem that is useful for the proof of the constructive converse theorem.

Theorem 3.1. *Let $\mathcal{V} \subset \mathbb{R}^n$ be a domain, $[t_0, t_1]$ be an interval, $-\infty < t_0 < t_1 < +\infty$, and $\mathbf{g}, \mathbf{h} : \mathcal{V} \longrightarrow \mathbb{R}^n$ be continuously differentiable functions. Suppose there are positive real constants M and L, such that*

$$\|\mathbf{g}(t,\mathbf{x}) - \mathbf{h}(t,\mathbf{x})\|_2 \leq M$$

and

$$\|[\nabla_{\mathbf{x}}\mathbf{g}](t,\mathbf{x})\|_2 \leq L$$

for all $t \in [t_0, t_1]$ and all $\mathbf{x} \in \mathcal{V}$. Let $t \mapsto \mathbf{y}(t)$ be the solution of the initial value problem

$$\dot{\mathbf{x}} = \mathbf{g}(t,\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{y}_0$$

and $t \mapsto \mathbf{z}(t)$ be the solution of the initial value problem

$$\dot{\mathbf{x}} = \mathbf{h}(t,\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{z}_0.$$

Then

$$\|\mathbf{y}(t) - \mathbf{z}(t)\|_2 \leq \|\mathbf{y}_0 - \mathbf{z}_0\|_2 e^{L(t-t_0)} + M\frac{e^{L(t-t_0)} - 1}{L}$$

for all $t \in [t_0, t_1]$.

Proof:
See, for example, Theorem 2.5 in [3].

■

We now state and prove the main theorem of this work.

Theorem 3.2 (Constructive converse theorem on exponential stability)**.** *Consider the system (1.1) and assume there are constants $\alpha > 0$, $m \geq 1$, and $a > 0$ such that $[-ma, ma]^n \subset \mathcal{U}$ and the inequality $\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2 \leq me^{-\alpha t}\|\boldsymbol{\xi}\|_2$ is satisfied for all $t \geq 0$ and all $\boldsymbol{\xi} \in [-a,a]^n$. Then, for every neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin, we can choose the constants d, N, and $y_0, y_1, \ldots, y_N$ in the linear program in Section 2, such that $y_N = a$, $]-y_d, y_d[^n \subset \mathcal{N}$, and such that the linear program has a feasible solution, i.e., it will succeed in parameterizing a* CPWA *Lyapunov function for the system.*

More exactly, let T, η, a_{ij}, and b'_{ijk}, be strictly positive real constants fulfilling

$$\begin{aligned}
T &> \frac{1}{\alpha}\ln(m), \\
\eta &< 1 - m^2 e^{-2\alpha T}, \\
a_{ij} &\geq \sup_{\boldsymbol{\xi}\in[-ma,ma]^n} \left|\frac{\partial f_i}{\partial x_j}(\boldsymbol{\xi})\right| \quad \textit{for } i,j = 1,2,\ldots,n, \\
b'_{ijk} &\geq \sup_{\boldsymbol{\xi}\in[-ma,ma]^n} \left|\frac{\partial^2 f_i}{\partial x_k \partial x_j}(\boldsymbol{\xi})\right| \quad \textit{for } i,j,k = 1,2,\ldots,n,
\end{aligned}$$

and set

$$\mathtt{A} := \left\| \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \right\|_2 ,$$

$$B_{rs} := \max_{i=1,2,\dots,n} b'_{irs} \quad \text{for } r,s = 1,2,\dots,n,$$

$$b_{ij} := \left(\sum_{k=1}^{n} {b'_{ijk}}^2 \right)^{\frac{1}{2}} \quad \text{for } i,j = 1,2,\dots,n,$$

and

$$\mathtt{B} := \left\| \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \right\|_2 .$$

Define the constants C, D, E, and F by

$$C := 2nm\mathtt{B} \left[\frac{e^{(3\mathtt{A}-\alpha)T} - 1}{3\mathtt{A} - \alpha} - \frac{e^{(2\mathtt{A}-\alpha)T} - 1}{2\mathtt{A} - \alpha} \right],$$

$$D := n(e^{2\mathtt{A}T} - 1),$$

$$E := 1 - m^2 e^{-2\alpha T},$$

$$F := 2nm \frac{e^{(\mathtt{A}-\alpha)T} - 1}{\mathtt{A} - \alpha} \sum_{r,s=1}^{n} B_{rs},$$

and let $c > 0$ be a constant, such that $]-c,c[^n \subset \mathcal{N}$ and

$$c < a \sqrt{\frac{\alpha}{nm^2 \mathtt{A}} \cdot \frac{1 - e^{-2\mathtt{A}T}}{1 - e^{-2\alpha T}}}.$$

Let $h^d > 0$ be a real constant, such that

$$h^d \leq \frac{-(Cc + D) + \sqrt{(Cc + D)^2 + 4(E - \eta)c(1 + \sqrt{n})F}}{2(1 + \sqrt{n})F}$$

and such that

$$d := \frac{c}{h^d}$$

is an integer. Set

$$y_i := ih^d \quad \text{for } i = 0,1,\dots,d$$

and choose y_{i+1} for $i = d, d+1, \dots, N-1$, such that

$$y_i < y_{i+1} \leq y_i + \min \left\{ y_i, \frac{-(Cy_i + D) + \sqrt{(Cy_i + D)^2 + 4(E - \eta)y_i(1 + \sqrt{n})F}}{2(1 + \sqrt{n})F} \right\}$$

and

$$y_N = a,$$

where N is a large enough integer. Define the function

$$\mathbf{PS} : [-N,N]^n \longrightarrow [-y_N, y_N]^n$$

through the constants $y_0, y_1, \ldots, y_N$ as in Section 2. Finally, set

$$B_{rs}^{(\mathbf{z},\mathcal{J})} := B_{rs}$$

for all $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$ and all $r, s = 1, 2, \ldots, n$. Then the linear program in Section 2 has a feasible solution.

PROOF:
Assign values to the constants ε and δ by the formulas

$$\varepsilon := h^d \cdot \min\left\{\eta, \frac{1-e^{-2\mathtt{A}T}}{2\mathtt{A}}\right\}$$

and

$$\delta := \frac{1-e^{-2\mathtt{A}T}}{2\mathtt{A}} y_N^2 - nm^2 \frac{1-e^{-2\alpha T}}{2\alpha} c^2.$$

If there is a feasible solution for these particular $\varepsilon > 0$ and $\delta > 0$, then there is a feasible solution for all $\varepsilon > 0$ and $\delta > 0$. Let $W : [-a,a]^n \longrightarrow \mathbb{R}$ be the Lyapunov function from Proposition 1.2,

$$W(\boldsymbol{\xi}) := \int_0^T \|\boldsymbol{\phi}(\tau,\boldsymbol{\xi})\|_2^2 d\tau$$

for all $\boldsymbol{\xi} \in [-a,a]^n$. Then the inequalities

$$\frac{1-e^{-2\mathtt{A}T}}{2\mathtt{A}} \|\boldsymbol{\xi}\|_2^2 \le W(\boldsymbol{\xi}) \le m^2 \frac{1-e^{-2\alpha T}}{2\alpha} \|\boldsymbol{\xi}\|_2^2 \tag{3.5}$$

and

$$\nabla W(\boldsymbol{\xi}) \cdot \mathbf{f}(\boldsymbol{\xi}) \le -(1 - m^2 e^{-2\alpha T}) \|\boldsymbol{\xi}\|_2^2$$

are satisfied for all $\boldsymbol{\xi} \in [-a,a]^n$ and all $\boldsymbol{\xi} \in]-a,a[^n$ respectively.

We assign values to the variables $\Psi[x_i]$, $\Gamma[x_i]$, and $V[\mathbf{x}]$, and we successively show that the linear constraints LC1-LC4 of the linear program are satisfied when the variables have these values.

Assign

$$\Psi[x_i] := \frac{1-e^{-2\mathtt{A}T}}{2\mathtt{A}} x_i^2$$

and

$$\Gamma[x_i] := \eta x_i^2$$

for $i = 1, 2, \ldots, K$, where $x_1, x_2, \ldots, x_K$ are the elements of $\mathcal{X}^{\|\cdot\|_2}$ in an increasing order. Set

$$V[\mathbf{x}] := W[\mathbf{x}]$$

for all $\mathbf{x} \in \mathcal{G}$ and set

$$C[\{\mathbf{x},\mathbf{y}\}] := 2m \frac{e^{(\mathtt{A}-\alpha)T} - 1}{\mathtt{A} - \alpha} \max\{\|\mathbf{x}\|_2, \|\mathbf{y}\|_2\}$$

for all $\{\mathbf{x},\mathbf{y}\} \in \mathcal{Y}$.

LC1: The equality $\Psi[x_1] = \Gamma[x_1] = 0$ is trivial. Because of $x_2 = h^d$ and the definition of ε, $\varepsilon x_2 \le \Psi[x_2]$ and $\varepsilon x_2 \le \Gamma[x_2]$ follow immediately. To see that

$$\frac{\Psi[x_i] - \Psi[x_{i-1}]}{x_i - x_{i-1}} \le \frac{\Psi[x_{i+1}] - \Psi[x_i]}{x_{i+1} - x_i}$$

and

$$\frac{\Gamma[x_i]-\Gamma[x_{i-1}]}{x_i-x_{i-1}} \le \frac{\Gamma[x_{i+1}]-\Gamma[x_i]}{x_{i+1}-x_i}$$

just note that they are equivalent to

$$x_i+x_{i-1} = \frac{x_i^2-x_{i-1}^2}{x_i-x_{i-1}} \le \frac{x_{i+1}^2-x_i^2}{x_{i+1}-x_i} = x_{i+1}+x_i,$$

which is obvious.

LC2: By (3.5),

$$\Psi[\|\mathbf{x}\|_2] \le V[\mathbf{x}]$$

for all $\mathbf{x}\in\mathcal{G}$. Let $\mathbf{y}\in\mathcal{G}\cap\{-y_d,-y_{d-1},\dots,y_0,y_1,\dots,y_d\}^n$. It follows from (3.5) and the definition of c and δ, that

$$\begin{aligned} V[\mathbf{y}] &\le m^2\frac{1-e^{-2\alpha T}}{2\alpha}\|\mathbf{y}\|_2^2 \le m^2\frac{1-e^{-2\alpha T}}{2\alpha}nc^2 \\ &= \frac{1-e^{-2\mathtt{A}T}}{2\mathtt{A}}y_N^2-\delta \le \Psi[y_N]-\delta \end{aligned}$$

LC3: Let $\{\mathbf{x},\mathbf{y}\}\in\mathcal{Y}$. From the definition of $\mathcal{Y}$ it follows, that there is a constant $h\in\mathbb{R}$ and an $i\in\{1,2,\dots,n\}$, such that $\mathbf{x}-\mathbf{y}=h\mathbf{e}_i$. Further, there is a $\mathbf{z}$ on the line segment between $\mathbf{x}$ and $\mathbf{y}$, such that

$$\frac{V[\mathbf{x}]-V[\mathbf{y}]}{\|\mathbf{x}-\mathbf{y}\|_\infty} = \text{sign}(h)\frac{\partial W}{\partial x_i}(\mathbf{z}).$$

Because $\mathbf{f}$ is a class $\mathcal{C}^2$ function, then so is $\boldsymbol{\phi}$ (see, for example, Theorem 1.4 in [5]) and

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial\boldsymbol{\phi}}{\partial\xi_i}(t,\boldsymbol{\xi})\right) &= \frac{\partial}{\partial\xi_i}\left(\frac{\partial\boldsymbol{\phi}}{\partial t}(t,\boldsymbol{\xi})\right) \\ &= \frac{\partial}{\partial\xi_i}\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \\ &= [\nabla\mathbf{f}](\boldsymbol{\phi}(t,\boldsymbol{\xi}))\cdot\frac{\partial\boldsymbol{\phi}}{\partial\xi_i}(t,\boldsymbol{\xi}) \end{aligned}$$

implies that the functions

$$\tau\mapsto\mathbf{0} \quad\text{and}\quad \tau\mapsto\frac{\partial\boldsymbol{\phi}}{\partial\xi_i}(\tau,\mathbf{z})$$

are the solutions of the initial value problems

$$\dot{\mathbf{x}} = [\nabla\mathbf{f}](\boldsymbol{\phi}(t,\mathbf{z}))\mathbf{x}, \quad \mathbf{x}(0)=\mathbf{0},$$

and

$$\dot{\mathbf{x}} = [\nabla\mathbf{f}](\boldsymbol{\phi}(t,\mathbf{z}))\mathbf{x}, \quad \mathbf{x}(0)=\mathbf{e}_i,$$

respectively. Hence, we get by Theorem 3.1 that

$$\|\frac{\partial\boldsymbol{\phi}}{\partial\xi_i}(\tau,\mathbf{z})\|_2 \le e^{\mathtt{A}\tau}$$

and then

$$\begin{aligned}\left|\frac{V[\mathbf{x}]-V[\mathbf{y}]}{\|\mathbf{x}-\mathbf{y}\|_\infty}\right| &= \left|\frac{\partial W}{\partial \xi_i}(\mathbf{z})\right| \\ &= \left|2\int_0^T \boldsymbol{\phi}(\tau,\mathbf{z})\cdot\frac{\partial \boldsymbol{\phi}}{\partial \xi_i}(\tau,\mathbf{z})d\tau\right| \\ &\le 2\int_0^T \|\boldsymbol{\phi}(\tau,\mathbf{z})\|_2\|\frac{\partial \boldsymbol{\phi}}{\partial \xi_i}(\tau,\mathbf{z})\|_2 d\tau \\ &\le 2\int_0^T me^{-\alpha\tau}\|\mathbf{z}\|_2 e^{\mathtt{A}\tau}d\tau \\ &= 2m\|\mathbf{z}\|_2\frac{e^{(\mathtt{A}-\alpha)T}-1}{\mathtt{A}-\alpha} \\ &\le C[\{\mathbf{x},\mathbf{y}\}].\end{aligned}$$

LC4: Let $(\mathbf{z},\mathcal{J})\in\mathcal{Z}$, $\sigma\in\mathrm{Sym}_n$, and define

$$h := \max_{i=1,2,\dots,n}\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}-\mathbf{y}_{\sigma,i+1}^{(\mathbf{z},\mathcal{J})}\|_\infty.$$

Note that

$$\begin{aligned}&\sum_{j=1}^n\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}]-V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}-\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\ &\quad=\sum_{j=1}^n\left(\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}]-V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}-\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}-\frac{\partial W}{\partial \xi_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\right)f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\ &\qquad+\nabla W(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\cdot\mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}).\end{aligned}$$

Now, for every $j=1,2,\dots,n$ there is a $\mathbf{z}_j$ on the line segment between $\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}$ and $\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}$, such that

$$\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}]-V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}-\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}=\frac{\partial W}{\partial \xi_{\sigma(j)}}(\mathbf{z}_j).$$

This means that

$$\begin{aligned}&\left|\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}]-V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}-\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}-\frac{\partial W}{\partial \xi_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\right|=\left|\frac{\partial W}{\partial \xi_{\sigma(j)}}(\mathbf{z}_j)-\frac{\partial W}{\partial \xi_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\right| \\ &=2\left|\int_0^T\left[\boldsymbol{\phi}(\tau,\mathbf{z}_j)\cdot\frac{\partial \boldsymbol{\phi}}{\partial \xi_{\sigma(j)}}(\tau,\mathbf{z}_j)-\boldsymbol{\phi}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\cdot\frac{\partial \boldsymbol{\phi}}{\partial \xi_{\sigma(j)}}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\right]d\tau\right| \\ &=2\left|\int_0^T\left[\boldsymbol{\phi}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\cdot\left(\frac{\partial \boldsymbol{\phi}}{\partial \xi_{\sigma(j)}}(\tau,\mathbf{z}_j)-\frac{\partial \boldsymbol{\phi}}{\partial \xi_{\sigma(j)}}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\right)\right.\right. \\ &\qquad\left.\left.+\ \frac{\partial \boldsymbol{\phi}}{\partial \xi_{\sigma(j)}}(\tau,\mathbf{z}_j)\cdot\left(\boldsymbol{\phi}(\tau,\mathbf{z}_j)-\boldsymbol{\phi}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\right)\right]d\tau\right| \\ &\le 2\int_0^T\left[\|\boldsymbol{\phi}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2\|\frac{\partial \boldsymbol{\phi}}{\partial \xi_{\sigma(j)}}(\tau,\mathbf{z}_j)-\frac{\partial \boldsymbol{\phi}}{\partial \xi_{\sigma(j)}}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2\right. \\ &\qquad\left.+\|\frac{\partial \boldsymbol{\phi}}{\partial \xi_{\sigma(j)}}(\tau,\mathbf{z}_j)\|_2\|\boldsymbol{\phi}(\tau,\mathbf{z}_j)-\boldsymbol{\phi}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2\right]d\tau.\end{aligned}$$

By the exponential stability

$$\|\boldsymbol{\phi}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2 \le me^{-\alpha\tau}\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2.$$

By Theorem 3.1

$$\|\boldsymbol{\phi}(\tau,\mathbf{z}_j)-\boldsymbol{\phi}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2 \le \|\mathbf{z}_j-\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 e^{\mathtt{A}\tau} \le h\sqrt{n}e^{\mathtt{A}\tau}.$$

Because

$$\tau \mapsto \mathbf{0}, \quad \tau \mapsto \frac{\partial\boldsymbol{\phi}}{\partial\xi_{\sigma(j)}}(\tau,\mathbf{z}_j), \quad \text{and } \tau \mapsto \frac{\partial\boldsymbol{\phi}}{\partial\xi_{\sigma(j)}}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})$$

are the solutions of the initial value problems

$$\dot{\mathbf{x}} = [\nabla\mathbf{f}](\boldsymbol{\phi}(t,\mathbf{z}_j))\mathbf{x}, \quad \mathbf{x}(0)=\mathbf{0},$$
$$\dot{\mathbf{x}} = [\nabla\mathbf{f}](\boldsymbol{\phi}(t,\mathbf{z}_j))\mathbf{x}, \quad \mathbf{x}(0)=\mathbf{e}_{\sigma(j)},$$

and

$$\dot{\mathbf{x}} = [\nabla\mathbf{f}](\boldsymbol{\phi}(t,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}))\mathbf{x}, \quad \mathbf{x}(0)=\mathbf{e}_{\sigma(j)},$$

respectively, we get by the same theorem, that

$$\|\frac{\partial\boldsymbol{\phi}}{\partial\xi_{\sigma(j)}}(\tau,\mathbf{z}_j)\|_2 \le e^{\mathtt{A}\tau}$$

and

$$\begin{aligned}
&\|\frac{\partial\boldsymbol{\phi}}{\partial\xi_{\sigma(j)}}(\tau,\mathbf{z}_j)-\frac{\partial\boldsymbol{\phi}}{\partial\xi_{\sigma(j)}}(\tau,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2 \\
&\le \sup_{\substack{\|\mathbf{x}\|_2\le e^{A\tau}\\ t\in[0,\tau]}} \|([\nabla\mathbf{f}](\boldsymbol{\phi}(t,\mathbf{z}_j))-[\nabla\mathbf{f}](\boldsymbol{\phi}(t,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})))\mathbf{x}\|_2 \frac{e^{\mathtt{A}\tau}-1}{\mathtt{A}} \\
&\le \sup_{t\in[0,\tau]} \|[\nabla\mathbf{f}](\boldsymbol{\phi}(t,\mathbf{z}_j))-[\nabla\mathbf{f}](\boldsymbol{\phi}(t,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}))\|_2 e^{A\tau}\frac{e^{\mathtt{A}\tau}-1}{\mathtt{A}} \\
&= e^{A\tau}\frac{e^{\mathtt{A}\tau}-1}{\mathtt{A}} \sup_{t\in[0,\tau]} \left\| \left(\frac{\partial f_r}{\partial x_s}(\boldsymbol{\phi}(t,\mathbf{z}_j))-\frac{\partial f_r}{\partial x_s}(\boldsymbol{\phi}(t,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}))\right)_{(r,s)\in\{1,2,\dots,n\}^2}\right\|_2 \\
&\le e^{A\tau}\frac{e^{\mathtt{A}\tau}-1}{\mathtt{A}} \sup_{t\in[0,\tau]} \left\| \left(b_{rs}\|\boldsymbol{\phi}(t,\mathbf{z}_j))-\boldsymbol{\phi}(t,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2\right)_{(r,s)\in\{1,2,\dots,n\}^2}\right\|_2 \\
&\le e^{A\tau}\frac{e^{\mathtt{A}\tau}-1}{\mathtt{A}}\mathtt{B} \sup_{t\in[0,\tau]} \|\boldsymbol{\phi}(t,\mathbf{z}_j))-\boldsymbol{\phi}(t,\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2 \\
&\le h\mathtt{B}\sqrt{n}e^{2\mathtt{A}\tau}\frac{e^{\mathtt{A}\tau}-1}{\mathtt{A}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}]-V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}-\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}-\frac{\partial W}{\partial\xi_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\right| \\
&\le 2\int_0^T \left(me^{-\alpha\tau}\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 h\mathtt{B}\sqrt{n}e^{2\mathtt{A}\tau}\frac{e^{\mathtt{A}\tau}-1}{\mathtt{A}}+e^{\mathtt{A}\tau}h\sqrt{n}e^{\mathtt{A}\tau}\right)d\tau \\
&= 2h\sqrt{n}\left(\frac{m\mathtt{B}}{\mathtt{A}}\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2\left[\frac{e^{(3\mathtt{A}-\alpha)T}-1}{3\mathtt{A}-\alpha}-\frac{e^{(2\mathtt{A}-\alpha)T}-1}{2\mathtt{A}-\alpha}\right]+\frac{e^{2\mathtt{A}T}-1}{2\mathtt{A}}\right) \\
&= \frac{h}{\mathtt{A}\sqrt{n}}(C\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2+D),
\end{aligned}$$

from which

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\
&\quad \leq \| \sum_{j=1}^{n} \left(\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} - \frac{\partial W}{\partial \xi_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \right) \mathbf{e}_{\sigma(j)} \|_2 \|\mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2 \\
&\qquad - (1 - m^2 e^{-2\alpha T}) \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2^2 \\
&\quad \leq \frac{h}{\mathtt{A}} (C\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 + D)\mathtt{A}\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 - E\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2^2 \\
&\quad = h(C\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 + D)\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 - E\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2^2
\end{aligned}$$

follows.

Because

$$\begin{aligned}
&\frac{1}{2} \sum_{r,s=1}^{n} B_{rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} \left(A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} \right) \sum_{j=1}^{n} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}] \\
&\qquad \leq \frac{1}{2} \sum_{r,s=1}^{n} B_{rs} h(h+h) \sum_{j=1}^{n} 2m \frac{e^{(\mathtt{A}-\alpha)T} - 1}{\mathtt{A} - \alpha} \max\{\|\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}\|_2, \|\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\|_2\} \\
&\qquad \leq h^2 2nm \frac{e^{(\mathtt{A}-\alpha)T} - 1}{\mathtt{A} - \alpha} \left(\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 + h\sqrt{n} \right) \sum_{r,s=1}^{n} B_{rs} \\
&\qquad \leq h^2 F \left(\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 + h\sqrt{n} \right),
\end{aligned}$$

$$\begin{aligned}
-\Gamma[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2] \geq & \sum_{j=1}^{n} \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\
& + \frac{1}{2} \sum_{r,s=1}^{n} B_{rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} \left(A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} \right) \sum_{j=1}^{n} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}],
\end{aligned} \tag{3.6}$$

if

$$\begin{aligned}
&-\eta\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2^2 \geq \\
&\quad h(C\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 + D)\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 - E\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2^2 + h^2 F \left(\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 + h\sqrt{n} \right).
\end{aligned}$$

The last inequality is equivalent to

$$E - \eta \geq h \left(C + \frac{D}{\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2} \right) + \frac{h^2 F}{\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2} \left(1 + \frac{h\sqrt{n}}{\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2} \right)$$

and because for every component z_l of $\mathbf{z}$,

$$\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 \geq y_{z_l}$$

for all $i = 1, 2, \ldots, n+1$, (3.6) is satisfied if

$$E - \eta \geq h \left(C + \frac{D}{y_{z_k}} \right) + \frac{h^2 F}{y_{z_k}} \left(1 + \frac{h\sqrt{n}}{y_{z_k}} \right)$$

for some $k \in \{1, 2, \ldots, n\}$.

Now, let $k \in \{1, 2, \ldots, n\}$ be such that

$$h = y_{z_k+1} - y_{z_k}.$$

There are two cases that must be considered.
The case $z_k \geq d$. Then

$$\begin{aligned} h &= y_{z_k+1} - y_{z_k} \\ &\leq \min\left\{ y_{z_k}, \frac{-(Cy_{z_k}+D) + \sqrt{(Cy_{z_k}+D)^2 + 4(E-\eta)y_{z_k}(1+\sqrt{n})F}}{2(1+\sqrt{n})F} \right\}. \end{aligned}$$

Because $h \leq y_{z_k}$,

$$E - \eta \geq h\left(C + \frac{D}{y_{z_k}}\right) + \frac{h^2 F}{y_{z_k}}\left(1 + \frac{h\sqrt{n}}{y_{z_k}}\right)$$

is satisfied if

$$E - \eta \geq h\left(C + \frac{D}{y_{z_k}}\right) + \frac{h^2 F}{y_{z_k}}\left(1 + \sqrt{n}\right).$$

That this inequality is satisfied, follows from

$$h \leq \frac{-(Cy_{z_k}+D) + \sqrt{(Cy_{z_k}+D)^2 + 4(E-\eta)y_{z_k}(1+\sqrt{n})F}}{2(1+\sqrt{n})F}.$$

The case $z_k < d$. Then $h = h^d$ and because at least one component of $\mathbf{z}$ must be larger than or equal to d, (3.6) is satisfied if

$$E - \eta \geq h\left(C + \frac{D}{c}\right) + \frac{h^2 F}{c}\left(1 + \frac{h\sqrt{n}}{c}\right),$$

which follows from

$$h = h^d \leq \frac{-(Cc+D) + \sqrt{(Cc+D)^2 + 4(E-\eta)c(1+\sqrt{n})F}}{2(1+\sqrt{n})F}.$$

■

The proof does not work for $d = 0$, but because $\mathcal{N}$ can be taken arbitrary small no information is lost. Every solution of (1.1) that starts in a compact connected component of a set of the form $\{\mathbf{x} \in \mathcal{U} \big| V^{Lya}(\mathbf{x}) \leq c\} \cup [-y_d, y_d]^n$, where $c > 0$ is an arbitrary positive constant, is driven into the set $\mathcal{N}$ by the dynamics of the system. This means that if $\mathcal{N}$ is contained in the region of attraction, which can always be taken care of by Lyapunov's indirect method, then every such a solution is asymptotically attracted to the equilibrium. Further, it should be noted that there is no need to a-priori calculate the constants d, N, and the grid steps (the y_i). It makes more sense to initially try some values for d ($d = 0$ inclusive), the y_i, and set $N := d + 1$, and then, in case the linear program has no feasible solution, make the y_i smaller and d larger. After having found a feasible solution, one can experiment with $N := d + 2$ and so on. Theorem 3.2 secures the success of such a procedure in a finite number of steps. In the next section we give examples of Lyapunov functions generated with the linear program.

4. Examples.

In this section we give three examples of CPWA Lyapunov functions generated by the linear program. The open source linear solver GLPK ¶ was used to solve the linear programs with the simplex method. We choose

$$\sum_{\mathbf{x} \in \mathcal{G}} (V[\mathbf{x}] - \Psi[\|\mathbf{x}\|_2])$$

¶GNU Linear Programming Kit, available at http://www.gnu.org/software/glpk/glpk.html.

as the (optional) objective in all the examples. It does not optimize the Lyapunov function in any specific way, but it leads to reasonable looking ones.

In these examples, we want to compare the basins of attraction secured by the CPWA Lyapunov functions parameterized by the linear program in Section 2 with the Lyapunov functions from the indirect method of Lyapunov, presented in Section 1. To do this we first derive a general formula for the size of a cubic region, in which the inequality (1.2) holds. Let $r > 0$ be a constant and let b_{ijk}, $i, j, k = 1, 2, \ldots, n$, be upper bounds of the second-order partial derivatives of the components f_i of the function $\mathbf{f}$ from the system (1.1) on the set $[-r, r]^n$,

$$b_{ijk} \geq \sup_{\boldsymbol{\xi} \in [-r,r]^n} \left| \frac{\partial^2 f_i}{\partial x_k \partial x_j}(\boldsymbol{\xi}) \right|.$$

Assume a is a constant such that $0 \leq a \leq r$ and consider the right-hand side of the inequality (1.2) for $\|\mathbf{x}\|_\infty = a$. Then, by Taylor's theorem,

$$\begin{aligned}\|\mathbf{f}(\mathbf{x}) - A\mathbf{x}\|_2 &\leq \|\frac{1}{2} \sum_{i,j,k=1}^{n} x_j x_k b_{ijk} \mathbf{e}_i\|_2 \\ &\leq \frac{a^2}{2} \sqrt{\sum_{i=1}^{n} \left(\sum_{j,k=1}^{n} b_{ijk} \right)^2}.\end{aligned}$$

Because $a = \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2$, it follows that the inequality (1.2) is satisfied for all $\|\mathbf{x}\|_\infty = a$ if

$$a < \frac{1}{\|P\|_2 \sqrt{\sum_{i=1}^{n} \left(\sum_{j,k=1}^{n} b_{ijk} \right)^2}}. \tag{4.7}$$

This means that the inequality (1.2) is satisfied for all $\mathbf{x}$ in the set

$$\mathcal{L} := \left] \frac{-1}{\|P\|_2 \sqrt{\sum_{i=1}^{n} \left(\sum_{j,k=1}^{n} b_{ijk} \right)^2}} \, , \, \frac{1}{\|P\|_2 \sqrt{\sum_{i=1}^{n} \left(\sum_{j,k=1}^{n} b_{ijk} \right)^2}} \right[^n \cap [-r, r]^n,$$

so $\mathcal{L}$ it is a valid domain for the Lyapunov function $V(\mathbf{x}) := \mathbf{x}^T P \mathbf{x}$ for the system (1.1). By these calculations the set $\mathcal{L}' := \{\mathbf{y} \in \mathbb{R}^n \big| \; \mathbf{y}^T P \mathbf{y} < c\}$, where $c > 0$ is (uniquely) chosen such that $\mathcal{L}' \subset \mathcal{L}$ and $\partial\mathcal{L}' \cap \partial\mathcal{L} \neq \emptyset$, is the best lower bound of the region of attraction of the equilibrium this Lyapunov function delivers.

Example 1. The first example is a CPWA Lyapunov function

$$V^{Lya} : [-1.056, 1.056]^2 \longrightarrow \mathbb{R}$$

for (1.1), where

$$\mathbf{f}(x, y) = \begin{pmatrix} -y \\ x - y(1 - x^2 + 0.1x^4) \end{pmatrix}, \tag{4.8}$$

$d = 0$, $N = 8$, and $y_0 = 0$, $y_1 = 0.078$, $y_2 = 0.280$, $y_3 = 0.510$, $y_4 = 0.696$, $y_5 = 0.842$, $y_6 = 0.961$, $y_7 = 1.024$, $y_8 = 1.056$. It is drawn in Figure 1.

The indirect method of Lyapunov delivers $V_{ind}^{Lya}(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$, where

$$P = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

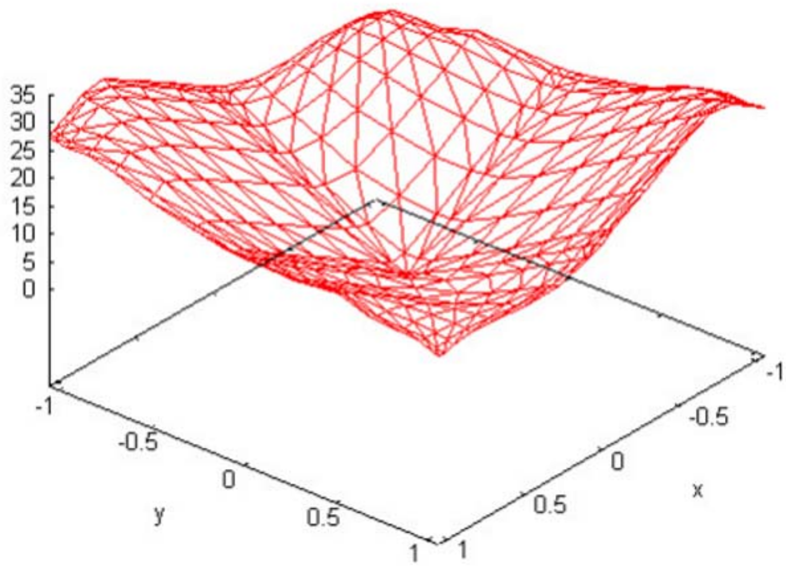

FIGURE 1. CPWA Lyapunov function for (4.8).

The only non-zero second-order partial derivatives of the components of $\mathbf{f}$ are

$$\frac{\partial^2 f_2}{\partial x \partial x}(x,y) = 2y - 1.2x^2 y \quad \text{and} \quad \frac{\partial^2 f_2}{\partial x \partial y}(x,y) = 2x - 0.4x^3,$$

so, with $r = y_8$, the only non-zero b_{ijk} are $b_{211} = 2.112$ and $b_{212} = b_{221} = 1.641$. The formula (4.7) gives

$$a < \frac{1}{\frac{5+\sqrt{5}}{4}\sqrt{(2.112 + 1.641 + 1.641)^2}} = 0.1025.$$

From this, it follows that V_{ind}^{Lya} is a Lyapunov function for the system (1.1) on the domain $[-0.102, 0.102]^2$ and $\{\mathbf{x} \in \mathbb{R}^2 \big| \ \mathbf{x}^T P \mathbf{x} < 0.00867\}$ is its best lower bound of the equilibrium's region of attraction. In Figure 2 the regions of attraction secured by the CPWA Lyapunov function V^{Lya} and the Lyapunov function V_{ind}^{Lya} from Lyapunov's indirect method are compared graphically.

Example 2. The second example is a CPWA Lyapunov function

$$V^{Lya} : [-1.686, 1.686]^2 \longrightarrow \mathbb{R}$$

for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where

$$\mathbf{f}(x,y) = \begin{pmatrix} y \\ -x + \frac{1}{3}x^3 - y \end{pmatrix}, \tag{4.9}$$

$d = 0$, $N = 8$, and $y_0 = 0$, $y_1 = 0.156$, $y_2 = 0.513$, $y_3 = 0.880$, $y_4 = 1.204$, $y_5 = 1.427$, $y_6 = 1.580$, $y_7 = 1.662$, $y_8 = 1.686$. It is drawn in Figure 3. This systems has further equilibria at $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$. It is interesting to compare this Lyapunov function to Figure 3.9 in [3], which contains a phase portrait of

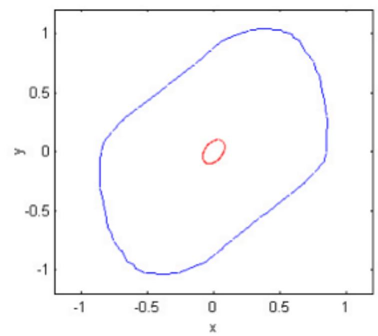

FIGURE 2. Lower bounds of the region of attraction for (4.8) secured by V_{ind}^{Lya} (the small ellipse) and the CPWA Lyapunov function V^{Lya}.

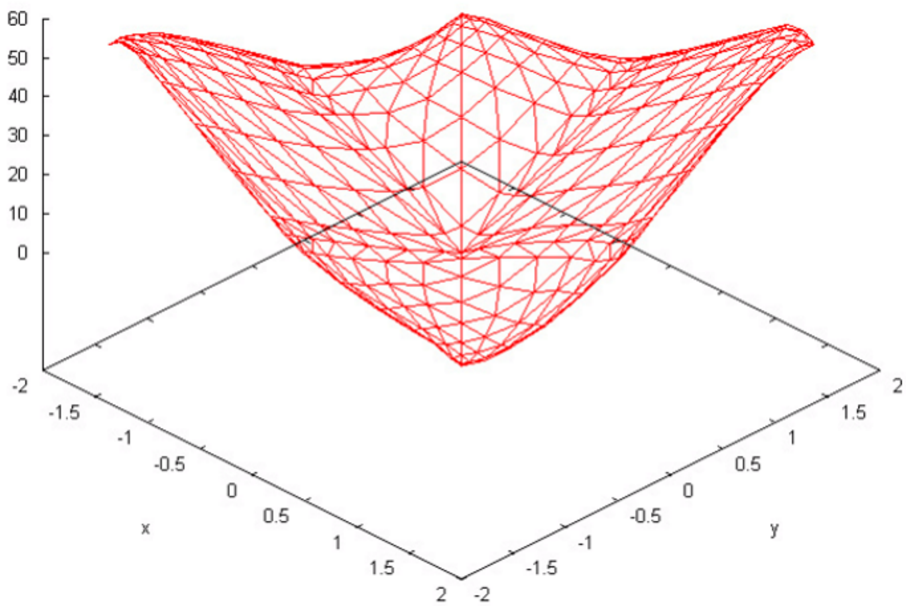

FIGURE 3. CPWA Lyapunov function for (4.9).

the same system. The indirect method of Lyapunov delivers the same Lyapunov function $V_{ind}^{Lya}(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$ as in Example 1. The only non-zero second-order partial derivative of the components of $\mathbf{f}$ is

$$\frac{\partial^2 f_2}{\partial x \partial x}(x, y) = 2y,$$

so the formula (4.7) gives

$$a < \frac{1}{\frac{5+\sqrt{5}}{4}\sqrt{(2a)^2}},$$

i.e., we can set $a = 0.52$, where we improved the estimate given by (4.7) by taking advantage of the simple algebraic forms of the b_{ijk}. ¿From this it follows that V_{ind}^{Lya} is a Lyapunov function for the system (1.1) on the domain $[-0.52, 0.52]^2$ and

$\{\mathbf{x} \in \mathbb{R}^2 \big| \ \mathbf{x}^T P\mathbf{x} < 0.225\}$ is its best lower bound of the equilibrium's region of attraction. In Figure 4 the regions of attraction secured by the CPWA Lyapunov function V^{Lya} and the Lyapunov function V^{Lya}_{ind} from Lyapunov's indirect method are compared graphically.

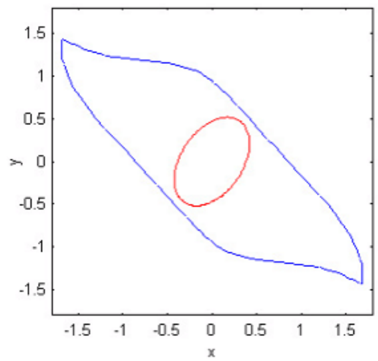

FIGURE 4. Lower bounds of the region of attraction for (4.8) secured by V^{Lya}_{ind} (the ellipse) and the CPWA Lyapunov function V^{Lya}.

Example 3. The last example is a little different to the first two. It is a CPWA Lyapunov function

$$V^{Lya} : \left([-23.967, 23.968]^2 \setminus \mathbb{R}^2_{>0}\right) \bigcup [0, 1.647]^2 \longrightarrow \mathbb{R}$$

for (1.1), where

$$\mathbf{f}(x, y) = \begin{pmatrix} -2x + xy \\ -y + xy \end{pmatrix}, \tag{4.10}$$

$d = 0$, $N = 17$, and $y_0 = 0$, $y_1 = 0.353$, $y_2 = 0.755$, $y_3 = 1.131$, $y_4 = 1.437$, $y_5 = 1.647$, $y_6 = 2.600$, $y_7 = 3.732$, $y_8 = 5.164$, $y_9 = 6.820$, $y_{10} = 8.734$, $y_{11} = 10.903$, $y_{12} = 13.072$, $y_{13} = 15.241$, $y_{14} = 17.410$, $y_{15} = 19.579$, $y_{16} = 21.748$, $y_{17} = 23.917$. It is drawn in Figure 5. It is a simple task to generalize the linear program from Section 2 for such a domain. For the details see [5]. Note, that it is not possible to use the linear program to parameterize a Lyapunov function on $[-23.967, 23.968]^2$ because the system has a further equilibrium (saddle point) at the point $(1, 2)$.

The indirect method of Lyapunov delivers $V^{Lya}_{ind}(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$, where

$$P := \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Instead of using the formula (4.7) to estimate its domain, we refer to Example 3.21 in [3] where a better estimate, $\{\mathbf{x} \in \mathbb{R}^2 \big| \ \mathbf{x}^T P\mathbf{x} \le 0.79\}$ instead of $\{\mathbf{x} \in \mathbb{R}^2 \big| \ \mathbf{x}^T P\mathbf{x} < 0.5\}$ by formula (4.7), is derived on the lower bound of the region of attraction for the equilibrium at the origin for this particular ODE. In Figure 6 the regions of attraction secured by the CPWA Lyapunov function V^{Lya} and the Lyapunov function V^{Lya}_{ind} from Lyapunov's indirect method are compared graphically. It is interesting to compare the lower bound of the region of attraction delivered by the CPWA Lyapunov function in Figure 6 with Figure 3.12 in [3], where the trajectory-reversing method [1] is used to estimate the region of attraction for the same system.

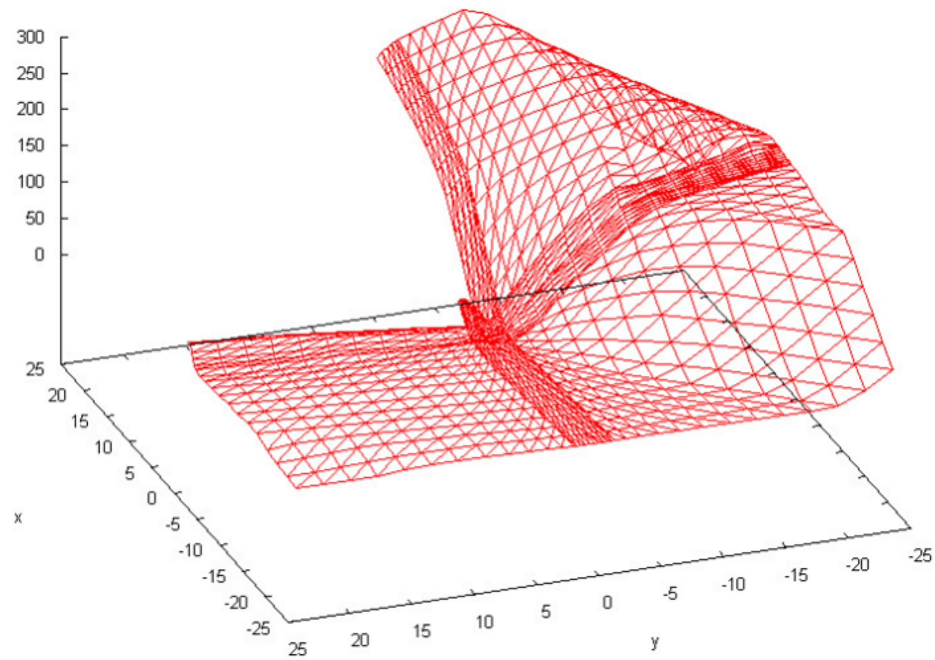

FIGURE 5. CPWA Lyapunov function for (4.10).

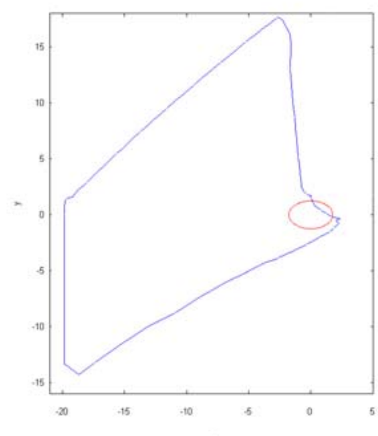

FIGURE 6. Lower bounds of the region of attraction for (4.10) secured by V_{ind}^{Lya} (the small ellipse) and the CPWA Lyapunov function V^{Lya}.

The lower bound delivered by the CPWA Lyapunov function is by far better than the best estimate from the trajectory-reversing method.

5. **Conclusions.** In this paper a constructive converse theorem on exponential stability is proved for class $\mathcal{C}^2$ autonomous ODEs. The Lyapunov function from Proposition 1.2, which is a non-constructive converse theorem, is used to assign values to the variables of the linear programming problem introduced in [5] and [6] and defined in Section 2 here. We prove that the linear constraints of the

linear programming problem are satisfied by these values. It follows that the linear programming problem can be used to generate a Lyapunov function, which can be used to estimate the basin of attraction of the corresponding equilibrium point.

Software, written in the C++ programming language, to generate arbitrary dimensional CPWA Lyapunov functions is available on the internet at the URL http://www.traffic.uni-duisburg.de/∼hafstein. It was used for the examples presented in this work. The complexity of this method to generate Lyapunov functions via linear programming is determined by the complexity of finding a feasible solution of the associated linear programming problem. We consider the complexity as a function of the number of elements $|\mathcal{G}|$ in $\mathcal{G}$, i.e., the number of the points, at which we calculate the value of the Lyapunov function (see Section 2), and the dimension n of the domain of $\mathbf{f}$ from (1.1). It is easy to see that for every point in $\mathcal{G}$ the number of variables introduced to the linear programming problem is $\mathcal{O}(n)$ (the $C[\{\mathbf{x}, \mathbf{y}\}]$) and the number of constraints introduced is $\mathcal{O}(n!)$ (LC4). Because we have to solve the linear program (2.4) to find a feasible solution to our original problem, we are interested in the complexity of solving (2.3) when C is a $\mathcal{O}(|\mathcal{G}|n!) \times \mathcal{O}(|\mathcal{G}|n!)$-matrix. The complexity of solving linear programming problems is not a closed problem. However, the average running time of our problem should be $\mathcal{O}((|\mathcal{G}|n!)^4)$ when solved with the simplex method according to [8]. The CPWA Lyapunov functions in the examples presented here were generated in a few seconds (examples 1 and 2) and approximately 2 minutes (Example 3) on a PC with a 2 GHz CPU.

It is the belief of the author, that this general method to numerically generate Lyapunov functions for (nonlinear) ODEs might lead to advantages in the stability theory of ODEs, the stability theory of continuous dynamical systems, and control theory.

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