Lyapunov Functions for Interconnected Systems

Fabian Wirth

Institute of Mathematics, University of Würzburg

Workshop on Algorithms for Dynamical Systems and Lyapunov Functions

joint work with:
Sergey Dashkovskiy, Björn Rüffer, Roman Geiselhart
ISS and Lyapunov functions
  Comparison functions

Input-to-state stability and interconnections
  Large-scale interconnections

Lyapunov functions
  The small-gain condition
  $\Omega$
  ISS-Lyapunov function construction

Simplicial fixed point algorithms
  The Algorithm
  Convergence
  Example: Quasi-monotone systems
Stability of networks of ISS subsystems

\[ \dot{x}_1 = f_1(x_1, \ldots) \]

\[ \dot{x}_2 = f_2(x_1, \ldots) \]

\[ \dot{x}_3 = f_3(x_1, \ldots) \]

\[ \dot{x}_n = f_n(x_1, \ldots) \]

What are conditions for input-to-state stability of such a network?

How can we construct Lyapunov functions?
Input-to-state stability (ISS) — Lyapunov version

\[
\begin{align*}
\dot{x} &= f(x, u) \\
\Sigma : \dot{x} &= f(x, u)
\end{align*}
\]

Definition

A locally Lipschitz continuous function \( V : \mathbb{R}^N \to \mathbb{R}_+ \) is an **ISS Lyapunov function** for \( \Sigma \) if there exist \( \psi_1, \psi_2, \gamma \in \mathcal{K}_\infty \) and a positive definite function \( \alpha \) such that

\[
\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|)
\]

and for a.a. \( x \in \mathbb{R}^N \)

\[
V(x) \geq \gamma(\|u\|) \implies \nabla V(x) \cdot f(x, u) \leq -\alpha(V(x)).
\]
Input-to-state stability (ISS) — Lyapunov version

\[ \{ x : V(x) \leq c \} \]

\[ \{ x : V(x) \leq \gamma(\|u\|) \} \]

\[ \dot{V} < 0 \]

\[ V > \gamma(\|u\|) \implies \dot{V} < 0 \]
Interconnections: First steps

Consider a simple feedback loop

\[ \Sigma_1 : \dot{x}_1 = f_1(x_1, x_2, u) \]
\[ \Sigma_2 : \dot{x}_2 = f_2(x_1, x_2, u) \]

where

\[ f_i : \mathbb{R}^{N_1+N_2+N_u} \rightarrow \mathbb{R}^{N_i} \]

with

\[ V_1(x_1) > \max \left\{ \gamma_{12}(V_2(x_2)), \gamma_u(\|u\|) \right\} \Rightarrow \dot{V}_1 < -\alpha_1(\|x_1\|) \]
\[ V_2(x_2) > \max \left\{ \gamma_{21}(V_1(x_1)), \gamma_u(\|u\|) \right\} \Rightarrow \dot{V}_2 < -\alpha_2(\|x_2\|) \]
Two systems in feedback interconnection

**Theorem [Jiang, Mareels, Wang 1996]**
If there exist $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2$ such that

$$\gamma_{12} \circ \gamma_{21} \leq \text{id},$$

then

$$\dot{x} = f(x, u)$$

with

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad f(x, u) = \begin{pmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{pmatrix}$$

is input-to-state stable from $u$ to $(x_1, x_2)$.
See also [Jiang, Teel, Praly (1994)] [Grüne 2002].
Two systems in feedback interconnection

The small gain condition is

$$\gamma_{12} \circ \gamma_{21} \leq \text{id},$$

Introduce the matrix

$$\Gamma := \begin{pmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{pmatrix}.$$

$\Gamma$ may be interpreted as an operator

$$\Gamma : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+ \quad \left( \begin{array}{c} s_1 \\ s_2 \end{array} \right) \mapsto \left( \begin{array}{c} \gamma_{12}(s_2) \\ \gamma_{21}(s_1) \end{array} \right).$$

With this interpretation, the small gain condition is equivalent to

$$\Gamma(s) \nless s \quad \text{for all } s \in \mathbb{R}^2_+, s \neq 0.$$
The small gain condition

The condition for the existence of Lyapunov functions will turn out to be

\[ \Gamma(s) \not\succ s, \quad \forall s \in \mathbb{R}^2_+ \setminus \{0\} \]
In two dimensions

$$\Gamma(s) \preceq s \quad \text{means} \quad \Gamma(s) = \begin{bmatrix} \gamma_{12}(s_2) \\ \gamma_{21}(s_1) \end{bmatrix} \preceq \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$
In two dimensions

\[ \Gamma(s) \not\geq s \text{ means } \Gamma(s) = \begin{bmatrix} \gamma_{12}(s_2) \\ \gamma_{21}(s_1) \end{bmatrix} \not\geq \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \]
In two dimensions

\[ \Gamma(s) \not\preceq s \quad \text{means} \quad \Gamma(s) = \begin{bmatrix} \gamma_{12}(s_2) \\ \gamma_{21}(s_1) \end{bmatrix} \not\preceq \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \]
By the small gain conditions for some $s \in \mathbb{R}^2^+$

$$\Gamma(s) = \begin{bmatrix} \gamma_{12}(s_2) \\ \gamma_{21}(s_1) \end{bmatrix} \ll \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$
By the small gain conditions for some \( s \in \mathbb{R}^2_+ \)

\[
\Gamma(s) = \begin{bmatrix}
\gamma_{12}(s_2) \\
\gamma_{21}(s_1)
\end{bmatrix} \ll \begin{bmatrix}
s_1 \\
 s_2
\end{bmatrix}
\]

Construction of the Lyapunov function: Find a strictly increasing, unbounded path \( \sigma \) in the orange region. Set

\[
V(x) = \max\{\sigma_1^{-1}(V_1(x_1)), \sigma_2^{-1}(V_2(x_2))\} 
\]
Large-scale interconnections

\[ \Sigma_i : \dot{x}_i = f_i(x_1, \ldots, x_n, u), \]

with \( f_i : \mathbb{R}^{\sum N_j + M} \rightarrow \mathbb{R}^{N_i} \), such that each system satisfies

\[ V_i(x_i) \geq \mu \{ \gamma_{ij}(V_j(x_j)), \gamma_i(\|u\|) \} \implies \nabla V_i(x_i) \cdot f_i(x, u) \leq -\alpha_i(V_i(x_i)) \]

where \( \gamma_{ij}, \gamma_{ij} \in \mathcal{K}_\infty \) or constantly zero.
Monotone Aggregation Functions

A continuous function $\mu : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$ is called a monotone aggregation function if the following two properties hold:

1. **Positive definiteness**: $\mu(0) = 0$, $\mu(s) > 0$ for all $s \in \mathbb{R}^n_+, s \neq 0$;
2. **Monotonicity**: if $x < y$, then $\mu(x) < \mu(y)$. 
The induced monotone operator

The interconnection structure given by the gain functions $\gamma_{ij}$ defines a matrix

$$\Gamma = (\gamma_{ij}) \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}.$$  

Such a matrix together with monotone aggregation functions $\mu_1, \ldots, \mu_n$ define a monotone operator $\Gamma_\mu : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\Gamma_\mu(s) = \left(\mu_1(\gamma_{1j}(s_j)) \quad \ldots \quad \mu_n(\gamma_{nj}(s_j))\right)^T, \quad s \in \mathbb{R}_+^n.$$  

Here and in the following we assume $\gamma_{ii} \equiv 0$ for all $i$. 
The small-gain condition

The condition for the existence of Lyapunov functions is the existence of a $D = \text{diag}(\text{id} + \alpha_i), \alpha_i \in \mathcal{K}_\infty$ such that

$$D \circ \Gamma_\mu(s) \not\geq s, \quad \forall s \in \mathbb{R}_+^n \setminus \{0\}$$
The small-gain condition

In the rest of the talk $D$ will be ignored.

The condition for the existence of Lyapunov functions is

$$\Gamma_\mu(s) \nleq s, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}$$
The small-gain condition

The condition for the existence of Lyapunov functions is

\[ \Gamma_{\mu}(s) \not\geq s, \quad \forall s \in \mathbb{R}_+^n \setminus \{0\} \]

Why is this condition interesting?
We want again a path in the set \( \{ \Gamma_{\mu}(s) \ll s \} \).
Consider the sets

\[ \Omega_i := \{ s \in \mathbb{R}_+^n \mid \Gamma(s)_i < s_i \} \quad i = 1, \ldots, n \]

If \( \Gamma_{\mu}(s) \not\geq s \) then \( \{0\} \cup \bigcup_i \Omega_i = \mathbb{R}_+^n \) and

\[ \Omega := \bigcap_i \Omega_i \neq \emptyset \]
The small-gain condition

**Theorem**

(Knaster-Kuratowski-Mazurkiewicz 1929)

Let $S$ be an $n - 1$-dimensional simplex with extremal points $e_1, \ldots, e_n$. Let $A_1, \ldots, A_n$ be open sets in $S$ such that for all subsets $\emptyset \neq J \subset \{1, \ldots, n\}$ we have

$$\text{conv}\{e_j \mid j \in J\} \subset \bigcup_{j \in J} A_j,$$

then

$$\bigcap_{i=1}^{n} A_i \neq \emptyset.$$
Properties of $\Omega$

\[ \Omega = \{ s \in \mathbb{R}_+^n : (\Gamma_\mu(s))_i < s_i \ \forall i \} \]

- Knaster-Kuratowski-Masurkiewicz-Theorem ensures that decay set intersected with $S_r$ is nonempty for any $r > 0$.
- $\Gamma_\mu$ is strictly decreasing on $\Omega$.
- The decay set contains a backward invariant set that is unbounded in every component and pathwise connected to the origin.
The final construction

**Theorem** (Dashkovskiy, Rüffer, W. 2010)

If the small-gain condition

\[ D \circ \Gamma(s) \not\geq s \quad \forall s \in \mathbb{R}_+^n \setminus \{0\} \]

is satisfied, then an unbounded sufficiently regular path \( \sigma \) exists in \( \Omega \) and an ISS Lyapunov function for \( \Sigma \) is given by

\[ V(x) = \max_{i=1,...,n} \sigma_i^{-1}(V_i(x_i)). \quad (1) \]
Properties of $\Omega$

Given a point $s \in \Omega$, the $\Omega$-path can be constructed locally by iteration and linear interpolation.

The only problem is to find $s \in \Omega$, i.e., a point such that

$$\Gamma_\mu(s) \ll s$$

$$\Omega = \{ s \in \mathbb{R}_+^n : (\Gamma(s))_i < s_i \ \forall i \}$$
Where are the algorithms?

For a numerical procedure we are left with the problem of finding a decrease point \( s \) for a monotone operator \( \Gamma : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+ \), i.e. a point satisfying

\[
\Gamma(s) \ll s
\]

given that a small-gain condition holds.
Step 1: Change the problem to a fixed-point problem

Consider

$$
\phi(v) = \Gamma_\mu(v) \left( 1 + \min \left\{ 0, \frac{\kappa_\Gamma - 2\|v\|}{\|v\| + \kappa_0} \right\} \right) + \max \left\{ 0, \kappa_h - 2\|v\| \right\} e.
$$

Here $\kappa_0 > 0$, $\kappa_\Gamma > \kappa_h > 0$ and $e := \sum_{i=1}^{N} e_i$. 

The diagram shows the function $\phi(v)$ plotted against $r$ with the following key points:

- $\kappa_h$
- $1 + \min\{0, \frac{\kappa_\Gamma - 2r}{r + \kappa_0}\}$
- $\max\{0, \kappa_h - 2r\}$

The graph illustrates the behavior of the function for different values of $r$. The points indicate the intersections and critical values of the function.
Step 1: Change the problem to a fixed-point problem

Consider

$$\phi(v) = \Gamma_{\mu}(v) \left(1 + \min \left\{ 0, \frac{\kappa_{\Gamma} - 2\|v\|}{\|v\| + \kappa_0} \right\} \right) + \max \left\{ 0, \kappa_h - 2\|v\| \right\} e.$$  

Lemma

If $\phi$ has a fixed point $s^*$, then this is a decay point of $\Gamma_{\mu}$. We can regulate the location of the fixed point by choice of the constants.
Step 2: Use homotopy to find a fixed point

**Definition**

Let $f, g : C \to D$ be continuous. We call $f, g$ homotopic, if there exists a continuous mapping

$$\vartheta : C \times [0, 1] \to D$$

with

$$\vartheta(s, 0) = f(s) \quad \text{and} \quad \vartheta(s, 1) = g(s) \quad \forall \ s \in C$$
Step 2: Use homotopy to find a fixed point

- Fix a triangulation of $\mathbb{R}^n_+ \times [0, 1]$ with corner points only $\mathbb{R}^n_+ \times \{0, 1\}$. Choose some $c \in \mathbb{R}^n_+$ such that $(c, 0)$ lies in the interior of a facet in $\mathbb{R}^n_+ \times \{0\}$.
- Consider the homotopy

$$\vartheta(v, t) = (1 - t)c + t\phi(v)$$

- For $t = 0$ the unique fixed point is obviously $c$.
- Given a sequence $t_k \to 1$ there are fixed points $s_{t_k}$ of $\vartheta(\cdot, t_k)$.
- The cluster points of the set of fixed points $s_{t_k}$ of are just the fixed points of $\vartheta(\cdot, 1) = \phi$. 
The Simplicial Fixed Point Algorithm

**The Algorithm**

**Step (0)** \( \tau^0 \): unique \( N \)-simplex containing \((c, 0)\).  
\( \eta^0 \): the unique \((N + 1)\)-simplex which has \( \tau^0 \) as its facet.  
\( y^+ \): the vertex of \( \eta^0 \) that is not a vertex of \( \tau^0 \).

**Step (1)** \( k \rightarrow k + 1 \)  
Compute unique “complete” facet \( \tau^{k+1} \) of \( \eta^k \) not equal to \( \tau^k \).  
This is possible, unless \( \tau^k \subset \mathbb{R}^N \times \{1\} \).

**Step (2)**  
Find unique simplex \( \eta^{k+1} \) sharing the facet \( \tau^{k+1} \) with \( \eta^k \) and let \( y^+ \) be the vertex \( \eta^{k+1} \) not being a vertex of \( \tau^{k+1} \). Set \( k = k + 1 \) and return to Step (1).
Convergence

**Theorem** (Geiselhart, W. 2012)
Let $\Gamma$ be irreducible and $\Gamma_\mu$ satisfy the small gain condition. Then there exists an explicit formula for the mesh size below which the simplicial algorithm converges in finitely many steps to a decay point of $\Gamma_\mu$. 
Test example

Rüffer, Dower, Ito, 2010: System construction: $P \in \mathbb{R}^{n \times n}_+, \rho(P) < 1$, $A := -I + P$.

$$S(v)_i = \begin{cases} e^{v_i-1} & \text{if } v_i > 1 \\ v_i & \text{if } v_i \in [-1, 1] \\ -e^{-v_i-1} & \text{if } v_i < -1 \end{cases}.$$  

Compute decay points for

$$\dot{v} = S'(S^{-1}(v))AS^{-1}(v) =: g(v).$$
Example: Quasi-monotone systems

<table>
<thead>
<tr>
<th>N</th>
<th>run time</th>
<th># iterations</th>
<th>simulations</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.0277s</td>
<td>20.9</td>
<td>100</td>
</tr>
<tr>
<td>10</td>
<td>0.0415s</td>
<td>34.5</td>
<td>100</td>
</tr>
<tr>
<td>15</td>
<td>0.0618s</td>
<td>72.3</td>
<td>100</td>
</tr>
<tr>
<td>25</td>
<td>0.1710s</td>
<td>187.8</td>
<td>100</td>
</tr>
<tr>
<td>50</td>
<td>1.180s</td>
<td>688.4</td>
<td>100</td>
</tr>
<tr>
<td>100</td>
<td>13.22s</td>
<td>2711.9</td>
<td>50</td>
</tr>
<tr>
<td>150</td>
<td>78.35s</td>
<td>6614.3</td>
<td>10</td>
</tr>
<tr>
<td>200</td>
<td>273.6s</td>
<td>11243.8</td>
<td>10</td>
</tr>
</tbody>
</table>
We have seen a general procedure for the construction of an (ISS) Lyapunov function for large-scale networks of interconnected systems. Our result extends the existing result of Jiang, Mareels, Wang (1996) to arbitrarily many systems. Local construction of Lyapunov functions can be performed numerically. An algorithm extending an approach by Merrill can be used to determine a point in $\Omega$. The rest of $\sigma$ is the obtained by iterating $\Gamma$. 


Conclusions