A differential Lyapunov framework for contraction analysis

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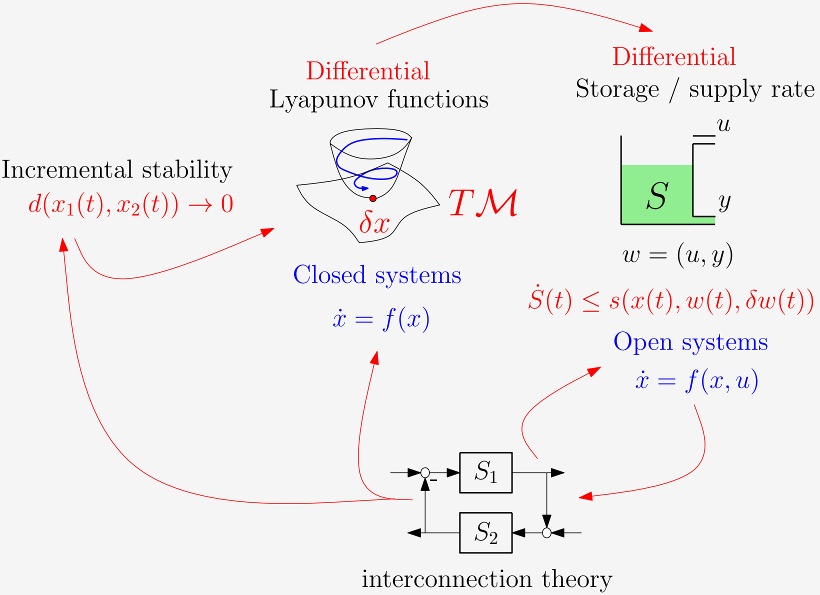
Reykjavik, July 19th, 2013

Outline

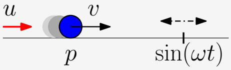


Differential
Lyapunov functions
TM
δx
Closed systems
$\dot{x} = f(x)$
Differential
Storage / supply rate
u
S
y
$w = (u, y)$
$\dot{S}(t) \leq s(x(t), w(t), \delta w(t))$
Open systems
$\dot{x} = f(x, u)$
Incremental stability
$d(x_1(t), x_2(t)) \rightarrow 0$
S_1
S_2
interconnection theory

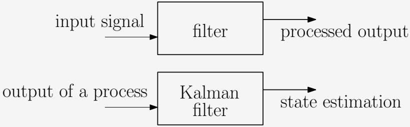
Why? regulation, observer design, synchronization

Regulation:
from $p(t) \to p_e$ to $p(t) \to \sin(\omega t)$

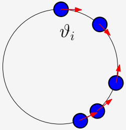
$$\begin{cases} \dot{p} & = & v \\ \dot{v} & = & u \end{cases}$$



Filtering:
filters forget initial conditions



Synchronization:



Linear systems: incremental stability $\equiv$ stability

$$\dot{x} = Ax \qquad \dot{z} = Az$$

$$e := x - z$$

$$\dot{e} = Ae$$

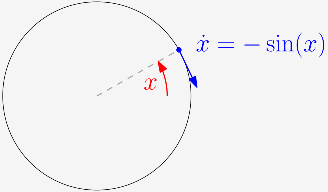
if $e \to 0$ then $d(x, z) \to 0$

Use Lyapunov on the error dynamics e:

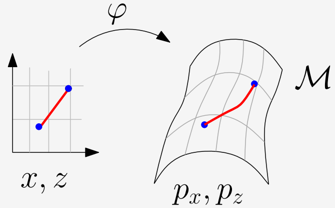
$$V = (x - z)^T P(x - z) = e^T Pe$$

But the error in nonlinear spaces/dynamics...

$$\dot{x} = -\sin(x)\ , \quad \dot{z} = -\sin(z)$$



$$\begin{aligned} e &:= x - z \\ \dot{e} &= -\sin(x) + \sin(z) \\ &\neq f(e) \end{aligned}$$



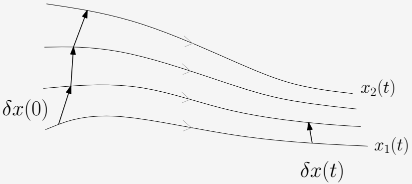
$$e = x - z?\ , \quad V(x, z)$$

The variational system replaces the error dynamics

$\delta x := x - z$ for $z \to x$ (infinitesimal variation).

$$\dot{\delta x} \text{ or } \delta x^+ \; = \; f(x + \delta x) - f(x) \simeq \frac{\partial f(x)}{\partial x}\delta x$$

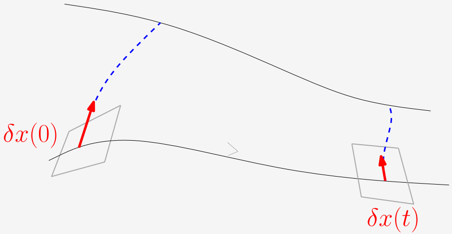
If $\delta x \to 0$ then incremental stability



Time-varying linear system and tangent vectors (intrinsic!)

$$\begin{cases} \dot{x} &= f(x) \\ \dot{\delta x} &= \frac{\partial f(x)}{\partial x}\delta x \end{cases} \qquad \begin{cases} x^+ &= f(x) \\ \delta x^+ &= \frac{\partial f(x)}{\partial x}\delta x \end{cases}$$

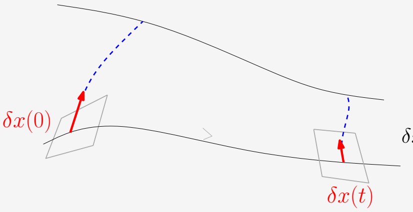
- Time-varying linear dynamics of the "error" δx along any given solution $x(t)$, since $\dot{\delta x} = \frac{\partial f(x(t))}{\partial x}\delta x$.
- $\delta x(t)$ is a tangent vector in the tangent space $T_{x(t)}\mathcal{M}$

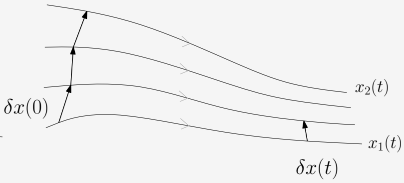


From δx to the distance between solutions by integration

$$\begin{cases} \dot{x} = f(x) \\ \dot{\delta x} = \frac{\partial f(x)}{\partial x}\delta x \end{cases} \qquad \begin{cases} x^+ = f(x) \\ \delta x^+ = \frac{\partial f(x)}{\partial x}\delta x \end{cases}$$

- $x_s(0) = \gamma(s)$
- $x_s(t) =$ time evolution of γ along the flow
- $\delta x_s(t) := \frac{\partial x_s(t)}{\partial s}$
- $\text{length}(t) = \int |\delta x_s(t)| ds$





We need two ingredients:

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) \\ \dot{\delta x} & = & \frac{\partial f(x)}{\partial x}\delta x \end{array} \right. \qquad \left\{ \begin{array}{rcl} x^+ & = & f(x) \\ \delta x^+ & = & \frac{\partial f(x)}{\partial x}\delta x \end{array} \right.$$

A metric for δx

A way to infer $\delta x(t) \to 0$

A metric for δx: Finsler structures

How to measure $\delta x \to 0$?

$$\delta x \in T_x\mathcal{M} \qquad\qquad |\delta x|_x \text{ Finsler structure}$$

- $\forall \delta x \in T_x\mathcal{M}, \qquad |\lambda \delta x|_x = \lambda |\delta x|_x$
- $\forall \delta x_1, \delta x_2 \in T_x\mathcal{M} \qquad |\delta x_1 + \delta x_2|_x \leq |\delta x_1|_x + |\delta x_2|_x$
- Example: $\sqrt{\delta x^T P(x) \delta x} \qquad P(x) > 0$

A way to infer $\delta x(t) \to 0$: Lyapunov theory in $T\mathcal{M}$

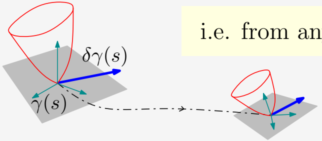
$$\begin{cases} \dot{x} &= f(x) \\ \dot{\delta x} &= \frac{\partial f(x)}{\partial x}\delta x \end{cases}$$

$$c_1|\delta x|_x^p \leq V(x,\delta x) \leq c_2|\delta x|_x^p$$

$$\dot{V} \leq -\lambda V$$

$$\dot{V} = \frac{\partial V(x,\delta x)}{\partial x} f(x) + \frac{\partial V(x,\delta x)}{\partial x}\frac{\partial f(x)}{\partial x}\delta x$$

$$|\delta x(t)|_{x(t)} \leq k e^{-\lambda t}|\delta x(0)|_{x(0)}$$



i.e. from any $(x_0, \delta x_0)$

A way to infer $\delta x(t) \to 0$: Lyapunov theory in $T\mathcal{M}$

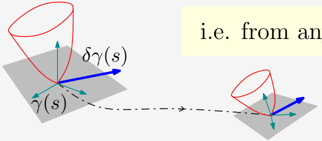
$$\begin{cases} x^+ &= f(x) \\ \delta x^+ &= \frac{\partial f(x)}{\partial x}\delta x \end{cases}$$

$$c_1|\delta x|_x^p \leq V(x,\delta x) \leq c_2|\delta x|_x^p$$

$$V^+ \leq \lambda V$$

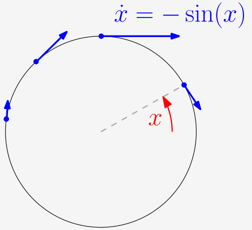
$$V^+ = V(f(x), \tfrac{\partial f(x)}{\partial x}\delta x)$$

$$|\delta x(t)|_{x(t)} \leq k\lambda^t|\delta x(0)|_{x(0)}$$



i.e. from any $(x_0, \delta x_0)$

Example: $\dot{x} = -\sin(x)$, contraction in $(-\pi, \pi)$



$$\begin{cases} \dot{x} &= -\sin(x) \\ \dot{\delta x} &= -\cos(x)\delta x \end{cases}$$

$$V(x, \delta x) = \frac{1}{2}\delta x^2$$

$$\dot{V}(x, \delta x) = -\cos(x)\delta x^2$$

$$\Rightarrow \text{ contraction in } (-\frac{\pi}{2}, \frac{\pi}{2})$$

same as Lyapunov on $e := x - z$

$$V(x, \delta x) = \frac{\delta x^2}{1 + \cos(x)}$$

$$\dot{V}(x, \delta x) = -\delta x^2$$

$$\Rightarrow \text{ contraction in } (-\pi, \pi)$$

What about contraction and inputs?

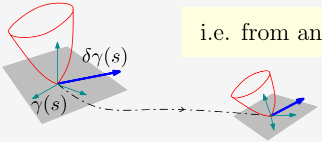
Nonlinear filters: uniform contraction w.r.t. input signal

$$\begin{cases} \dot{x} = f(x) \\ \dot{\delta x} = \frac{\partial f(x)}{\partial x}\delta x \end{cases} \rightarrow \begin{cases} \dot{x} = f(x,t) \\ \dot{\delta x} = \frac{\partial f(x,t)}{\partial x}\delta x \end{cases} \rightarrow \begin{cases} \dot{x} = f(x,u(t)) \\ \dot{\delta x} = \frac{\partial f(x,u(t))}{\partial x}\delta x \end{cases}$$

$$c_1|\delta x|_x^p \leq V(x,\delta x) \leq c_2|\delta x|_x^p \quad \rightarrow \quad c_1|\delta x|_x^p \leq V(t,x,\delta x) \leq c_2|\delta x|_x^p$$

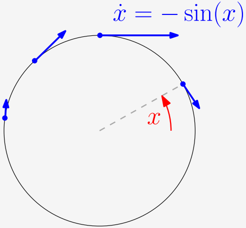
$$\dot{V} \leq -\lambda V$$

$$|\delta x(t)|_{x(t)} \leq k e^{-\lambda t}|\delta x(0)|_{x(0)}$$



i.e. from any $(x_0, \delta x_0)$

Example: $\dot{x} = -\sin(x) + u$



$$\begin{cases} \dot{x} &= -\sin(x) + u \\ \dot{\delta x} &= -\cos(x)\delta x \end{cases}$$

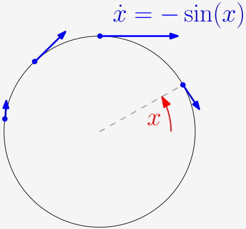
$$V(x, \delta x) = \frac{1}{2}\delta x^2$$

$$\dot{V}(x, \delta x) = -\cos(x)\delta x^2$$

uniform contraction in $(-\frac{\pi}{2}, \frac{\pi}{2})$

Solutions converge towards each other provided that $u(t)$ preserves the invariance of the contraction region.

Example: $\dot{x} = -\sin(x) + u$



$$\begin{cases} \dot{x} & = & -\sin(x) + u \\ \dot{\delta x} & = & -\cos(x)\delta x \end{cases}$$

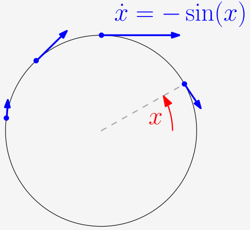
$$V(x, \delta x) = \frac{\delta x^2}{1 + \cos(x)}$$

$$\dot{V}(x, \delta x) = -\delta x^2 + \frac{\delta x^2 \sin(x) u}{(1 + \cos(x))^2}$$

no uniform contraction in $(-\pi, \pi)$

Solutions converge towards each other only for suitable inputs $u(t)$

Example: $\dot{x} = -\sin(x) + \cos(\frac{x}{2})u$



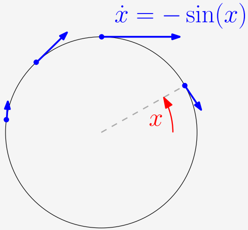
$$\begin{cases} \dot{x} &= -\sin(x) + \cos(\frac{x}{2})u \\ \dot{\delta x} &= \left[-\cos(x) - \frac{1}{2}\sin(\frac{x}{2})u\right]\delta x \end{cases}$$

$$V(x, \delta x) = \frac{\delta x^2}{1 + \cos(x)}$$

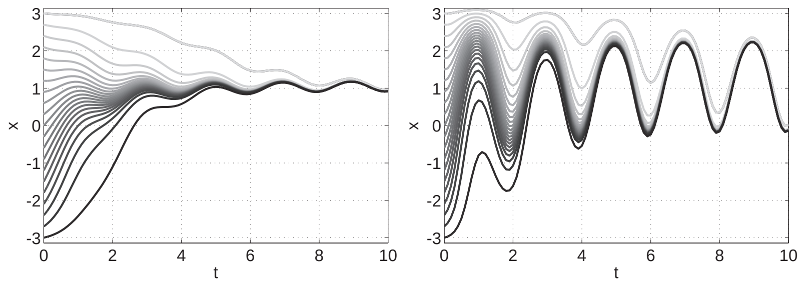
$$\dot{V}(x, \delta x) = -\delta x^2 + \delta x^2 \, \underbrace{u\left(-\frac{\cos(\frac{x}{2})}{(1+\cos(x))^2} - \frac{\sin(\frac{x}{2})}{2(1+\cos(x))}\right)}_{=0!}$$

uniform contraction in $(-\pi, \pi)$

Example: $\dot{x} = -\sin(x) + \cos(\frac{x}{2})u$



$$\begin{cases} \dot{x} = -\sin(x) + \cos(\frac{x}{2})u \\ \dot{\delta x} = \left[-\cos(x) - \frac{1}{2}\sin(\frac{x}{2})u\right]\delta x \end{cases}$$



What we gain from a differential Lyapunov theory?

Lyapunov (stability) vs Finsler-Lyapunov (incr. stability)

- $\dot{x} = f(x)$
- $\dot{X} = F(X)$
  $X := \begin{bmatrix} x \\ \delta x \end{bmatrix}$, $F(X) := \begin{bmatrix} f(x) \\ \frac{\partial f(x)}{\partial x}\delta x \end{bmatrix}$

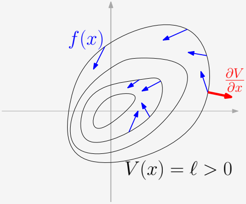
- Find Lyapunov $V(x)$

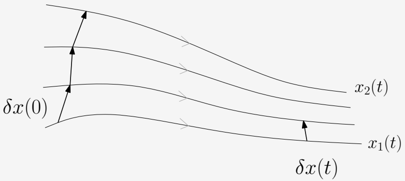
$$\frac{\partial V(x)}{\partial x}^T f(x) \leq -\lambda V(x)$$

- Find Finsler-Lyapunov $V(x, \delta x)$

$$\frac{\partial V(X)}{\partial X}^T F(X) \leq -\lambda V(X)$$

- Then, $x(t) \rightarrow x_e$.
- Then, $\delta x(t) \rightarrow 0$.





Finsler-Lyapunov unifies many contraction approaches

- $V = |\delta x|$
  $|A| := \sup_{x=1} |Ax|$, $\mu(A) := \lim_{h\to 0^+} \frac{|I+hA|-1}{h}$, $\exists c > 0,\ \forall x$

$$\mu(\frac{\partial f(x)}{\partial x}) < -c$$

- $V = \delta x^T P \delta x$
  $\exists P = P^T > 0, \exists Q = Q^T > 0,\ \forall x$

$$\frac{\partial f(x)}{\partial x}^T P + P\frac{\partial f(x)}{\partial x} < -Q$$

- $V = \delta x^T M(x) \delta x$,
  $\exists M(x) = M(x)^T > 0,\ \forall x$

$$\frac{\partial f(x)}{\partial x}^T M(x) + M(x)\frac{\partial f(x)}{\partial x} + [\dot{M}(x)] < -cM(x)$$

IS, IAS, IES

$$X := \left[\begin{smallmatrix} x \\ \delta x \end{smallmatrix}\right] \ , \ F(X) := \left[\begin{array}{c} f(x) \\ J(x)\delta x \end{array}\right] \ , \ \dot{X} = F(X)$$

Let $V : T\mathcal{M} \to \mathbb{R}_{\geq 0}$ be a Finsler-Lyapunov function s.t. $\forall X$

- $\frac{\partial V(X)}{\partial X}^T F(X) \leq 0$
  $\Rightarrow$ Incremental Stability.
- $\frac{\partial V(X)}{\partial X}^T F(X) \leq -\alpha(V(X)) \quad \alpha \in \mathcal{K}$
  $\Rightarrow$ Incremental Asymptotic Stability.
  $\lim_{t\to\infty} d(\varphi(t,x_0),\varphi(t,x_1)) \leq \lim_{t\to\infty} \int_0^1 V(...)ds = 0$
- $\frac{\partial V(X)}{\partial X}^T F(X) \leq -cV(X) \quad c > 0$
  $\Rightarrow$ Incremental Exponential Stability.
  $d(\varphi(t,x_0),\varphi(t,x_1)) \leq \int_0^1 e^{-ct} ... \, ds = e^{-ct} \int_0^1 ... \, ds$

Advantages: relaxations, design, I/O

- **Relaxations (LaSalle):**

  from $\dot{V} \leq -\alpha(V)$ to $\dot{V} \leq -W(x, \delta x) \leq 0$

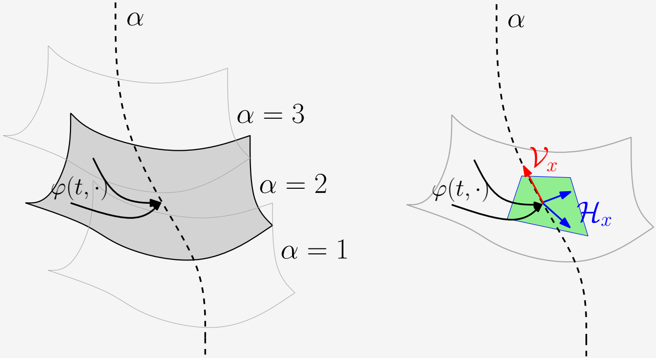
- **Design**:

  $u = k(x)$ such that $\dot{V} \leq -\alpha(V)$

- open systems...

What about systems that contract only partially?

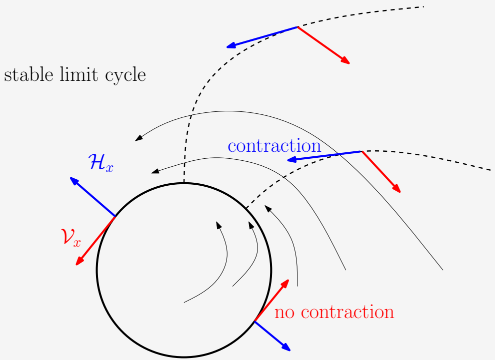
Horizontal contraction - symmetries or first integrals



$$V(x,\delta x) = V(x,\delta x_h) \qquad \delta x = \delta x_h + \delta x_v$$

(*Cooperative irreducibile systems of ordinary differential equations with first integral*, Mierczynski)

Horizontal contraction - attractors - Bendixon's criterion

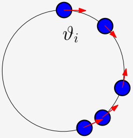


no limit cycles if the system contracts in the direction of the vector field $f(x)$ - connections with Bendixon-like criteria.

Example: phase synchronization

(*Stabilization of planar collective motion: All-to-all communication*, Sepulchre, Paley, and Leonard)

$$\dot{\theta}_k = \frac{1}{n}\sum_{j=1}^{n}\sin(\theta_j - \theta_k)$$

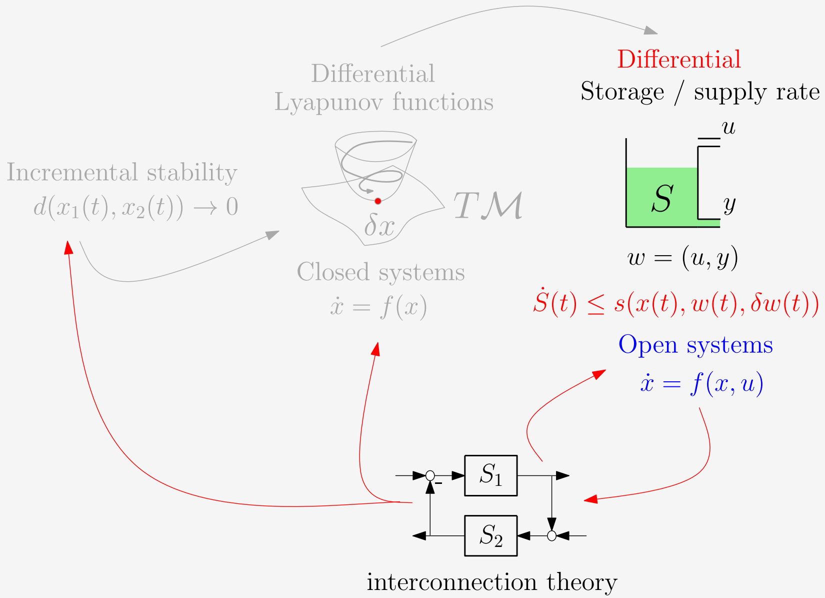


$$\dot{\theta} = \underbrace{\frac{1}{n}\begin{bmatrix} 0 & s_{21} & \cdots & s_{n1} \\ s_{12} & 0 & \cdots & s_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & 0 \end{bmatrix}}_{=:\mathbf{S}(\theta)}\mathbf{1} \quad \Rightarrow \quad \mathbf{S}(\theta) = \mathbf{S}(\theta + \alpha\mathbf{1})$$

Contraction in any forward invariant compact set $\mathcal{C}$ not containing unstable and saddle points

A glimpse into open systems

Open systems - differential dissipativity



Variational open systems

$$\begin{cases} \dot{x} & = & f(x,u) \\ \dot{y} & = & h(x,u) \end{cases}$$

$$\Rightarrow$$

$$\begin{aligned} \dot{\delta x} &= \underbrace{A(x,u)}_{\frac{\partial f(x,u)}{\partial x}} \delta x + \underbrace{B(x,u)}_{\frac{\partial f(x,u)}{\partial u}} \delta u \\ \dot{\delta y} &= \underbrace{C(x,u)}_{\frac{\partial h(x,u)}{\partial x}} \delta x + \underbrace{D(x,u)}_{\frac{\partial h(x,u)}{\partial u}} \delta u \end{aligned}$$

Storage and supply rate lifted to $T\mathcal{M}$

$$S : T\mathcal{M} \to \mathbb{R} \quad \leftarrow \text{differential storage}$$

To make the connection to incremental stability we need

$$c_1 |\delta x|_x^p \leq S(x, \delta x) \leq c_2 |\delta x|_x^p$$

$$\dot{S}(x, \delta x) \leq s(x, w, \delta w) \quad \leftarrow \text{differential supply rate}$$

$$w := (u, y) \qquad \delta w = (\delta u, \delta y)$$

- Differential passivity: $s(x, w, \delta w) = \delta y^T \delta u$
- "Quadratic" differential supply: $s(x, w, \delta w) = \delta w^T Q(x) \delta w$

Linear systems: differential passivity $\equiv$ passivity

$$\left\{\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} \dot{\delta x} & = & A\delta x + B\delta u \\ \delta y & = & C\delta x \end{array}\right.$$

$$S = \frac{1}{2}\delta x^T P \delta x \quad \Rightarrow \quad \dot{S} \leq \delta y^T \delta u \quad \text{if}$$

$$\delta u = 0 \quad \Rightarrow \quad A^T P + PA \leq 0$$

$$\delta y^T \delta u \quad \Rightarrow \quad PB = C^T$$

Differential passivity if uniform contraction w.r.t. u.

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases} \Rightarrow \begin{cases} \dot{\delta x} &= \frac{\partial (f(x)+g(x)u)}{\partial x}\delta x + g(x)\delta u \\ \delta y &= \frac{\partial h(x)}{\partial x}\delta x \end{cases}$$

For differential passivity, let $S = \delta x^T P(x) \delta x$. Then, $\dot{S} \leq \delta y^T \delta u$ if

$$\delta u = 0 \quad \Rightarrow \quad P(x)\frac{\partial (f(x) + g(x)u)}{\partial x} + \dot{P}(x) \leq 0$$

$$\delta y^T \delta u \quad \Rightarrow \quad P(x)g(x) = \frac{\partial h(x)}{\partial x} \leq 0$$

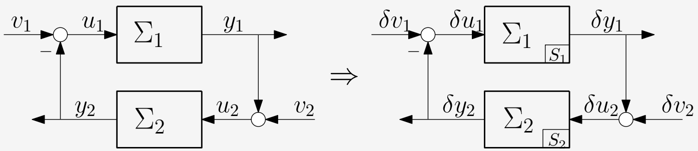
Differential passivity, incremental stability, interconnections

Thm1: differential passivity $\Rightarrow$ incremental stability.

Proof: $\dot{S} \leq \delta y^T \delta u \Rightarrow \dot{S} \leq 0 \; (\delta u = 0) \Rightarrow |\delta x(t)|_{x(t)} \leq K |\delta x(0)|_{x(0)}$

Thm2: differentially passivity + feedback $\Rightarrow$ differentially passivity

Proof: $S = S_1 + S_2 \quad \dot{S} = -\delta y_1^T \delta y_2 + \delta y_2^T \delta y_1 + \begin{bmatrix} \delta y_1 \\ \delta y_2 \end{bmatrix}^T \begin{bmatrix} \delta v_1 \\ \delta v_2 \end{bmatrix}$



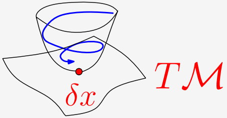
Contraction and differential dissipativity

Incremental stability

$d(x_1(t), x_2(t)) \to 0$

Differential

Lyapunov functions

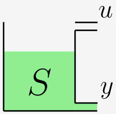


Closed systems

$\dot{x} = f(x)$

Differential

Storage / supply rate

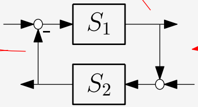


$w = (u, y)$

$\dot{S}(t) \leq s(x(t), w(t), \delta w(t))$

Open systems

$\dot{x} = f(x, u)$



interconnection theory

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