Geometric construction of polytopic invariant sets for constrained linear systems

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Outline

• Setting
• Problem
• Existing approaches
• Proposed solution
• Examples
Systems

**autonomous systems**

\[
\begin{align*}
\dot{x}(t) &= Ax(t) & t \in \mathbb{R}_+ \\
x(t + 1) &= Ax(t) & t \in \mathbb{N} \\
x(t) &\in \mathbb{R}^n
\end{align*}
\]

**systems with inputs**

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) & x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \\
x(t + 1) &= Ax(t) + Bu(t) & x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m
\end{align*}
\]

**systems with uncertainties**

\[
\begin{align*}
\dot{x}(t) &\in \Phi(x(t), u(t)) & \Phi : \mathbb{R}^n \times \mathbb{R}^m \Rightarrow \mathbb{R}^n \\
x(t + 1) &\in \Phi(x(t), u(t))
\end{align*}
\]

\[
\Phi(x, u) = \{ Ax + Bu : A \in \mathcal{A}, B \in \mathcal{B} \}
\]

\[
\mathcal{A} = \text{conv}(\{ A_i \}_{i \in \mathbb{N}_{[1, q_A]}}) \quad \mathcal{B} = \text{conv}(\{ B_i \}_{i \in \mathbb{N}_{[1, q_B]}})
\]
Constraints

state constraints

\[ x(t) \in X \subset \mathbb{R}^n \quad t \in \mathbb{R}_+ \]
\[ t \in \mathbb{N} \]

input constraints

\[ u(t) \in U \subset \mathbb{R}^m \quad t \in \mathbb{R}_+ \]
\[ t \in \mathbb{N} \]
Invariance

**admissible positively invariant sets**

\[ \dot{x}(t) = \Phi(x(t)) \quad x(t) \in \mathbb{X} \subset \mathbb{R}^n \]
\[ x(t + 1) = \Phi(x(t)) \]

\[ x(0) \in \mathcal{S} \subseteq \mathbb{X} \quad \Rightarrow \quad x(t) \in \mathcal{S}, \quad t \in \mathbb{R}_+ \]
\[ \quad t \in \mathbb{N} \]

**admissible \( \lambda \)-contractive sets**

\[ x(0) \in \mathcal{S} \subseteq \mathbb{X} \quad \Rightarrow \quad \exists \quad \lambda \geq 0 \text{ such that } x(t) \in e^{-\lambda t} \mathcal{S}, \quad t \in \mathbb{R}_+ \]
\[ \exists \quad 0 \leq \lambda \leq 1 \text{ such that } x(t) \in \lambda^t \mathcal{S}, \quad t \in \mathbb{N} \]
Invariance

\[ \dot{x}(t) = \Phi(x(t), u(t)) \quad x(t) \in \mathbb{X} \subset \mathbb{R}^n \]
\[ x(t + 1) = \Phi(x(t), u(t)) \quad u(t) \in \mathbb{U} \subset \mathbb{R}^m \]

\[ x(0) \in \mathcal{S} \subseteq \mathbb{X} \quad \Rightarrow \quad \exists \quad f : \mathbb{X} \to \mathbb{U} \text{ such that} \]
\[ \mathcal{S} \text{ is positively invariant w.r.t.} \quad \dot{x}(t) = \Phi(x(t), f(x(t))) \]
\[ x(t + 1) = \Phi(x(t), f(x(t))) \]

\[ x(0) \in \mathcal{S} \subseteq \mathbb{X} \quad \Rightarrow \quad \exists \quad f : \mathbb{X} \to \mathbb{U} \text{ such that} \]
\[ \exists \quad \lambda \geq 0 \text{ such that} \quad \mathcal{S} \text{ is } \lambda\text{-contractive w.r.t.} \quad \dot{x}(t) = \Phi(x(t), f(x(t))) \]
\[ \exists \quad 0 \leq \lambda \leq 1 \text{ such that} \quad \mathcal{S} \text{ is } \lambda\text{-contractive w.r.t.} \quad x(t + 1) = \Phi(x(t), f(x(t))) \]
Problem

a. Compute the maximal $\lambda$-contractive set / the maximal controlled $\lambda$-contractive set

b. Compute a $\lambda$-contractive / controlled $\lambda$-contractive set of a non-trivial size and of a specified complexity
Polyhedral sets

Polytopic sets \( S = \{x \in \mathbb{R}^n : Px \leq 1_p\} \)

Half-space description

1. \( P \in \mathbb{R}^{p \times n} \) has at least \( n + 1 \) rows

2. \( S \) is in general non-symmetric

3. \( P \) is of full row-rank and \( S \) includes the origin in its interior
Polyhedral sets

**Polytopic sets**

\[ S = \operatorname{conv}(\{[V]_i\}_{i \in \mathbb{N}_{[1,q]}}) \]

**Vertex description**

1. \( V \in \mathbb{R}^{n \times q} \) has at least \( n + 1 \) columns
2. \( S \) is in general non-symmetric
3. \( V \) is of full column-rank and \( S \) includes the origin in its interior
Why search for polytopes?

✓ necessary to exist for stable (stabilizable) linear systems (and uncertain, switched systems)

✓ non-conservative for approximating the region of attraction, region of stabilizability
Existing approaches

- Algebraic necessary and sufficient conditions of existence
- Set iterations
  - Inverse reachability from state constraint set
  - Inverse reachability from singleton equilibrium point \{0\}
- Spectral properties
- Norm properties
- Conic partitions
- Trajectory propagation
Existing approaches

Algebraic nec. and suff. conditions of existence of \( \lambda \)-contractive sets


Existing approaches

\[ V(x) = \max_i \{[Px]_i\} \]

\[ S = \{ x \in \mathbb{R}^n : Px \leq 1_p \} = \text{conv} (\{[V]_i\}_{i \in \mathbb{N}_{[1,q]}}) \]

Conditions for existence of a \( \lambda \)-contractive set \( S \) w.r.t. a linear system

**Half-space description**

**continuous-time:**

\[ PA = HP \]

\[ H_{ij} \geq 0, \ (i,j) \in \mathbb{N}_{[1,p]} \times \mathbb{N}_{[1,p]}, \ i \neq j \]

\[ H1_p \leq -\lambda 1_p \]

**discrete-time:**

\[ PA = HP, \quad H \in \mathbb{R}^{p \times p} \]

\[ H \geq 0 \]

\[ H1_p \leq \lambda 1_p \]
Existing approaches

\[ V(x) = \max_i[[Px]_i] \]

\[ S = \{ x \in \mathbb{R}^n : Px \leq 1 \} = \text{conv}(\{[V]_i\}_{i \in \mathbb{N}_{[1,q]}}) \]

Conditions for existence of a \( \lambda \)-contractive set \( S \) w.r.t. a linear system

**Vertex description**

- **Continuous-time:**
  \[ AV = VH \]
  \[ H_{ij} \geq 0, \ (i,j) \in \mathbb{N}_{[1,q]} \times \mathbb{N}_{[1,q]}, \ i \neq j \]
  \[ 1_q^\top H \leq -\lambda 1_q^\top \]

- **Discrete-time:**
  \[ AV = VH, \quad H \in \mathbb{R}^{q \times q} \]
  \[ H \geq 0 \]
  \[ 1_q^\top H \leq \lambda 1_q^\top \]

difficulty to solve
Existing approaches

Set iterations


Existing approaches

\[ S_{i+1} = \{ x \in \mathbb{X} : (\exists u \in \mathbb{U} : Ax + Bu \in S_i) \} \]

\[ S_0 = \{0\} \quad \text{inner approximation} \quad \text{high complexity (not scalable)} \]
\[ S_0 = \mathbb{X} \quad \text{outer approximation} \quad + \text{only last element is contractive} \]
Why search for polytopes?

✓ necessary to exist for stable (stabilizable) linear systems (and uncertain, switched systems)

✓ non-conservative for approximating the region of attraction, region of stabilizability

However,

✗ algebraic necessary and sufficient conditions cannot be used directly to provide polytopic sets of non-trivial size

✗ set iteration methods usually explode

✗ existing methods do not account for other specifications such as complexity or other geometrical aspects of the resulting sets
Proposed approach

Problem: Given a $\lambda$-contractive set $S = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}}$.
Proposed approach

Problem: Given a $\lambda$-contractive set $\mathcal{S} = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}}$; add a vector $v^*$ to its convex hull
Proposed approach

Problem: Given a $\lambda$–contractive set $\mathcal{S} = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}}$, add a vector $v^*$ to its convex hull such that the resulting set $\mathcal{S}^* = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}, v^*}$ is also $\lambda$–contractive.
Proposed approach

\[ x(t + 1) = Ax(t) \]

**discrete-time case**

**Result:** Given a \( \lambda \)-contractive set \( \mathcal{S} = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}} \) and a vector \( v^* \), the set \( \mathcal{S}^* = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}, v^*} \) is \( \lambda \)-contractive if and only if there exist a vector \( p^* \in \mathbb{R}^q \) and a scalar \( p^*_q > 1 \), such that

\[
Av^* = Vp^* + p^*_q v^*,
\]

\[
1 \quad \text{T} \quad p^* + p^*_q \leq \lambda,
\]

\[
p^* \geq 0,
\]

\[
p^*_q \geq 0.
\]
Proposed approach

\[ x(t + 1) = Ax(t) \]
\[ \mathcal{C}(\mathcal{S}) = \{ x \in \mathbb{R}^n : Ax \in \mathcal{S} \} \]

Result: Given a \( \lambda \)-contractive set \( \mathcal{S} = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}} \) and a vector \( v^* \), \( v^* \in \mathcal{C}(\mathcal{S}) \), the set \( \mathcal{S}^* = \text{conv}(\{[V]_i\}_{i \in \mathbb{N}_{[1,q]}}, v^*) \) is \( \lambda \)-contractive if and only if there exist a vector \( p^* \in \mathbb{R}^q \) such that

\[ Av^* = Vp^*; \]
\[ 1^T_q p^* + p^*_{q+1} \leq \lambda, \]
\[ p^* \geq 0, \]
\[ p^*_{q+1} \geq 0. \]
Proposed approach

\[ x(t+1) = Ax(t) + Bu(t) \]

discrete-time case + inputs

**Result:** Given a controlled \( \lambda \)-contractive set \( \mathcal{S} = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}} \) and a vector \( v^* \), the set \( \mathcal{S}^* = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}}, v^* \) is controlled \( \lambda \)-contractive if and only if there exist a vector \( p^* \in \mathbb{R}^q \), a scalar \( p_{q+1}^* \) and a vector \( u^* \in \mathbb{R}^m \), such that

\[
Av^* + Bu^* = Vp^* + p_{q+1}^* v^*,
\]

\[
1^T_q p^* + p_{q+1}^* \leq \lambda,
\]

\[
p^* \geq 0,
\]

\[
p_{q+1}^* \geq 0.
\]
Proposed approach

\[
x(t + 1) \in \Phi(x(t), u(t)) \quad \Phi(x) = \{ Ax : A \in \mathcal{A} \} \quad \mathcal{A} = \text{conv}(\{ A_i \}_{i \in \mathbb{N}_{[1,q_A]}})
\]

Result: Given a $\lambda$-contractive set $\mathcal{S} = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}}$ and a vector $v^*$, the set $\mathcal{S}^* = \text{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}}$ is $\lambda$-contractive if and only if there exist vectors \( p_i^*, \; i \in \mathbb{N}_{[1,q_A]} \) and scalars \( p_{i,q+1}^*, \; i \in \mathbb{N}_{[1,q_A]} \), such that

\[
A_i v^* = V p_i^* + p_{i,q+1}^* v^*,
\]

\[
1_{q_i} p_i^* + p_{i,q+1}^* \leq \lambda,
\]

\[
p_i^* \geq 0,
\]

\[
p_{i,q+1}^* \geq 0.
\]

for all $i \in \mathbb{N}_{[1,q_A]}$

+ polytopic uncertainties
Proposed approach

\[ \dot{x}(t) = Ax(t) \]

**continuous–time case**

Result: Given a $\lambda$–contractive set $\mathcal{S} = \text{conv}([V]_i)_{i \in \mathbb{N}[1,q]}$ and a vector $v^*$, the set $\mathcal{S}^* = \text{conv}([V]_i)_{i \in \mathbb{N}[1,q]}, v^*)$ is $\lambda$–contractive if and only if there exist a vector $p^* \in \mathbb{R}^q$ and a scalar $p^*_{q+1}$, such that

\[
Av^* = Vp^* + p^*_{q+1}v^*,
\]

\[
1_q^\top p^* + p^*_{q+1} \leq -\lambda,
\]

\[
p^* \geq 0.
\]
Proposed approach

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

**Result:** Given a controlled \( \lambda \)-contractive set \( S = \text{conv}([V], i \in \mathbb{N}_{[1,q]}) \) and a vector \( v^* \), the set \( S^* = \text{conv}(\{[V], i \in \mathbb{N}_{[1,q]}, v^*\}) \) is controlled \( \lambda \)-contractive if and only if there exist a vector \( p^* \in \mathbb{R}^q \), a scalar \( p_{q+1}^* \) and a vector \( u^* \in \mathbb{R}^m \), such that

\[
Av^* + Bu^* = Vp^* + p_{q+1}^*v^*,
\]

\[
1^\top_q p^* + p_{q+1}^* \leq -\lambda,
\]

\[
p^* \geq 0,
\]
Proposed approach

\[
\dot{x}(t) \in \Phi(x(t), u(t)) \quad \Phi(x) = \{Ax : A \in \mathcal{A}\} \quad \mathcal{A} = \text{conv}(\{A_i\}_{i \in \mathbb{N}_{[1,qA]}})
\]

Result: Given a $\lambda$-contractive set $\mathcal{S} = \text{conv}([V]_{i \in \mathbb{N}_{[1,q]}})$ and a vector $v^*$, the set $\mathcal{S}^* = \text{conv}(\{[V]_{i \in \mathbb{N}_{[1,q]}}, v^*\})$ is $\lambda$-contractive if and only if there exist vectors $p_i^*, \ i \in \mathbb{N}_{[1,qA]}$ and scalars $p_{i,q+1}^*, \ i \in \mathbb{N}_{[1,qA]}$, such that

\[
A_i v^* = V p_i^* + p_{i,q+1}^* v^*;
\]

\[
1^T_{q} p_i^* + p_{i,q+1}^* \leq -\lambda,
\]

\[
p_i^* \geq 0,
\]

for all $i \in \mathbb{N}_{[1,qA]}$.

+polytopic uncertainties
Region of attraction/ stabilizability

Linear system, polytopic state and input constraints

\[ x(t + 1) = Ax(t) + Bu(t) \]
\[ x(t) \in X, \quad u(t) \in U, \quad \forall t \in \mathbb{N} \]

\[ X = \{ x \in \mathbb{R}^n : P_x x \leq 1_{p_x} \} \]
\[ U = \{ x \in \mathbb{R}^n : P_u u \leq 1_{p_u} \} \]

Compute a sequence \( \{ S_i \}_{i \in \mathbb{N}} \) of controlled \( \lambda \)-contractive polytopes such that

1. \( S_i \subseteq S_{i+1} \)
2. \( S_i \subseteq X \)
3. \( S_i \) is a polytope
4. \( \exists f_i : S_i \rightarrow U \) such that \( S_i \) is \( \lambda \)-contractive w.r.t. \( x(t + 1) = Ax(t) + Bf_i(x(t)) \)
Region of attraction/ stabilizability

Compute a sequence \( \{S_i\}_{i \in \mathbb{N}} \) of controlled \( \lambda \)-contractive polytopes such that

1. \( S_i \subset S_{i+1} \) add vertices \( \{v_i^*\}_{i \in \mathbb{N}[1,p_i]} \) to convex hull of \( S_i \)

2. \( S_i \subset \mathbb{X} \) true if \( v_i^* \in \mathbb{X}, i \in \mathbb{N}[1,p_i] \) (linear ineqs)

3. \( S_i \) is a polytope from proposed approach

4. \( \exists f_i : S_i \rightarrow \mathbb{U} \) such that \( S_i \) is \( \lambda \)-contractive

\[
\begin{align*}
   x(t + 1) = Ax(t) + B f_i(x(t)) & \text{ from proposed approach} \\
   \text{true if } u_i^* \in \mathbb{U}, i \in \mathbb{N}[1,p_i] & \text{ (linear ineqs)}
\end{align*}
\]
Region of attraction/ stabilizability

\[ \min_{v^*, u^*, p^*, p_{q+1}} \{0\} \]

\[ A v^* + B u^* = V p^* + p_{q+1}^* v^* , \]
\[ 1^T p^* + p_{q+1}^* \leq \lambda , \]
\[ p^* \geq 0 , \]
\[ p_{q+1}^* \geq 0 , \]
\[ P_x v^* \leq 1_{p_x} , \]
\[ P_u u^* \leq 1_{p_u} . \]
Region of attraction/ stabilizability

$$\max_{v^*, u^*, p^*, p_{q+1}^*} \{ [P_i v^*]_j \}$$

$$Av^* + Bu^* = Vp^* + p_{q+1}^* v^*,$$
$$1^\top p^* + p_{q+1}^* \leq \lambda,$$
$$p^* \geq 0,$$
$$p_{q+1}^* \geq 0,$$
$$P_x v^* \leq 1_{p_x},$$
$$P_u u^* \leq 1_{p_u}$$
Region of attraction/ stabilizability

\[ \max_{v^*, u^*, p^*, p_{q+1}^*} \{[[P_i v^*]_j]\} \]

\[
Av^* + Bu^* = Vp^* + p_{q+1}^* v^*, \\
1^\top p^* + p_{q+1}^* \leq \lambda, \\
p^* \geq 0, \\
p_{q+1}^* \geq 0, \\
P_x v^* \leq 1_{p_x} \\
P_u u^* \leq 1_{p_u}
\]

There exists a vector \( v \) such that conv\( (S_i, v) \) is controlled \( \lambda \)-contractive if and only if there exists a non-trivial solution to the above problems.
Region of attraction/ stabilizability

\[
\max_{v^*,u^*,p^*,p_{q+1}^*} \{[P_i v^*]_1\} = c_1
\]

\[Av^* + Bu^* = V p^* + p_{q+1}^* v^*,\]

\[1_1^\top p^* + p_{q+1}^* \leq \lambda,\]

\[p^* \geq 0,\]

\[p_{q+1}^* \geq 0,\]

\[P_x v^* \leq 1_{px},\]

\[P_u u^* \leq 1_{pu}.\]
Region of attraction/ stabilizability

\[ \max_{v^*, u^*, p^*, p_{q+1}^*} \{ [P_i v^*]_2 \} = c_2 \]

\[ Av^* + Bu^* = Vp^* + p_{q+1}^* v^*; \]
\[ 1^T_q p^* + p_{q+1}^* \leq \lambda, \]
\[ p^* \geq 0, \]
\[ p_{q+1}^* \geq 0, \]
\[ P_x v^* \leq 1_{p_x} \]
\[ P_u u^* \leq 1_{p_u} \]
Region of attraction/ stabilizability

\[
\max_{v^*, u^*, p^*, p_{q+1}^*} \{ [P_i v^*]_3 \} = c_3
\]

\[
Av^* + Bu^* = Vp^* + p_{q+1}^* v^*,
\]

\[
1_q^\top p^* + p_{q+1}^* \leq \lambda,
\]

\[
p^* \geq 0,
\]

\[
p_{q+1}^* \geq 0,
\]

\[
P_x v^* \leq 1_{p_x}
\]

\[
P_u u^* \leq 1_{p_u}
\]
The convex hull of the vectors $v_i^*, \ i \in \mathbb{N}_{[1,p_i]}$ with $S_i$ is also controlled $\lambda$–contractive

The procedure continues until no further expansion can be made or another termination criterion is met

For discrete–time systems, the set sequence converges to the maximal controlled $\lambda$–contractive set
Specified complexity

We can compute sets of prespecified complexity if we choose to add vertices in specific regions of the state-space.

$S$ has 8 vertices
Specified complexity

We can compute sets of specified complexity if we choose to add vertices in specific regions of the state-space.

$S$ has 8 vertices

$S^*$ has 9 vertices
Specified complexity

We can compute sets of prespecified complexity if we choose to add vertices in specific regions of the state-space.

$\mathcal{S}$ has 8 vertices

$\mathcal{S}^*$ has 8 vertices
Specified complexity

We can compute sets of prespecified complexity if we choose to add vertices in specific regions of the state-space.

$S$ has 8 vertices

$S^*$ has 5 vertices
Specified complexity

We can compute sets of prespecified complexity if we choose to add vertices in specific regions of the state-space.

Key idea: Search for enlargements by ordering the regions according to the complexity induced.
Examples-domain of attraction (1)

\[ x(t + 1) = Ax(t) + Bu(t) \]

\[ X = \{ x \in \mathbb{R}^n : -25 \leq x_1 \leq 25, \quad -5 \leq x_2 \leq 5 \} \]

\[ U = \{ x \in \mathbb{R}^n : -1 \leq u \leq 1 \} \]

\[ A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix}, \]
Examples - domain of attraction (1)

\[ x(t + 1) = Ax(t) + Bu(t) \]
\[ X = \{ x \in \mathbb{R}^n : -25 \leq x_1 \leq 25, -5 \leq x_2 \leq 5 \} \]
\[ U = \{ x \in \mathbb{R}^n : -1 \leq u \leq 1 \} \]

\[ A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix}, \]

A. Algebraic nec. and suf. conditions:
1. place eigenvalues in unit rhombus.
2. construct set \( S \) by the left eigenvectors of the closed-loop matrix

existing approaches
Examples-domain of attraction (1)

\[ x(t + 1) = Ax(t) + Bu(t) \]

\[ X = \{ x \in \mathbb{R}^n : -25 \leq x_1 \leq 25, \quad -5 \leq x_2 \leq 5 \} \]

\[ U = \{ x \in \mathbb{R}^n : -1 \leq u \leq 1 \} \]

\[ A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix}, \]

existing approaches

**B.** Inverse reachability map: start from the equilibrium point \( S_0 = \{0\} \)

Convergence in 146 iterations
Examples-domain of attraction (1)

\[ x(t + 1) = Ax(t) + Bu(t) \]

\[ X = \{ x \in \mathbb{R}^n : -25 \leq x_1 \leq 25, \quad -5 \leq x_2 \leq 5 \} \]

\[ U = \{ x \in \mathbb{R}^n : -1 \leq u \leq 1 \} \]

\[
A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix}
\]

existing approaches

C. Inverse reachability map: start from the state constraint set \( S_0 = X \)

Convergence in 56 iterations
Examples - domain of attraction (1)

\[ x(t + 1) = Ax(t) + Bu(t) \]

\[ X = \{ x \in \mathbb{R}^n : -25 \leq x_1 \leq 25, \ -5 \leq x_2 \leq 5 \} \]

\[ U = \{ x \in \mathbb{R}^n : -1 \leq u \leq 1 \} \]

\[ A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix} \]

D. Proposed approach
$x(t + 1) = Ax(t) + Bu(t)$

$X = \{x \in \mathbb{R}^n : -25 \leq x_1 \leq 25, \; -5 \leq x_2 \leq 5\}$

$U = \{x \in \mathbb{R}^n : -1 \leq u \leq 1\}$

$A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix},$

**D. Proposed approach**
Examples-domain of attraction (1)

\[ x(t + 1) = Ax(t) + Bu(t) \]
\[ X = \{ x \in \mathbb{R}^n : -25 \leq x_1 \leq 25, \ -5 \leq x_2 \leq 5 \} \]
\[ U = \{ x \in \mathbb{R}^n : -1 \leq u \leq 1 \} \]

\[ A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{T_s^2}{2} \\ \frac{T_s}{2} \end{bmatrix}, \]

**D. Proposed approach**
Examples-domain of attraction (1)

\[ x(t + 1) = Ax(t) + Bu(t) \]

\[ X = \{ x \in \mathbb{R}^n : -25 \leq x_1 \leq 25, \ -5 \leq x_2 \leq 5 \} \]

\[ U = \{ x \in \mathbb{R}^n : -1 \leq u \leq 1 \} \]

\[ A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix}, \]

D. Proposed approach

Convergence in 19 iterations
\[ \dot{x}(t) \in \Phi(x(t)), \text{ where } \Phi : \mathbb{R}^n \to \mathbb{R}^n, \Phi(x) := \{ Ax : A \in \text{convh}(\{ A_i \}_{i \in \mathbb{N}_{[1,2]}}) \} \]

\[ A_1 = \begin{bmatrix} 0.3 & 0.7 \\ -2.3 & -2.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.8 & 1.0 \\ -0.8 & 0.1 \end{bmatrix}. \]

\[ \mathbf{x} = \text{conv}(\{[V_x]_i\}_{i \in \mathbb{N}_{[1,4]}}) \quad V_x = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 2 & -1 & 1 \end{bmatrix}. \]
Examples-domain of attraction (2)

\[ \dot{x}(t) \in \Phi(x(t)), \text{ where } \Phi : \mathbb{R}^n \to \mathbb{R}^n, \Phi(x) := \{ Ax : A \in \text{convh}(\{A_i\})_{i \in \mathbb{N}_{[1,2]}} \} \]

\[ A_1 = \begin{bmatrix} 0.3 & 0.7 \\ -2.3 & -2.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.8 & 1.0 \\ -0.8 & 0.1 \end{bmatrix}. \]

\[ X = \text{conv}(\{[V_x]_i\}_{i \in \mathbb{N}_{[1,4]}}) \quad V_x = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 2 & -1 & 1 \end{bmatrix}. \]

Termination criterion: Hausdorff distance between two consecutive sets \( S_{i-1} \) and \( S_i \) is less than \( d = 10^{-3} \)

\[ d_H(S_i, S_{i-1}) \leq 0.001 \]
Examples—specified complexity (1)

double integrator, discretized

Table 1. Complexity and set coverage for the computed sets.

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complexity</td>
<td>10</td>
<td>18</td>
<td>32</td>
<td>106</td>
</tr>
<tr>
<td>Coverage of the set $S_{\text{max}}$(%)</td>
<td>92</td>
<td>98</td>
<td>99.63</td>
<td>100</td>
</tr>
</tbody>
</table>
Examples- specified complexity (2)

triple integrator

Table 2. Complexity and set coverage for the computed sets.

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
<th>$S_5$</th>
<th>$S_{EAS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complexity</td>
<td>10</td>
<td>16</td>
<td>24</td>
<td>44</td>
<td>62</td>
<td>356</td>
</tr>
<tr>
<td>Coverage of the set $S_{EAS}$ (%)</td>
<td>38</td>
<td>45</td>
<td>73.5</td>
<td>90</td>
<td>94</td>
<td>100</td>
</tr>
</tbody>
</table>

$S_{EAS}$

$S_4$
Conclusions

1. Construction of invariant/contractive sets, based on geometric properties of polytopes

2. Directly applicable to both discrete-time and continuous-time linear systems

3. Simple implementation

4. Other types of specifications can be addressed (such as complexity constraints)
Relevant works


Acknowledgments

Marie Curie IEF: “Set-Induced Comparison Principles for Complex Systems” (REA No 302345).

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Mircea Lazar, Eindhoven University of Technology, the Netherlands

Thank you!