WORKSHOP ON ALGORITHMS FOR DYNAMICAL SYSTEMS AND LYAPUNOV FUNCTIONS

Stability and the basin of attraction of periodic orbits of nonsmooth differential equations

Pascal Stiefenhofer
Department of Mathematics, University of Sussex

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Introduction

- We consider a nonautonomous periodic ODE with discontinuous right hand side.
- For the 1 dimensional case we have a theory for existence, uniqueness, stability, and the basin of attraction of a periodic orbit.
- We want to know whether and how we can generalize this theory to 2 and more dimensions.
- The idea of this talk is to consider the 1 dimensional case and then to discuss expected problems associated with adding one more dimension.
- This is work at early stage. Hence we can only provide a discussion of the problem.

ODE with Discontinuous Right Hand Side, Filippov (1988)

Nonautonomous periodic ODE

$$\dot{x} = f(t,x), \tag{1}$$

where $f(t+T,x) = f(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, with $T > 0$, and f is a discontinuous function.

$$f(t,x) = f^{\pm}(t,x) = \begin{cases} f^{+}(t,x) & \textit{if} \quad x_2 > 0 \\ f^{-}(t,x) & \textit{if} \quad x_2 < 0, \end{cases}$$

Switching surface $\Sigma := \{(t,x) \in \mathbb{R} \times \mathbb{R}^2 : h(t,x) = 0\}$, where $h(t,x) = x_2$. We are interested in

- existence and uniqueness
- stability
- basin of attraction of periodic orbit.

Conditions: 2-Dimensional Case

Conditions which guarantee existence, uniqueness, and continuous dependence on the initial conditions of solutions of

$$\dot{x} = f(t, x).$$

- $f \in C^1(\mathbb{R} \times (\mathbb{R}^2 \backslash \Sigma), \mathbb{R}^2)$.
- $f(t + T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, and $T > 0$.
- $f^{\pm}(t, x) := f(t, x)$ for either $x_2 > 0$, or $x_2 < 0$ can be extended to a continuous function $f^{\pm}(t, x)$ up to $x_2 = 0$.
- $D_x f^+(t, x)$ with $x_2 > 0$, and $D_x f^-(t, x)$ with $x_2 < 0$ can be extended to a continuous function $D_x f^{\pm}(t, x)$ up to $x_2 = 0$.
- At $x_2 = 0$, let $f^+(t, x_1, 0) - f^-(t, x_1, 0)$ be a C^1 function w.r.t time t and x_1.
- For all $t \in [0, T]$ either $\nabla h(t, x_1, 0) \cdot f^+(t, x_1, 0) < 0$ or $\nabla h(t, x_1, 0) \cdot f^-(t, x_1, 0) > 0$.

Let the conditions hold. Then, according to Filippov's **convexification** method we can define a set valued function $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{P}$, where $\mathbb{P}$ is the idempotent set, by

$$F(t,x) := \{f(t,x)\}, \text{ for } x \neq \Sigma,$$

and by a linear combination of $f^{\pm}$

$$F(t,x) := \{\lambda f^-(t,x) + (1-\lambda) f^+(t,x) : \lambda \in [0,1]\}, \text{ for } x \in \Sigma, \ (x_2 = 0),$$

Filippov defines a solution of $\dot{x} = f(t,x)$ to be an absolutely continuous function

$$x : [a,b] \to \mathbb{R}^2$$

which for almost all $t \in [a,b]$ satisfies the differential inclusion

$$\dot{x} \in F(t, x(t)).$$

For all $x \notin \Sigma$, $x(t)$ is a solution of the smooth differential equation $\dot{x} = f(t,x)$.

Filippov's Theorem:(Existence and uniqueness)

Assume that above conditions hold and that if in addition at least one of the strict inequalities $\nabla h(t, x_1, 0) \cdot f^+(t, x_1, 0) < 0$ or $\nabla h(t, x_1, 0) \cdot f^-(t, x_1, 0) > 0$ holds. Then for initial data $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^2$ a solution of the initial value problem

$$\dot{x}(t) \in F(t, x(t)), \; x(t_0) = x_0$$

exists for almost all t on an interval $t \in (t_0, b)$ with $b > 0$. Moreover the solution is unique in forward time, $t \geq 0$, and continuously depending on the initial value (t_0, x_0).

Stability and the Basin of Attraction

Sufficient Condition (Giesl; Smooth ODE's, 2004): if all adjacent solutions within a positively invariant set move towards each other with respect to a Riemannian metric,

- then there exists a unique periodic orbit,
- the periodic orbit is exponentially asymptotically stable and its positively invariant set is a subset of the basin of attraction, $K \subset A(\Omega)$.

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$$L_{M(t,x)} := \max_{v \in \mathbb{R}^2, v^T M(t,x) v = 1} v^T \left[M(t,x) D_x f(t,x) + \frac{1}{2} M'(t,x) \right] v < 0,$$

where $M(t,x)$ is an Riemannian metric and $M'(t,x)$ its orbital derivative, and v the distance between solutions.

Definition (Riemannian metric)

The matrix valued function $M = (M^{\pm}) \in C^1(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^{2\times 2})$ is called a Riemannian metric, if the matrix $M(t, x_1, x_2)$ is:

- symmetric
- positive definite for each $(t, x_1, x_2) \in K$
- and periodic such that for all $(t, x_1, x_2) \in \mathbb{R} \times \mathbb{R}^2$ $M(t + T, x_1, x_2) = M(t, x_1, x_2)$ holds.

Table: Example of M(t,x)

Case	Metric	Comment
1-Dimension	$M(t, x) = e^{W(t,x(t))} I$	Giesl (2005), not work in higher Dim

Metric is used to define the **evolution of the distance between solutions**:

$$\begin{aligned} A_W(t) &:= e^{W(t,x)} \mid y(t) - x(t) \mid \\ A_M(t) &:= \{[y(t) - x(t)]^T M(t, x_1, x_2)[y(t) - x(t)]\}^{1/2} \end{aligned}$$

Hence, for the non-smooth case, stability requires to formulate conditions such that :

- $L_{W(t,x(t))} < 0$ in the 1-Dim case (provided by Giesl (2005))
- $L_{M(t,x(t))} < 0$ in the 2-Dim case (to be found).

Theorem: Giesl (2005)

Let the conditions hold. Also assume that $W^{\pm} : \mathbb{R} \times \mathbb{R}_0^{\pm} \to \mathbb{R}$ are C^0 functions with $W^{\pm}(t+T, x) = W^{\pm}(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}_0^{\pm}$. Also assume that the orbital derivative $(W^{\pm})'$ exists and being a continuous function in $\mathbb{R} \times \mathbb{R}^{\pm}$ and continuously extendable to $\mathbb{R} \times \mathbb{R}_0^{\pm}$. Let $K \subset S_T^1 \times \mathbb{R}$ be a nonempty, connected, compact, and positively invariant set, such that

1. $f_x(t,x) + W'(t,x) \leq -\nu < 0$ for all $(t,x) \in K$ with $x \neq 0$
2. $\frac{f^-(t,0)}{f^+(t,0)} e^{W^- - W^+} \leq e^{-\epsilon} < 1$ for all $(t,0) \in K$ with $f^- < 0$
3. $\frac{f^+(t,0)}{f^-(t,0)} e^{W^+ - W^-} \leq e^{-\epsilon} < 1$ for all $(t,0) \in K$ with $f^+ > 0$

hold with $\nu, \epsilon > 0$. Then there exists a unique periodic orbit Ω with period T in K which is exponentially asymptotically stable with exponent $-\nu$. Also $K \subset A(\Omega)$.

The proof of this theorem relies on the following lemma.

Main lemma

Let the conditions hold and $K \subset S^1_T \times \mathbb{R}$ be a nonempty, connected, compact, and positively invariant set. Also assume that for each $(t_0, x_0) \in K$ there are constants $\delta, \nu > 0$, and $C \geq 1$ such that for all y_0 with $\mid y_0 - x_0 \mid \leq \delta$

$$\mid (y(t) - x(t)) \mid \leq Ce^{-\nu t} \mid y_0 - x_0 \mid$$

holds for all $t \geq 0$ with $\nu, \epsilon > 0$. Then there exists a unique periodic orbit Ω with period T in K which is exponentially asymptotically stable with exponent $-\nu$. For its basin of attraction we have the inclusion $K \subset A(\Omega)$.

The proof requires to show that the evolution path $A_W(t)$ (weighted distance between two solutions) decreases over time. Hence let

$$A_W(t) := e^{W(t,x)} \mid y(t) - x(t) \mid .$$

We need to show that

$$A_W(t) \leq ce^{(-\nu+\iota)(t-t_0)} A(t_0) \text{ for all } t \geq 0$$

for all cases listed below, where $\mathcal{X}^{\pm} \subset \mathbb{R} \backslash \{0\}$ is the state space and Σ the switching line:

1. both solutions have the same sign $(x, y \in \mathcal{X}^{+}; x, y \in \mathcal{X}^{-})$
2. solutions have opposite sign $(x \in \mathcal{X}^{+}, y \in \mathcal{X}^{-};$ $x \in \mathcal{X}^{-}, y \in \mathcal{X}^{+})$
3. $(x \in \Sigma,\ y \in \mathcal{X}^{+}; x \in \Sigma,\ y \in \mathcal{X}^{-})$
4. $(x \in \mathcal{X}^{+}, y \in \Sigma\ ; x \in \mathcal{X}^{-}, y \in \Sigma,\ )$
5. sliding $(x \in \Sigma, y \in \Sigma)$.

Let $\mathcal{X}^{\pm}$ be the state space and Σ the switching manifold of the 1 or 2 Dim case.

Table: Comparison between 1 and 2 Dimensional Cases

1-Dim	Case	2-Dim
$L_{W^+(t,x)} < 0$	$x, y \in \mathcal{X}^+$	$L_{M^+(t,x)} < 0$
$L_{W^-(t,x)} < 0$	$x, y \in \mathcal{X}^-$	$L_{M^-(t,x)} < 0$
trivial	$x, y \in \Sigma$	$L_{M^s(t,x_1,0)} < 0$, M^s ?
$\frac{f^-}{f^+} e^{W^- - W^+} < 1$	$x \in \mathcal{X}^+$, $y \in \mathcal{X}^-$	jump
$\frac{f^+}{f^-} e^{W^+ - W^-} < 1$	$x \in \mathcal{X}^-$, $y \in \mathcal{X}^+$	jump
$L_{W^+(t,x)} < 0$	$x \in \Sigma$,$y \in \mathcal{X}^+$	$L_{M^+} < 0$, $L_{M^s} < 0$?
$L_{W^-(t,x)} < 0$	$x \in \mathcal{X}^-$,$y \in \Sigma$	$L_{M^-} < 0$, $L_{M^s} < 0$?

Conclusion

1. At variance to the 1-Dimensional case where the evolution path is defined by a weighted function $A(t) := e^{W(t,x)} \mid y(t) - x(t) \mid$ we need to consider an evolution path where the weighted function is replaced by a **Riemannian metric**.
2. At variance to the 1-Dimensional model where contraction of solutions on the sliding manifold are trivial (solutions coincide), we need to show that in the 2-Dimensional case **sliding solutions contract**. For that, we need to formulate a sliding equation f^s and define M^s.

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