Construction of Lyapunov functions and Contraction Metrics to determine the Basin of Attraction

Peter Giesl

University of Sussex, UK – Department of Mathematics

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Overview

1. Basin of Attraction of Equilibria
   - Lyapunov Function
   - Radial Basis Functions

2. Basin of Attraction of Periodic Orbits
   - Contraction criterion with Riemannian metric
   - Semidefinite Optimization
1. Basin of Attraction of Equilibria

System of autonomous ordinary differential equations

\[
\begin{align*}
\dot{x} &= f(x) \\ x(0) &= \xi
\end{align*}
\]

\(x \in \mathbb{R}^n, \ f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)\) where \(\sigma \geq 1, \ n \in \mathbb{N}\)

Flow \(S_t \xi := x(t)\), solution of (1)

Assumptions

- \(x_0\) is equilibrium \((f(x_0) = 0)\)
- \(x_0\) is exponentially asymptotically stable
  (real parts of all eigenvalues of \(Df(x_0)\) are negative)

Definition (Basin of attraction) The basin of attraction of \(x_0\) is

\[
A(x_0) := \{ \xi \in \mathbb{R}^n \mid S_t \xi \xrightarrow{t \to \infty} x_0 \}.
\]

Goal: Determine basin of attraction \(A(x_0)\) using a Lyapunov function
1.1 Lyapunov function

**Theorem (Lyapunov 1893)**

1. \( v \in C^1(\mathbb{R}^n, \mathbb{R}) \)
2. \( K \subset \mathbb{R}^n \) compact set
3. \( v'(x) = \frac{d}{dt} v(x(t)) \big|_{t=0} = \langle \nabla v(x), f(x) \rangle < 0 \) for all \( x \in K \setminus \{x_0\} \)
   (orbital derivative = derivative along a solution)
4. \( K = \{ x \in \mathbb{R}^n \mid v(x) \leq R \} \) sublevel set of \( v \)

Then \( K \subset A(x_0) \).

**Existence of Lyapunov functions**

“Converse Theorems” (Massera 1949) etc. – but not constructive!

**Goal:** explicit calculation of Lyapunov function
Example

\[
\begin{align*}
\dot{x} &= -x + x^3 \\
\dot{y} &= -\frac{1}{2}y + x^2
\end{align*}
\]

\[v(x, y) = \frac{1}{2}x^2 + y^2\]

sign of \(v'(x, y)\)
Suitable linear PDE for $V$, e.g.

$$V'(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial V}{\partial x_i}(x) = -\|x - x_0\|^2$$

Approximation $v$ of $V$ using Radial Basis Functions

Error estimate ensures $v'(x) < 0$

The approximation $v$ itself is a Lyapunov function

Sublevel set of $v$ is a subset of $A(x_0)$

**Radial Basis Functions: advantages**

- meshless method: scattered data
- any dimension
- smooth approximation
1.2 Radial Basis Functions

- PDE $LV(x) = g(x)$, $L$ linear differential operator (orbital derivative)
  
  \[ Lu(x) = \sum_{|\alpha| \leq m} c_\alpha(x) D^\alpha u(x) \]

- $\Phi(x) = \psi_k(\|x\|)$ (Radial Basis Function), here:
  $\psi_k$ Wendland’s function (compact support)

- (Scattered) grid points $X_N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^n$

- (Symmetric) ansatz: $v(x) = \sum_{k=1}^N \beta_k (\delta x_k \circ L)^y \Phi(x - y)$

- Interpolation conditions: $Lv(x_j) = LV(x_j) = g(x_j)$, i.e.
  $(\delta x_j \circ L)^x v(x) = (\delta x_j \circ L)^x V(x)$ for all $j = 1, \ldots, N$

- Plug in the ansatz:
  \[ \sum_{k=1}^N \beta_k (\delta x_j \circ L)^x (\delta x_k \circ L)^y \Phi(x - y) = LV(x_j) = g(x_j) =: \gamma_j \]

- System of linear equations $A\beta = \gamma$, $A$ is symmetric

- $A$ is positive definite $\Rightarrow$ non-singular
Steps of the method

1. Find a suitable linear PDE, existence and smoothness of solution
   There exists a Lyapunov function $V \in C^\sigma(A(x_0), \mathbb{R})$ with
   
   $$V'(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial V}{\partial x_i}(x) = -\|x - x_0\|^2 \text{ for all } x \in A(x_0)$$

   (Zubov 1964): $V(x) = \int_0^\infty \|S_t x - x_0\|^2 dt$

2. Positive definiteness of matrix $A$
   Assumption: grid points are distinct and no equilibria
   Proof via positive Fourier-transform $\hat{\Phi}(\omega) > 0$

3. Error estimate on $v'$

4. Estimate of the level sets of $v$: method covers every compact subset of the basin of attraction

5. Solve local problem
Error Estimates

Theorem (Giesl 2007, Giesl & Wendland 2007)

Use Wendland’s compactly supported functions as Radial Basis Functions:

\[ \Phi(x) = \psi_k(\|x\|) \in C^{2k}(\mathbb{R}^n, \mathbb{R}), \quad k \in \mathbb{N} \] (Wendland 1998)

Let \( f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n) \) where \( \sigma \geq \frac{n+1}{2} + k \).

For each compact set \( K_0 \subset A(x_0) \) there is \( C \) such that

\[ |V'(x) - v'(x)| \leq C \hat{h}^{k-1/2} \text{ for all } x \in K_0 \]

\( h := \max_{y \in K_0} \min_{x \in X_N} \|x - y\|: \text{ fill distance} \)

Example of Wendland function:

\[ \psi_2(r) = \begin{cases} 
(1 - r)^6[35r^2 + 18r + 3] & \text{if } r < 1 \\
0 & \text{if } r \geq 1 
\end{cases} \]
Solution of the local problem

Approximation $v$ is (nearly) a Lyapunov function:

- $V'(x) = -\|x - x_0\|^2$
- $|v'(x) - V'(x)| \leq \epsilon$ error estimate

implies $v'(x) \leq V'(x) + \epsilon = -\|x - x_0\|^2 + \epsilon < 0$ except for $x$ near $x_0$

$v'(x)$ is in general positive near $x_0$

**Solution: Linearisation**

local Lyapunov function $V_{loc}(x) = (x - x_0)^T B(x - x_0)$ from linearised equation $\dot{x} = Df(x_0)x \Rightarrow$ local basin of attraction
Example

\[
\begin{align*}
\dot{x} &= -x + x^3 \\
\dot{y} &= -\frac{1}{2}y + x^2
\end{align*}
\]

Grid, \( v' = 0 \), sublevel sets: local (thin black) and calculated (thick black)
Goal: Basin of attraction $A(x_0)$

Instrument: Lyapunov function $v$ and sublevel set $K$

Construction: Approximation $v$ of $V$ using Radial Basis Functions

Result: $v'(x) < 0$ if grid dense enough

Global: every compact set within $A(x_0)$ can be covered

Local: use local Lyapunov function (linearisation)

Outlook

Grid refinement – talk by Najla Mohammed this afternoon
Combination with CPA method – talk by Sigurdur Hafstein
Variations

- different PDE, e.g. $V'(x) = -1$

- boundary conditions
  mixed approximation
  (given the values of $V'$ and $V$)

- discrete dynamical system:
  $x_{n+1} = g(x_n)$

- time-periodic ODEs
  (with Holger Wendland)
2. Basin of Attraction of Periodic Orbits

- \( \dot{x} = f(x), \ f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n) \): autonomous ODE
- periodic orbit \( \Omega = \{ S_t x \mid t \in [0, T) \} \) with \( x = S_T x \), minimal \( T > 0 \): period
- basin of attraction \( A(\Omega) = \{ \xi \in \mathbb{R}^n \mid \text{dist}(S_t \xi, \Omega) \xrightarrow{t \to \infty} 0 \} \)

Disadvantage of Lyapunov function:
- \( V(x) = 0 \) and \( V'(x) = 0 \) for all \( x \in \Omega \) and \( V(x) > 0 \) and \( V'(x) < 0 \) for all \( x \notin \Omega \), but position of the periodic orbit not (exactly) known
- local structure (solution of first variation equation) not known

Goal: Determine the basin of attraction \( A(\Omega) \) without (exact) knowledge of the periodic orbit via (local) contraction criterion
2.1 Contraction criterion

$\|v\| = 1, \ v \perp f(p)$

$L(p, v) = v^T Df(p) v$

$L(p) = \max_{\|v\|=1, v \perp f(p)} L(p, v)$

- $L(p, v) < 0$: Trajectories through $p$ and $p + \delta v$ ($\delta > 0$ small) approach each other
- $L(p) < 0$: Trajectories through $p$ and adjacent points approach each other
Sufficient condition

Theorem (Borg 1960)

- $\emptyset \neq K \subset \mathbb{R}^n$ includes no equilibria, is positively invariant, compact and connected
- $L(p) \leq -\nu < 0$ for all $p \in K$

Then

- Existence and uniqueness of an exponentially asymptotically stable periodic orbit $\Omega \subset K$
- $K \subset A(\Omega)$ (basin of attraction)
- $-\nu$ is upper bound of real parts of non-trivial Floquet exponents

Question: Is this condition also necessary?
Example \[
\begin{align*}
\dot{x} &= x(1 - x^2 - y^2) \left( x + \frac{1}{2} \right) - y \\
\dot{y} &= y(1 - x^2 - y^2) \left( x + \frac{1}{2} \right) + x
\end{align*}
\]

- Unit sphere \( \Omega \) is an exponentially asymptotically stable periodic orbit
- but \( L(p) < 0 \) does not hold for all \( p \in \Omega \)
- criterion is not necessary!

**Reason:** non-monotone approach of adjacent solutions

**Idea:** modify contraction condition using a weighted distance/different metric
Riemannian metric

**Definition (Riemannian metric)**

A matrix-valued function \( M \in C^1(\mathbb{R}^n, \mathbb{R}^{n \times n}) \) is called **Riemannian metric** if \( M(x) \) is a symmetric and positive definite matrix for each \( x \in \mathbb{R}^n \).

**Note:** Then \( \langle v, w \rangle_{M(x)} := v^T M(x) w \) defines a point-dependent scalar product for \( v, w \in \mathbb{R}^n \).

**Examples**

- \( M(x) = I \): Riemannian metric
- \( M(x) = e^{2W(x)} I \): weighted distance
Sufficient condition (with Riemannian metric)

**Theorem (Stenström 1962)**

- $\emptyset \neq K \subset \mathbb{R}^n$ includes no equilibria, is positively invariant, compact and connected
- **Riemannian metric** $M \in C^1(\mathbb{R}^n, \mathbb{R}^{n \times n})$
- $L_M(x) \leq -\nu < 0$ for all $x \in K$ where
  
  \[
  L_M(x) := \max_{v^T M(x) v = 1, v^T M(x) f(x) = 0} v^T \left[ M(x) Df(x) + \frac{1}{2} M'(x) \right] v
  \]

Then

- Existence and uniqueness of an **exponentially asymptotically stable periodic orbit** $\Omega \subset K$
- $K \subset A(\Omega)$ (**basin of attraction**)
- $-\nu$ is upper bound of real parts of non-trivial Floquet exponents

Example: $L_M(x) = L(x) + W'(x)$ for $M(x) = e^{2W(x)}I$

**Goal:** prove existence of $M$
Necessary condition

**Theorem (Giesl 2004)**

Consider \( \dot{x} = f(x), f \in C^1(\mathbb{R}^n, \mathbb{R}^n) \).

- \( \Omega \) exponentially asymptotically stable periodic orbit
- \( -\nu_0 < 0 \) largest real part of all non-trivial *Floquet exponents*
- \( K \subset A(\Omega) \) compact with \( \Omega \in \mathring{K} \)

Then for all \( -\nu > -\nu_0 \) there is a *Riemannian metric* \( M \) such that \( L_M(x) \leq -\nu \) for all \( x \in K \)

For \( n = 2 \) one can choose \( M(x) = e^{2W(x)}I \) (weight function), not true for \( n \geq 3 \!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\)

**Proof:**

- local: Floquet theory – need *matrix-valued* Riemannian metric \( M \)
- global: Lyapunov function – need only *scalar-valued* function
Related work

Other systems

Similar results for
- time-periodic systems: $\dot{x} = f(t, x)$ (Giesl 2004)
- time-almost periodic systems (Giesl/Rasmussen 2008)
- non-smooth systems, 1-d in space (Giesl 2005/2007) – jump conditions

Outlook
Non-smooth systems – talk by Pascal Stiefenhofer this afternoon
2.2 Construction of Riemannian metric

- **Construct matrix-valued function** $M : \mathbb{R}^n \to \mathbb{R}^{n \times n}$

- $M$ satisfies inequality (local construction condition) $L_M(x) < 0$ where $L_M(x) := \max_{vT \geq 1, v^T M(x) f(x) = 0} v^T \left[ M(x) Df(x) + \frac{1}{2} M'(x) \right] v$

Radial Basis Functions

- two-dimensional systems: $M(x) = e^{2W(x)} I$, characterisation of $W(x)$ by linear PDE, approximation by Radial Basis Functions (Giesl 2007)

- higher-dimensional systems: combination of Riemannian metric locally (obtained by numerical integration) with Lyapunov function (Giesl 2009)
CPA Construction of Riemannian metric

Idea (similar to construction of Lyapunov function – Hafstein 2004):
- Triangulate phase space into simplices
- Define Riemannian metric $M(x)$ as CPA (continuous piecewise affine) function, affine on each simplex – need values on vertices only
- Formulate contraction condition as semidefinite optimization problem

Challenges:
- Riemannian metric is not differentiable – notion of orbital derivative?
- show contraction property inside simplex using conditions on vertices only
- show equivalence: solution of semidefinite optimization problem $\iff$ contraction metric
- solve semidefinite optimization problem
Triangulation and CPA Riemannian metric

Time-periodic setting:

(2) \[ \dot{x} = f(t, x), \]

- \( f \) periodic, i.e. \( f(t, x) = f(t + T, x) \)
- phase space: cylinder \( S^1_T \times \mathbb{R}^n \)
- \( f \in C^2(S^1_T \times \mathbb{R}^n, \mathbb{R}^n) \)
- (unique) solution with initial value \( x(t_0) = x_0: (t + t_0 \mod T, x(t)) = S_t(t_0, x_0) \in S^1_T \times \mathbb{R}^n \)
- notation: \( \tilde{x} := (t, x) \)

Triangulation of \( \mathcal{C} \subset S^1_T \times \mathbb{R}^n: T_\nu \in \mathcal{T}_K^c \), \( K \) corresponds to number of simplices:
Theorem (CPA Riemannian metric)

- $M \in C^0(S^1_T \times \mathbb{R}^n, \mathbb{R}^{n \times n})$
- $M(t, x)$ symmetric and positive definite
- $M$ locally Lipschitz-continuous with respect to $x$
- forward orbital derivative exists

\[
M'_+(t, x) = \lim_{\theta \to 0^+} \frac{M(S_\theta(t, x)) - M(t, x)}{\theta}
\]

For CPA Riemannian metric $M$ define

\[
L_M(t, x) := \sup_{w \in \mathbb{R}^n, w^T M(t, x) w = 1} L_M(t, x; w)
\]

\[
L_M(t, x; w) := \frac{1}{2} w^T \left[ M(t, x) D_x f(t, x) + D_x f(t, x)^T M(t, x) + M'_+(t, x) \right] w.
\]
Theorem (Giesl/Hafstein 2013)

- $K \subset S^1_T \times \mathbb{R}^n$ connected, compact and positively invariant
- $M$ CPA Riemannian metric
- $L_M(t, x) \leq -\nu < 0$ for all $(t, x) \in K$

Then:

- Existence and uniqueness of periodic orbit $\Omega \subset K$
- Basin of attraction $K \subset A(\Omega)$
- $-\nu$ is upper bound of real parts of non-trivial Floquet exponents

Same proof as in smooth case, technical details.
Semidefinite Optimization problem

Variables

1. $M_{ij}(\tilde{x}_k) \in \mathbb{R}$ for all $1 \leq i \leq j \leq n$ and all vertices $\tilde{x}_k$ – values of the Riemannian metric at vertices

   **Periodicity:** $M_{ij}(0, x_k) = M_{ij}(T, x_k)$

   [note: triangulation respects periodicity]

2. $C \in \mathbb{R}_0^+$ – bound on $M$

3. $D \in \mathbb{R}_0^+$ – bound on gradient of $M$

$2 + \frac{1}{2}n(n + 1)v$ variables, where $v$ is number of vertices.

Constraints

- feasibility problem or minimize $C$ (to obtain bound on largest Floquet exponent)
- linear (3.) and semidefinite (1., 2., 4.) constraints ($n \times n$ positive semidefinite matrices)
- notation: $A \preceq B \iff B - A$ is positive semidefinite
Semidefinite Optimization problem: Statement

1. **Positive definiteness of** $M$: Fix $\epsilon_0 > 0$. $M(\tilde{x}_k) \succeq \epsilon_0 I$
   - all vertices $\tilde{x}_k$

2. **Bound on** $M$: $M(\tilde{x}_k) \preceq CI$
   - all vertices $\tilde{x}_k$

3. **Bound on derivative of** $M$:
   $$|(\nabla \tilde{x} M_{ij} |_{T_\nu}(\tilde{x}))_l| \leq \frac{D}{n+1} \text{ for all } l = 0, \ldots, n$$
   - all simplices $T_\nu \in T^C_K$
   - linear in $M_{ij}$
   - $\nabla \tilde{x} M_{ij} |_{T_\nu}(\tilde{x})$ same for all $\tilde{x}$ in simplex $T_\nu$

   $\nabla \tilde{x} M_{ij} |_{T_\nu}(\tilde{x}) := \begin{pmatrix} (\tilde{x}_1 - \tilde{x}_0)^T \\ (\tilde{x}_2 - \tilde{x}_0)^T \\ \vdots \\ (\tilde{x}_{n+1} - \tilde{x}_0)^T \end{pmatrix}^{-1} \begin{pmatrix} M_{ij}(\tilde{x}_1) - M_{ij}(\tilde{x}_0) \\ \vdots \\ M_{ij}(\tilde{x}_{n+1}) - M_{ij}(\tilde{x}_0) \end{pmatrix}$
Semidefinite Optimization problem: Statement (cont.)

Contraction of the metric:

\[
M(\tilde{x}_k)D_x f(\tilde{x}_k) + D_x f(\tilde{x}_k)^T M(\tilde{x}_k)
+ (\nabla \tilde{x} M_{ij} \big|_{T_{\nu}} (\tilde{x}_k) \cdot \tilde{f}(\tilde{x}_k))_{i,j=1,...,n} \leq -(E_\nu + 1) I
\]

- all simplices \( T_\nu \in \mathcal{T}_K^c \), \( \tilde{f}(\tilde{x}) = \begin{pmatrix} 1 \\ f(\tilde{x}) \end{pmatrix} \), where:
  
  \( E_\nu = h_\nu n B_\nu [\sqrt{n + 1} h_\nu D + 2n(n + 1)C] \)
  
  diameter of simplex \( h_\nu = \text{diam}(T_\nu) \)
  
  bound on second derivatives of \( f \)
  
  \( B_\nu := \max_{\tilde{x} \in T_\nu, i,j \in \{0,...,n\}} \left\| \frac{\partial^2 f(\tilde{x})}{\partial x_i \partial x_j} \right\|_{\infty} , \text{ where } x_0 := t \)
Feasible solution defines CPA contraction metric

Theorem (Giesl/Hafstein 2013)

- All constraints satisfied
- Define CPA metric $M_{ij}(\tilde{x})$ by affine interpolation on each simplex

Then:

1. $M(\tilde{x})$ is symmetric, positive definite and $T$-periodic: thus Riemannian metric
2. $L_M(\tilde{x}) \leq -\frac{1}{2C} < 0$

Hence, $M$ satisfies all assumptions of CPA Theorem.

Proof:

- Estimate difference between $M_{ij}(\tilde{x})D_xf(\tilde{x})$ and convex interpolation $\sum_{k=0}^{n+1} \lambda_k M_{ij}(\tilde{x}_k)D_xf(\tilde{x}_k)$, where $\tilde{x} = \sum_{k=0}^{n+1} \lambda_k \tilde{x}_k$ [also $M'(\cdot)$]
- Use $E_\nu$ to obtain estimate inside simplex
Theorem (Giesl/Hafstein 2013)

- $\dot{x} = f(t, x), \ f \in C^2$ has exponentially stable periodic orbit $\Omega$
- $\Omega \subset \mathcal{C} \subset A(\Omega)$ compact

Then there is a $K^* \in \mathbb{N}$ such that semidefinite optimization problem is feasible for all triangulations $\mathcal{T}_K^C$ with $K \geq K^*$.

Proof:

- Use smooth metric $M$
- Scale $M$ and assign values on vertices
- Choose $K$ large enough so that $h_\nu$ and thus $E_\nu$ is small
Summary – part 2

- basin of attraction of periodic orbit
- contraction metric: adjacent solutions approach each other
- construct suitable contracting Riemannian metric
- CPA (continuous piecewise affine) metric, affine on triangulation
- contraction properties as semidefinite optimization problem

Outlook

- solve semidefinite optimization problem (SDPA, PENSDP)
- periodic orbit in autonomous ODE
- equilibrium in autonomous ODE
Summary

Determination of Basin of Attraction

- Lyapunov function (position known)
- Contraction criterion (position unknown)

Calculation of Lyapunov function using

- suitable PDE
- Radial Basis Function approximation

Calculation of Riemannian metric using

- suitable triangulation
- semidefinite optimization

Outlook

- Combination of Riemannian metric locally and Lyapunov function globally
- Combination of Radial Basis Functions and CPA Optimization methods


