Construction of Lyapunov functions and Contraction Metrics to determine the Basin of Attraction

Peter Giesl

University of Sussex, UK – Department of Mathematics

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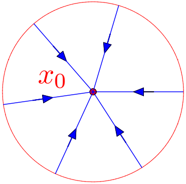
Overview

1. Basin of Attraction of Equilibria
   - Lyapunov Function
   - Radial Basis Functions

2. Basin of Attraction of Periodic Orbits
   - Contraction criterion with Riemannian metric
   - Semidefinite Optimization

System of autonomous ordinary differential equations



(1) $$\begin{cases} \dot{x} & = & f(x) \\ x(0) & = & \xi \end{cases}$$

$x \in \mathbb{R}^n$, $f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ where $\sigma \geq 1$, $n \in \mathbb{N}$
Flow $S_t\xi := x(t)$, solution of (1)

Assumptions

- x_0 is equilibrium ($f(x_0) = 0$)
- x_0 is exponentially asymptotically stable (real parts of all eigenvalues of $Df(x_0)$ are negative)

Definition (Basin of attraction) The basin of attraction of x_0 is

$$A(x_0) := \{\xi \in \mathbb{R}^n \mid S_t\xi \stackrel{t\to\infty}{\longrightarrow} x_0\}.$$

Goal: Determine basin of attraction $A(x_0)$ using a Lyapunov function

1.1 Lyapunov function

Theorem (Lyapunov 1893)

- $v \in C^1(\mathbb{R}^n, \mathbb{R})$
- $K \subset \mathbb{R}^n$ *compact set*

1. $v'(x) = \frac{d}{dt} v(x(t))\big|_{t=0} = \langle \nabla v(x), f(x) \rangle < 0$ *for all* $x \in K \setminus \{x_0\}$
   (orbital derivative = derivative along a solution)
2. $K = \{x \in \mathbb{R}^n \mid v(x) \leq R\}$ *sublevel set of* v

Then $K \subset A(x_0)$.

Existence of Lyapunov functions

"Converse Theorems" (Massera 1949) etc. – but not constructive!

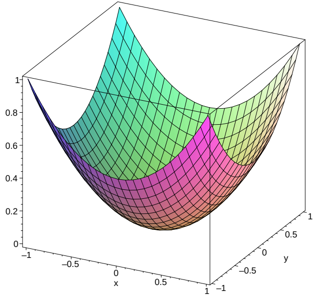
Goal: explicit calculation of Lyapunov function

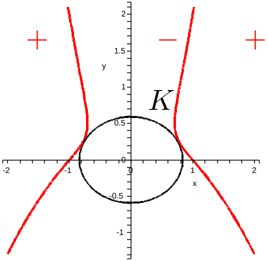
Example

$$\begin{cases} \dot{x} &= -x + x^3 \\ \dot{y} &= -\frac{1}{2}y + x^2 \end{cases}$$

$v(x,y) = \frac{1}{2}x^2 + y^2$

sign of $v'(x,y)$





Construction of a Lyapunov function (Giesl 2007 LNM)

- Suitable linear PDE for V, e.g.

$$V'(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial V}{\partial x_i}(x) = -\|x - x_0\|^2$$

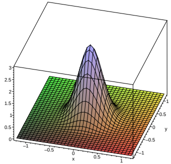
- Approximation v of V using Radial Basis Functions
- Error estimate ensures $v'(x) < 0$
- The approximation v itself is a Lyapunov function
- Sublevel set of v is a subset of $A(x_0)$

Radial Basis Functions: advantages

- meshless method: scattered data
- any dimension
- smooth approximation

1.2 Radial Basis Functions

- PDE $LV(x) = g(x)$, L linear differential operator (orbital derivative)
  $Lu(x) = \sum_{|\alpha| \leq m} c_\alpha(x) D^\alpha u(x)$



- $\Phi(x) = \psi_k(\|x\|)$ (Radial Basis Function), here:
  ψ_k Wendland's function (compact support)
- (Scattered) grid points $X_N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^n$
- (Symmetric) ansatz: $v(x) = \sum_{k=1}^N \beta_k (\delta_{x_k} \circ L)^y \Phi(x - y)$
- Interpolation conditions: $Lv(x_j) = LV(x_j) = g(x_j)$, i.e.
  $(\delta_{x_j} \circ L)^x v(x) = (\delta_{x_j} \circ L)^x V(x)$ for all $j = 1, \ldots, N$
- Plug in the ansatz:
  $\sum_{k=1}^N \beta_k \underbrace{(\delta_{x_j} \circ L)^x (\delta_{x_k} \circ L)^y \Phi(x - y)}_{= a_{jk}} = LV(x_j) = g(x_j) =: \gamma_j$
- System of linear equations $A\beta = \gamma$, A is symmetric
- A is positive definite $\Rightarrow$ non-singular

Steps of the method

1. Find a suitable linear PDE, existence and smoothness of solution
   There exists a Lyapunov function $V \in C^\sigma(A(x_0), \mathbb{R})$ with

$$V'(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial V}{\partial x_i}(x) = -\|x - x_0\|^2 \text{ for all } x \in A(x_0)$$

   (Zubov 1964): $V(x) = \int_0^\infty \|S_t x - x_0\|^2 \, dt$
2. Positive definiteness of matrix A
   Assumption: grid points are distinct and no equilibria
   Proof via positive Fourier-transform $\hat{\Phi}(\omega) > 0$
3. Error estimate on v'
4. Estimate of the level sets of v: method covers every compact subset of the basin of attraction
5. Solve local problem

Error Estimates

Theorem (Giesl 2007, Giesl & Wendland 2007)

Use Wendland's compactly supported functions as Radial Basis Functions:
$\Phi(x) = \psi_k(\|x\|) \in C^{2k}(\mathbb{R}^n, \mathbb{R})$, $k \in \mathbb{N}$ *(Wendland 1998)*
Let $f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ *where* $\sigma \geq \frac{n+1}{2} + k$.
For each compact set $K_0 \subset A(x_0)$ *there is* C *such that*

$$|V'(x) - v'(x)| \leq Ch^{k-1/2} \text{ for all } x \in K_0$$

$h := \max_{y \in K_0} \min_{x \in X_N} \|x - y\|$: *fill distance*

Example of Wendland function:

$$\psi_2(r) = \begin{cases} (1-r)^6[35r^2 + 18r + 3] & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \end{cases}$$

Solution of the local problem

Approximation v is (nearly) a Lyapunov function:

- $V'(x) = -\|x - x_0\|^2$
- $|v'(x) - V'(x)| \leq \epsilon$ error estimate

implies $v'(x) \leq V'(x) + \epsilon = -\|x - x_0\|^2 + \epsilon < 0$ except for x near x_0

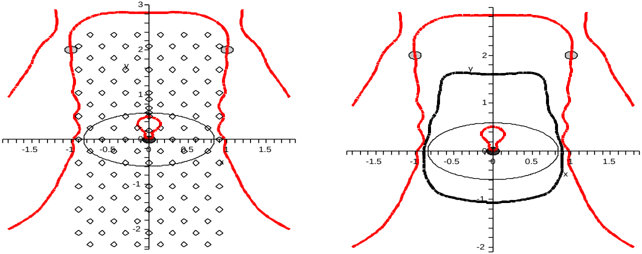
$v'(x)$ is in general positive near x_0

Solution: Linearisation
local Lyapunov function $V_{loc}(x) = (x - x_0)^T B(x - x_0)$ from linearised equation $\dot{x} = Df(x_0)x \Rightarrow$ local basin of attraction

Example

$$\begin{cases} \dot{x} &=& -x + x^3 \\ \dot{y} &=& -\frac{1}{2}y + x^2 \end{cases}$$

Grid, $v' = 0$, sublevel sets: local (thin black) and calculated (thick black)



Summary – part 1

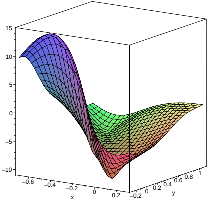
- **Goal:** Basin of attraction $A(x_0)$
- **Instrument:** Lyapunov function v and sublevel set K
- **Construction:** Approximation v of V using Radial Basis Functions
- **Result:** $v'(x) < 0$ if grid dense enough
- **Global:** every compact set within $A(x_0)$ can be covered
- **Local:** use local Lyapunov function (linearisation)

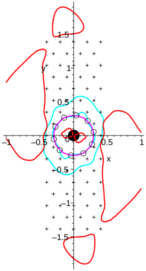
Outlook

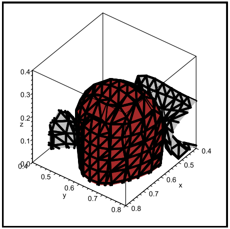
Grid refinement – talk by Najla Mohammed this afternoon
Combination with CPA method – talk by Sigurdur Hafstein

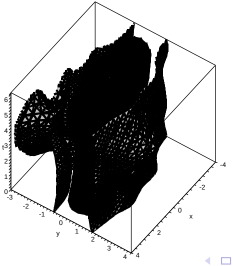
Variations

- different PDE, e.g. $V'(x) = -1$
- boundary conditions
  mixed approximation
  (given the values of V' and V)
- discrete dynamical system:
  $x_{n+1} = g(x_n)$
- time-periodic ODEs
  (with Holger Wendland)

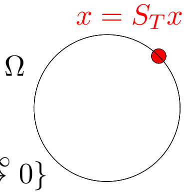








2. Basin of Attraction of Periodic Orbits



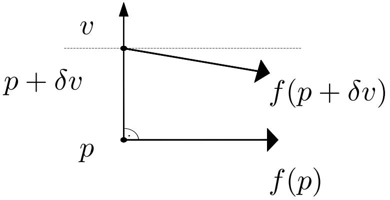
- $\dot{x} = f(x)$, $f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$: autonomous ODE
- periodic orbit $\Omega = \{S_t x \mid t \in [0, T)\}$ with $x = S_T x$
  minimal $T > 0$: period
- basin of attraction $A(\Omega) = \{\xi \in \mathbb{R}^n \mid \text{dist}(S_t \xi, \Omega) \stackrel{t \to \infty}{\longrightarrow} 0\}$

Disadvantage of Lyapunov function:

- $V(x) = 0$ and $V'(x) = 0$ for all $x \in \Omega$ and
  $V(x) > 0$ and $V'(x) < 0$ for all $x \notin \Omega$,
  but position of the periodic orbit not (exactly) known
- local structure (solution of first variation equation) not known

Goal: Determine the basin of attraction $A(\Omega)$ without (exact) knowledge of the periodic orbit via (local) contraction criterion

2.1 Contraction criterion



$$\|v\| = 1, \; v \perp f(p)$$

$$\begin{aligned} L(p,v) &= v^T Df(p)\, v \\ L(p) &= \max_{\|v\|=1, v \perp f(p)} L(p,v) \end{aligned}$$

- $L(p,v) < 0$:
  Trajectories through p and $p + \delta v$ ($\delta > 0$ small) approach each other
- $L(p) < 0$:
  Trajectories through p and adjacent points approach each other

Sufficient condition

Theorem (Borg 1960)

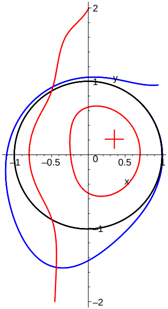
- $\varnothing \neq K \subset \mathbb{R}^n$ *includes no equilibria, is positively invariant, compact and connected*
- $L(p) \leq -\nu < 0$ *for all* $p \in K$

Then

- *Existence and uniqueness of an exponentially asymptotically stable periodic orbit* $\Omega \subset K$
- $K \subset A(\Omega)$ *(basin of attraction)*
- $-\nu$ *is upper bound of real parts of non-trivial Floquet exponents*

Question: Is this condition also necessary?

Example $\begin{cases} \dot{x} &= x(1-x^2-y^2)\left(x+\frac{1}{2}\right)-y \\ \dot{y} &= y(1-x^2-y^2)\left(x+\frac{1}{2}\right)+x \end{cases}$



- Unit sphere Ω is an exponentially asymptotically stable periodic orbit
- but $L(p) < 0$ does not hold for all $p \in \Omega$
- criterion is not necessary!

Reason: non-monotone approach of adjacent solutions

Idea: modify contraction condition using a weighted distance/different metric

Riemannian metric

Definition (Riemannian metric)

A matrix-valued function $M \in C^1(\mathbb{R}^n, \mathbb{R}^{n\times n})$ is called Riemannian metric if $M(x)$ is a symmetric and positive definite matrix for each $x \in \mathbb{R}^n$.

Note: Then $\langle v, w\rangle_{M(x)} := v^T M(x) w$ defines a point-dependent scalar product for $v, w \in \mathbb{R}^n$.

Examples

- $M(x) = I$: Riemannian metric
- $M(x) = e^{2W(x)} I$: weighted distance

Sufficient condition (with Riemannian metric)

Theorem (Stenström 1962)

- $\emptyset \neq K \subset \mathbb{R}^n$ *includes no equilibria, is positively invariant, compact and connected*
- *Riemannian metric* $M \in C^1(\mathbb{R}^n, \mathbb{R}^{n \times n})$
- $L_M(x) \leq -\nu < 0$ *for all* $x \in K$ *where*
  $L_M(x) := \max_{v^T M(x) v = 1, v^T M(x) f(x) = 0} v^T \left[M(x) Df(x) + \frac{1}{2} M'(x) \right] v$

Then

- *Existence and uniqueness of an exponentially asymptotically stable periodic orbit* $\Omega \subset K$
- $K \subset A(\Omega)$ (*basin of attraction*)
- $-\nu$ *is upper bound of real parts of non-trivial Floquet exponents*

Example: $L_M(x) = L(x) + W'(x)$ for $M(x) = e^{2W(x)} I$

Goal: prove existence of M

Necessary condition

Theorem (Giesl 2004)

Consider $\dot{x} = f(x)$, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

- Ω *exponentially asymptotically stable periodic orbit*
- $-\nu_0 < 0$ *largest real part of all non-trivial Floquet exponents*
- $K \subset A(\Omega)$ *compact with* $\Omega \in \overset{\circ}{K}$

Then for all $-\nu > -\nu_0$ *there is a Riemannian metric* M *such that* $L_M(x) \leq -\nu$ *for all* $x \in K$

For $n = 2$ one can choose $M(x) = e^{2W(x)} I$ (weight function), not true for $n \geq 3$!

Proof:

- local: Floquet theory – need matrix-valued Riemannian metric M
- global: Lyapunov function – need only scalar-valued function

Related work

Other systems

Similar results for

- time-periodic systems: $\dot{x} = f(t, x)$ (Giesl 2004)
- time-almost periodic systems (Giesl/Rasmussen 2008)
- non-smooth systems, 1-d in space (Giesl 2005/2007) – jump conditions

Outlook

Non-smooth systems – talk by Pascal Stiefenhofer this afternoon

- Construct matrix-valued function $M\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$
- M satisfies inequality (local construction condition) $L_M(x) < 0$ where $L_M(x) := \max_{v^T M(x) v = 1, v^T M(x) f(x) = 0} v^T \left[M(x)Df(x) + \frac{1}{2}M'(x)\right] v$

Radial Basis Functions

- two-dimensional systems: $M(x) = e^{2W(x)} I$, characterisation of $W(x)$ by linear PDE, approximation by Radial Basis Functions (Giesl 2007)
- higher-dimensional systems: combination of Riemannian metric locally (obtained by numerical integration) with Lyapunov function (Giesl 2009)

CPA Construction of Riemannian metric

Idea (similar to construction of Lyapunov function – Hafstein 2004):

- Triangulate phase space into simplices
- Define Riemannian metric $M(x)$ as CPA (continuous piecewise affine) function, affine on each simplex – need values on vertices only
- Formulate contraction condition as semidefinite optimization problem

Challenges:

- Riemannian metric is not differentiable – notion of orbital derivative?
- show contraction property inside simplex using conditions on vertices only
- show equivalence: solution of semidefinite optimization problem $\Leftrightarrow$ contraction metric
- solve semidefinite optimization problem

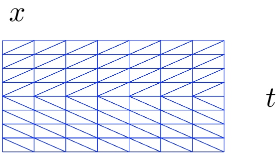
Triangulation and CPA Riemannian metric

Time-periodic setting:

(2) $$\dot{x} = f(t,x),$$

- f periodic, i.e. $f(t,x) = f(t+T,x)$
- phase space: cylinder $S^1_T \times \mathbb{R}^n$
- $f \in C^2(S^1_T \times \mathbb{R}^n, \mathbb{R}^n)$
- (unique) solution with initial value $x(t_0) = x_0$:
  $(t + t_0 \bmod T, x(t)) =: S_t(t_0, x_0) \in S^1_T \times \mathbb{R}^n$
- notation: $\tilde{x} := (t,x)$

Triangulation of $\mathcal{C} \subset S^1_T \times \mathbb{R}^n$: $T_\nu \in \mathcal{T}^{\mathcal{C}}_K$,
K corresponds to number of simplices:



CPA contraction metric and implications

Theorem (CPA Riemannian metric)

- $M \in C^0(S^1_T \times \mathbb{R}^n, \mathbb{R}^{n\times n})$
- $M(t,x)$ *symmetric and positive definite*
- M *locally Lipschitz-continuous with respect to* x
- *forward orbital derivative exists*

$$M'_+(t,x) = \lim_{\theta \to 0^+} \frac{M(S_\theta(t,x)) - M(t,x)}{\theta}$$

For CPA Riemannian metric M define

$$L_M(t,x) := \sup_{w\in\mathbb{R}^n, w^T M(t,x) w = 1} L_M(t,x;w)$$

$$L_M(t,x;w) := \frac{1}{2} w^T \left[M(t,x) D_x f(t,x) + D_x f(t,x)^T M(t,x) + M'_+(t,x) \right] w.$$

Theorem for CPA contraction metric

Theorem (Giesl/Hafstein 2013)

- $K \subset S^1_T \times \mathbb{R}^n$ *connected, compact and positively invariant*
- M *CPA Riemannian metric*
- $L_M(t,x) \leq -\nu < 0$ *for all* $(t,x) \in K$

Then:

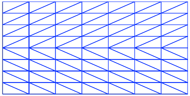
- *Existence and uniqueness of periodic orbit* $\Omega \subset K$
- *Basin of attraction* $K \subset A(\Omega)$
- $-\nu$ *is upper bound of real parts of non-trivial Floquet exponents*

Same proof as in smooth case, technical details.

Semidefinite Optimization problem

Variables

1. $M_{ij}(\tilde{x}_k) \in \mathbb{R}$ for all $1 \leq i \leq j \leq n$ and all vertices $\tilde{x}_k$ – values of the Riemannian metric at vertices
   Periodicity: $M_{ij}(0, x_k) = M_{ij}(T, x_k)$
   [note: triangulation respects periodicity]



2. $C \in \mathbb{R}_0^+$ – bound on M
3. $D \in \mathbb{R}_0^+$ – bound on gradient of M

$2 + \frac{1}{2}n(n+1)v$ variables, where v is number of vertices.

Constraints

- feasibility problem or minimize C (to obtain bound on largest Floquet exponent)
- linear (3.) and semidefinite (1., 2., 4.) contraints ($n \times n$ positive semidefinite matrices)
- notation: $A \preceq B \Leftrightarrow B - A$ is positive semidefinite

Semidefinite Optimization problem: Statement

1. **Positive definiteness of M:** Fix $\epsilon_0 > 0$. $M(\tilde{x}_k) \succeq \epsilon_0 I$
   – all vertices $\tilde{x}_k$
2. **Bound on M:** $M(\tilde{x}_k) \preceq CI$
   – all vertices $\tilde{x}_k$
3. **Bound on derivative of M:**
   $|(\nabla_{\tilde{x}} M_{ij}\big|_{T_\nu}(\tilde{x}))_l| \leq \frac{D}{n+1}$ for all $l = 0, \ldots, n$
   – all simplices $T_\nu \in \mathcal{T}_K^{\mathcal{C}}$
   – linear in M_{ij}
   – $\nabla_{\tilde{x}} M_{ij}\big|_{T_\nu}(\tilde{x})$ same for all $\tilde{x}$ in simplex T_ν

$$\nabla_{\tilde{x}} M_{ij}\big|_{T_\nu}(\tilde{x}) := \begin{pmatrix} (\tilde{x}_1 - \tilde{x}_0)^T \\ (\tilde{x}_2 - \tilde{x}_0)^T \\ \vdots \\ (\tilde{x}_{n+1} - \tilde{x}_0)^T \end{pmatrix}^{-1} \begin{pmatrix} M_{ij}(\tilde{x}_1) - M_{ij}(\tilde{x}_0) \\ \vdots \\ M_{ij}(\tilde{x}_{n+1}) - M_{ij}(\tilde{x}_0) \end{pmatrix}$$

4. **Contraction of the metric:**

$$M(\tilde{x}_k)D_x f(\tilde{x}_k) + D_x f(\tilde{x}_k)^T M(\tilde{x}_k)$$
$$+(\nabla_{\tilde{x}} M_{ij}\big|_{T_\nu}(\tilde{x}_k) \cdot \tilde{f}(\tilde{x}_k))_{i,j=1,\dots,n} \preceq -(E_\nu + 1)I$$

– all simplices $T_\nu \in \mathcal{T}_K^{\mathcal{C}}$, $\tilde{f}(\tilde{x}) = \begin{pmatrix} 1 \\ f(\tilde{x}) \end{pmatrix}$, where:

- $E_\nu = h_\nu n B_\nu[\sqrt{n+1} h_\nu D + 2n(n+1)C]$
- diameter of simplex $h_\nu = \mathrm{diam}(T_\nu)$
- bound on second derivatives of f
  $B_\nu := \max_{\tilde{x} \in T_\nu, i,j \in \{0,\dots,n\}} \left\| \frac{\partial^2 f(\tilde{x})}{\partial x_i \partial x_j} \right\|_\infty$, where $x_0 := t$

Feasible solution defines CPA contraction metric

Theorem (Giesl/Hafstein 2013)

- *All constraints satisfied*
- *Define CPA metric* $M_{ij}(\tilde{x})$ *by affine interpolation on each simplex*

Then:

1. $M(\tilde{x})$ *is symmetric, positive definite and* T*-periodic: thus Riemannian metric*
2. $L_M(\tilde{x}) \leq -\frac{1}{2C} < 0$

Hence, M *satisfies all assumptions of CPA Theorem.*

Proof:

- Estimate difference between $M_{ij}(\tilde{x})D_x f(\tilde{x})$ and convex interpolation $\sum_{k=0}^{n+1} \lambda_k M_{ij}(\tilde{x}_k) D_x f(\tilde{x}_k)$, where $\tilde{x} = \sum_{k=0}^{n+1} \lambda_k \tilde{x}_k$ [also $M'(\cdot)$]
- Use E_ν to obtain estimate inside simplex

Existence of feasible solution

Theorem (Giesl/Hafstein 2013)

- $\dot{x} = f(t,x)$, $f \in C^2$ *has exponentially stable periodic orbit* Ω
- $\Omega \subset \mathcal{C} \subset A(\Omega)$ *compact*

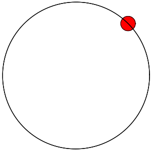
Then there is a $K^* \in \mathbb{N}$ *such that semidefinite optimization problem is feasible for all triangulations* $\mathcal{T}_K^{\mathcal{C}}$ *with* $K \geq K^*$*.*

Proof:

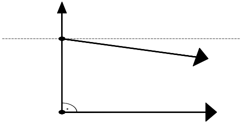
- Use smooth metric M
- Scale M and assign values on vertices
- Choose K large enough so that h_ν and thus E_ν is small

Summary – part 2

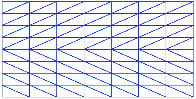
- basin of attraction of periodic orbit



- contraction metric: adjacent solutions approach each other



- construct suitable contracting Riemannian metric
- CPA (continuous piecewise affine) metric, affine on triangulation



- contraction properties as semidefinite optimization problem

Outlook

- solve semidefinite optimization problem (SDPA, PENSDP)
- periodic orbit in autonomous ODE
- equilibrium in autonomous ODE

Summary

Determination of Basin of Attraction

- Lyapunov function (position known)
- Contraction criterion (position unknown)

Calculation of Lyapunov function using

- suitable PDE
- Radial Basis Function approximation

Calculation of Riemannian metric using

- suitable triangulation
- semidefinite optimization

Outlook

- Combination of Riemannian metric locally and Lyapunov function globally
- Combination of Radial Basis Functions and CPA Optimization methods

References

- G. Borg: A condition for the existence of orbitally stable solutions of dynamical systems. Kungl. Tekn. Högsk. Handl. **153** (1960).
- P. Giesl: Necessary conditions for a limit cycle and its basin of attraction. *Nonlinear Anal.* **56** (2004) 5, 643–677.
- P. Giesl: On the Basin of Attraction of Limit Cycles in Periodic Differential Equations. *Z. Anal. Anwendungen* **23** (2004) 3, 547–576.
- P. Giesl, *Construction of Global Lyapunov Functions Using Radial Basis Functions*, Lecture Notes in Math. 1904, Springer, 2007.
- P. Giesl & S. Hafstein: Construction of a CPA contraction metric for periodic orbits using semidefinite optimization, *Nonlinear Anal.* **86** (2013), 114–134.
- P. Giesl & H. Wendland, *Meshless Collocation: Error Estimates with Application to Dynamical Systems*, SIAM J. Numer. Anal. 45 No. 4 (2007), 1723–1741.