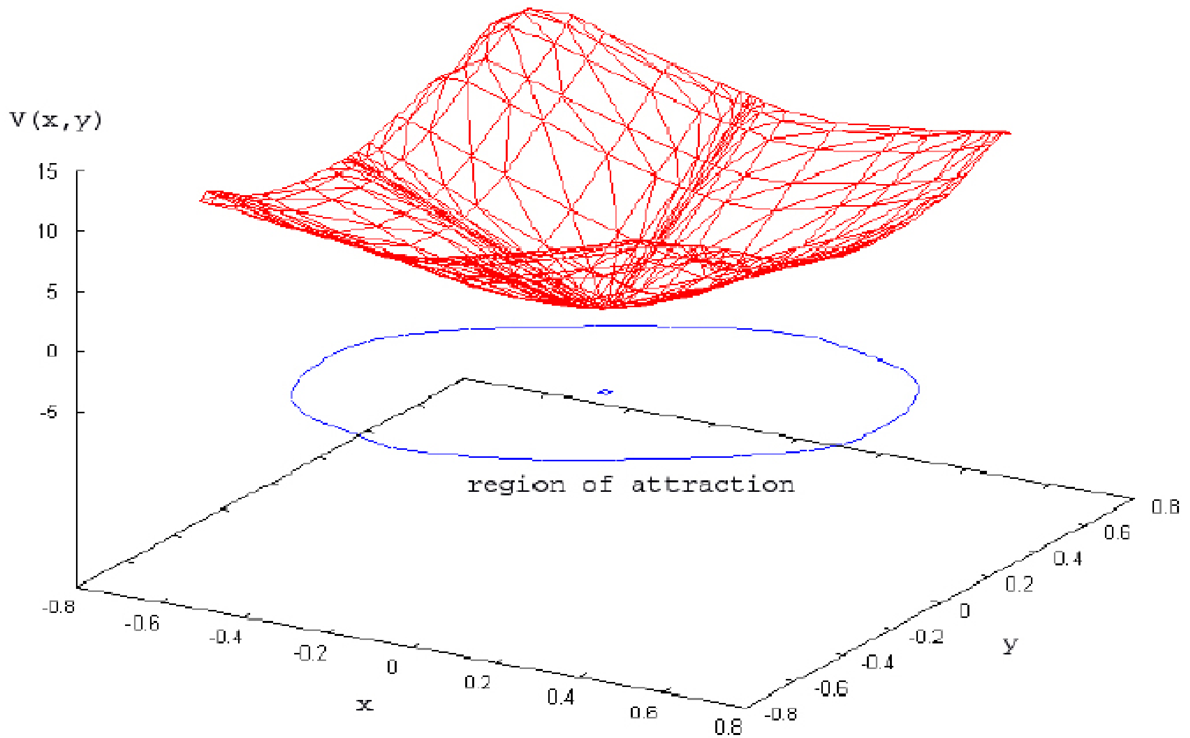
DIFFURJÖFNUR, VARÐVEISLULÖGMÁL OG NÚNINGSLÖGMÁL

SIGURÐUR FREYR HAFSTEIN, DÓSENT

TÆKNI- OG VERKFRÆÐIDEILD | FYRIRLESTRAMARAÞON HR

Construction of Lyapunov functions with linear optimization



$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}), \quad p \in \{1,2,3\}, \qquad \mathbf{f}_1(x,y) := \begin{pmatrix} -y \\ x - y(1-x^2+0.1x^4) \end{pmatrix},$$

$$\mathbf{f}_2(x,y) := \begin{pmatrix} -y + x(x^2+y^2-1) \\ x + y(x^2+y^2-1) \end{pmatrix}, \quad \mathbf{f}_3(x,y) := \begin{pmatrix} -1.5y \\ \frac{x}{1.5} + y\left(\left(\frac{x}{1.5}\right)^2 + y^2 - 1\right) \end{pmatrix}$$

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$$\dot{x}_1 = x_3 \cdot \cos^2(x_2^4) - \exp(x_1)$$
$$\dot{x}_2 = -x_1^2 \cdot \sin(x_2 - x_3) + 4x_4$$
$$\dot{x}_3 = x_2 + x_3 - x_4^2 + \cosh(x_1)$$
$$\dot{x}_4 = -\sin(x_2^2 - x_3^2) + \cos(x_1)$$

What can we do to get information about the solution?

- Analytical solution (almost never possible)
- Numerical solution (not applicable for the general solution, bad approximation for large times for special solutions)
- Search for traps in the phase-space ☺
  (trap = forward invariant set)

Dynamical systems

Let $\phi_{\xi}(t)$ be the solution to the idealized closed physical system

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x})$$

Then, by conservation of energy, we have

$$E(\phi_{\xi}(t_1)) = E(\phi_{\xi}(t_2)) \quad \forall t_1, t_2$$

or equivalently

$$\frac{d}{dt} E(\phi_{\xi}(t)) = 0$$

Dynamical systems

Let $\phi_{\xi}(t)$ be the solution to the non-idealized closed physical system

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x})$$

Then, by dissipation of energy, we have

$$E(\phi_{\xi}(t_1)) > E(\phi_{\xi}(t_2)) \quad \forall (t_1 < t_2)$$

or equivalently

$$\frac{d}{dt} E(\phi_{\xi}(t)) < 0$$

Energy vs. Lyapunov-functions

Real physical systems end up in a state where the energy of the system is at a local minimum. Such a state is called a stable equilibrium

$$\mathrm{f}(\mathrm{y}) = 0 \quad \text{and} \quad E(\mathrm{y}) \longrightarrow \text{local minimum}$$

If we have a differential equation that does not possess an energy, can we do something similar?

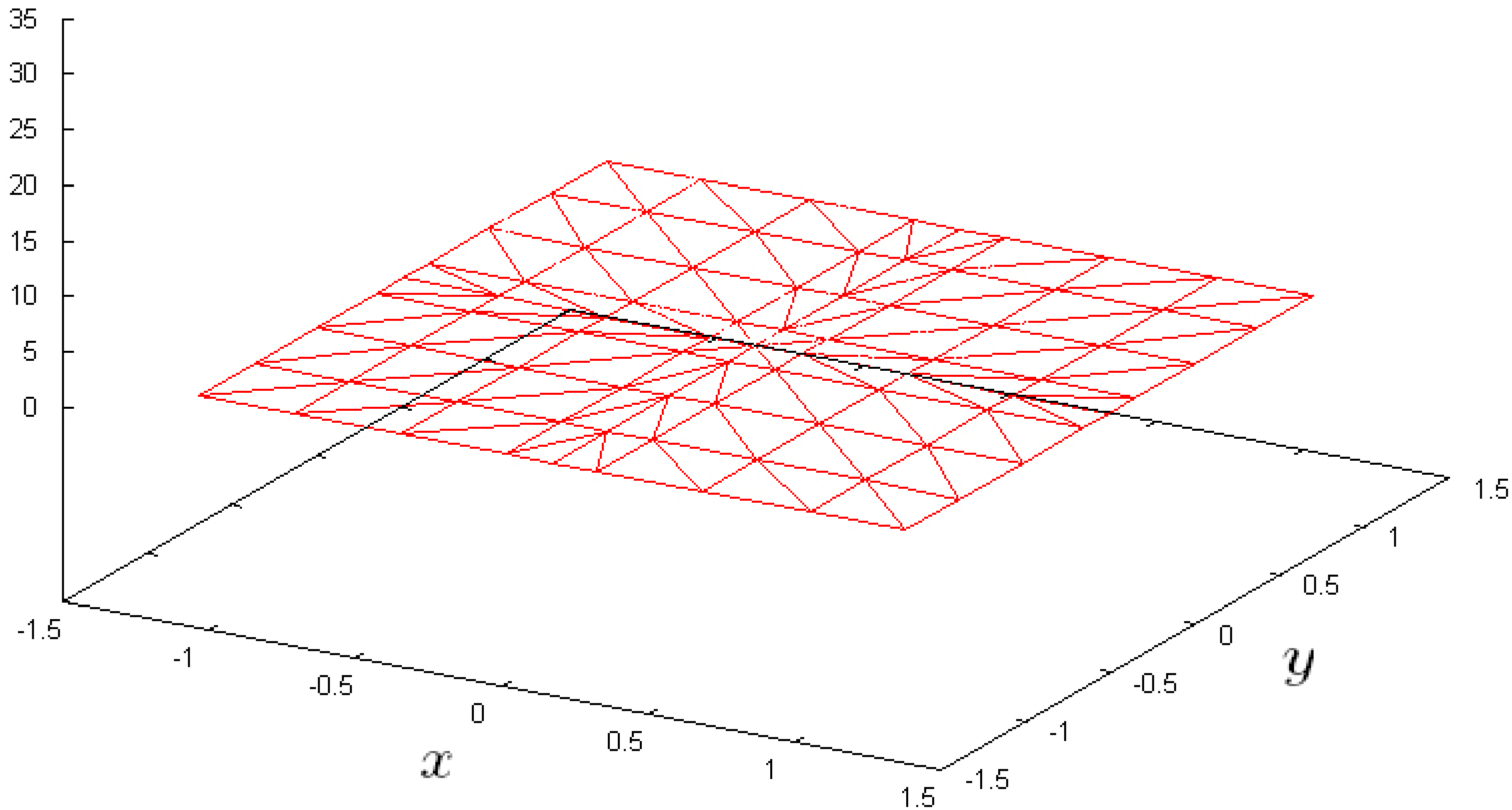
Answer by Lyapunov 1892: if similar to energy
Kurzweil/Massera 1950‘s: such an energy exists

YES !

Example:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \text{where} \quad \mathbf{f}(x,y) := \begin{pmatrix} y \\ -x + x^3/3 - y \end{pmatrix}$$

Partition of the domain of V:



Example:

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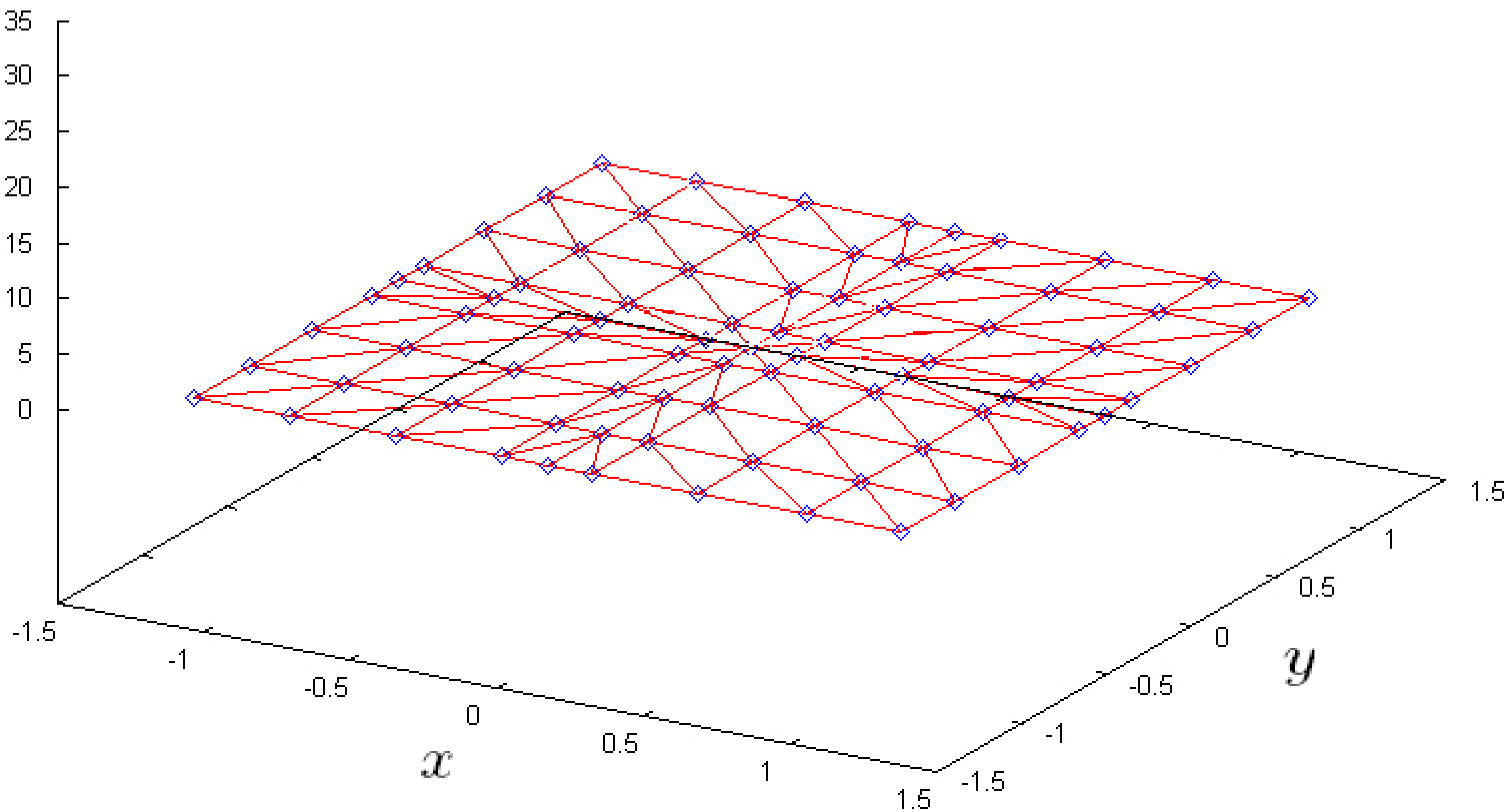
Grid $\mathcal{G}$: ⋄



Example:

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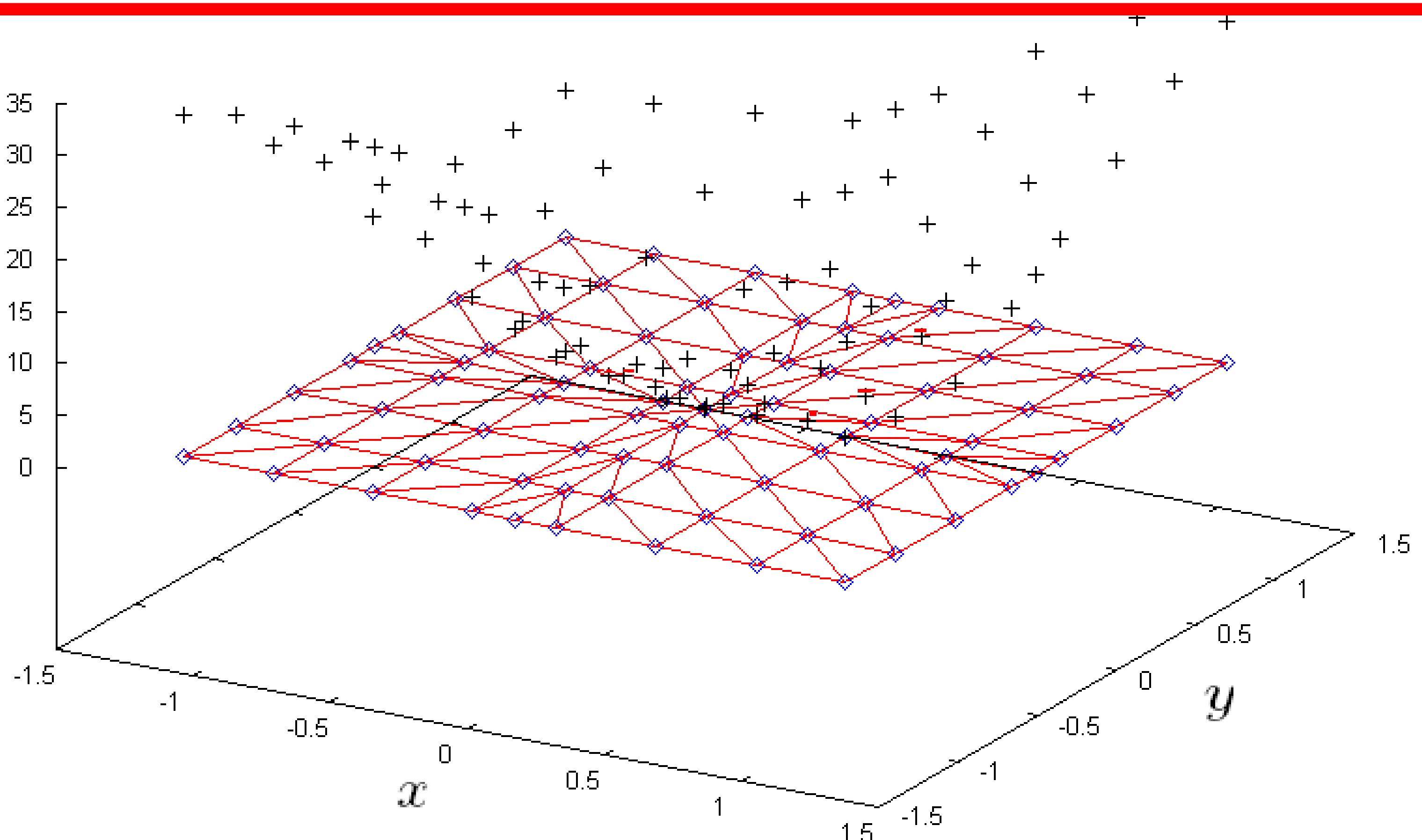
Values for $V[\mathbf{x}], \forall \mathbf{x} \in \mathcal{G}$, that fulfill the constraints



Example:

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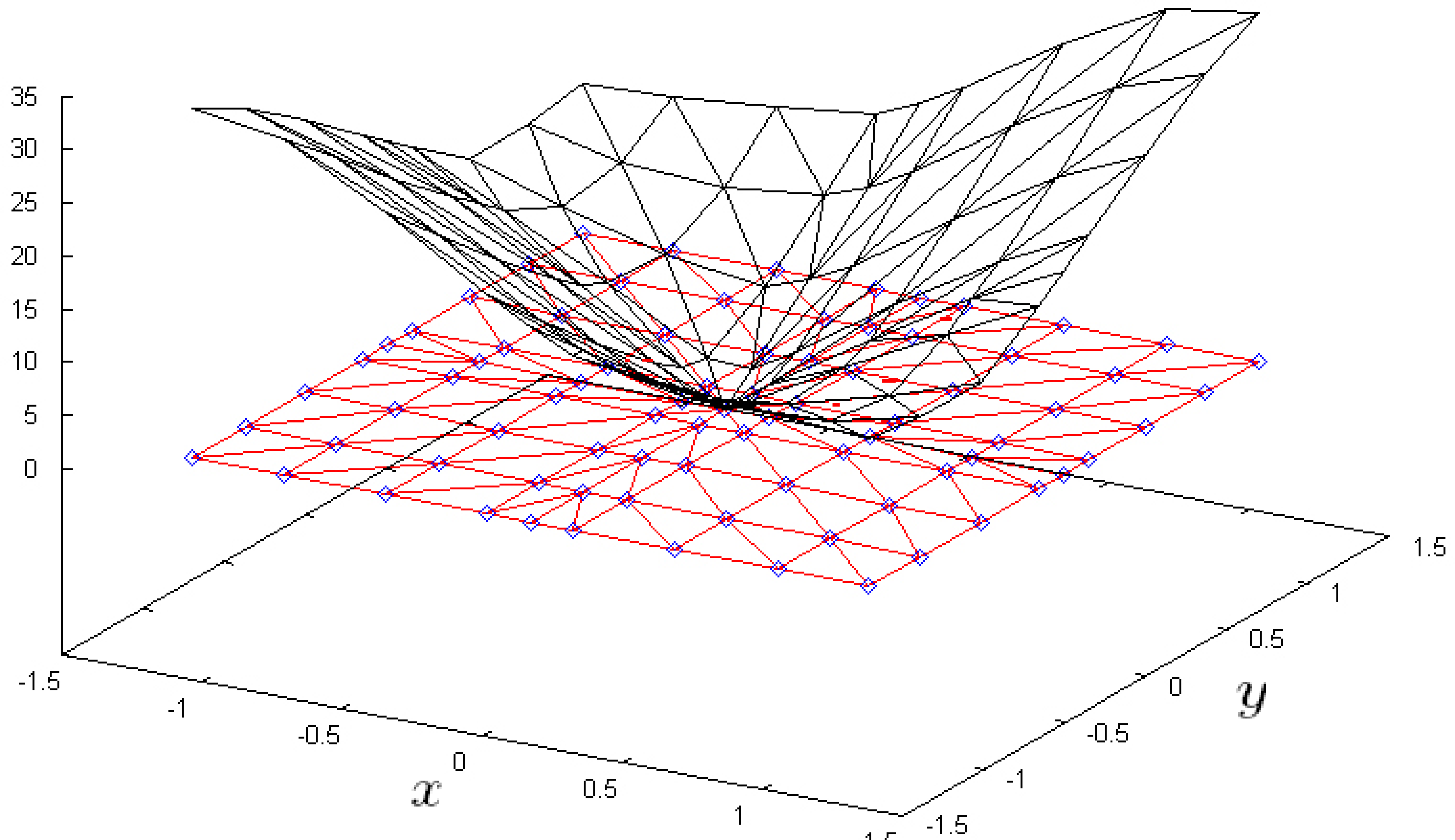
Convex interpolation delivers a Lyapunov-function



Example:

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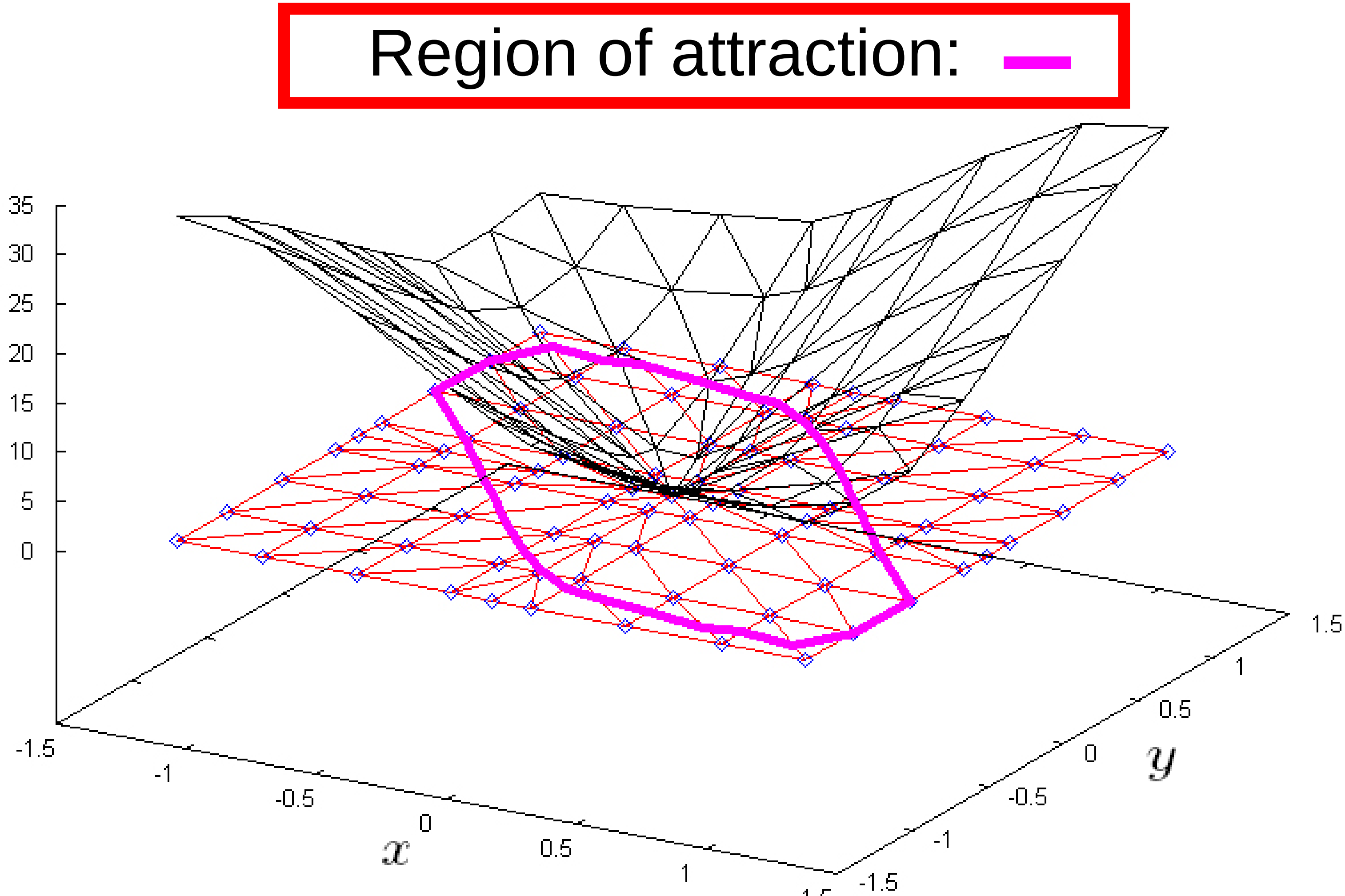
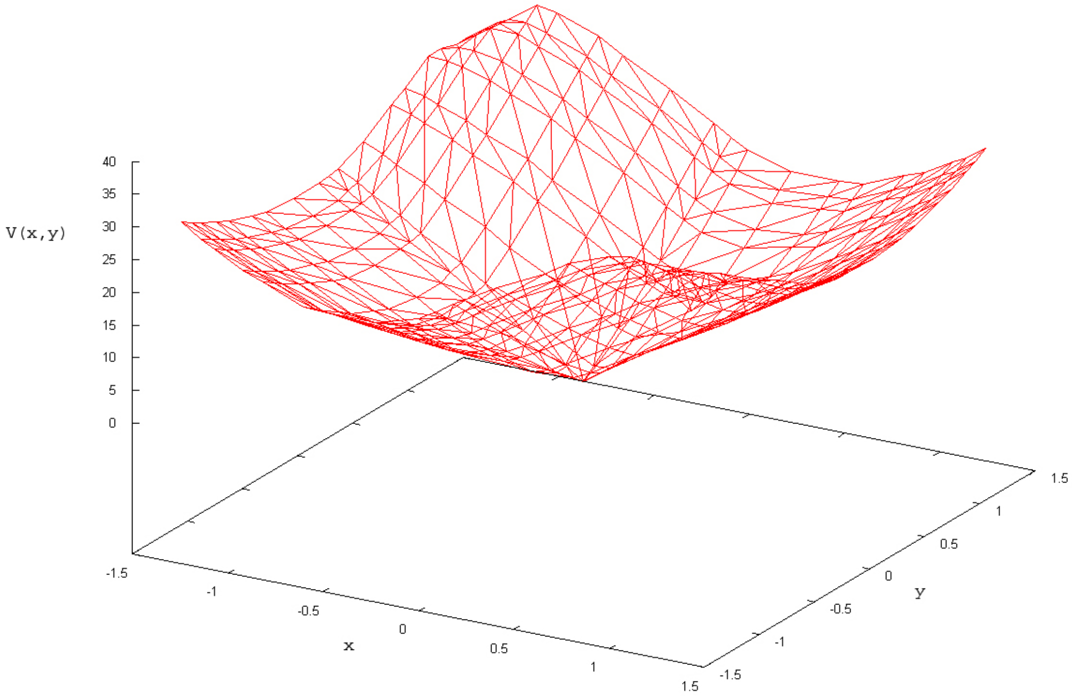
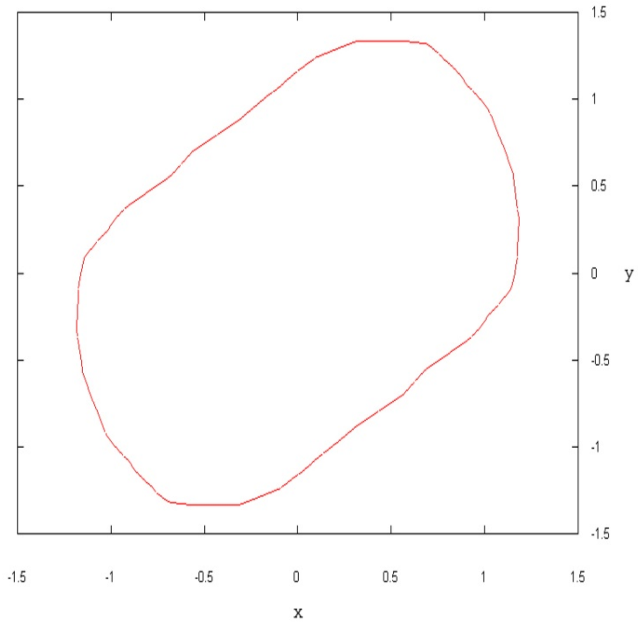


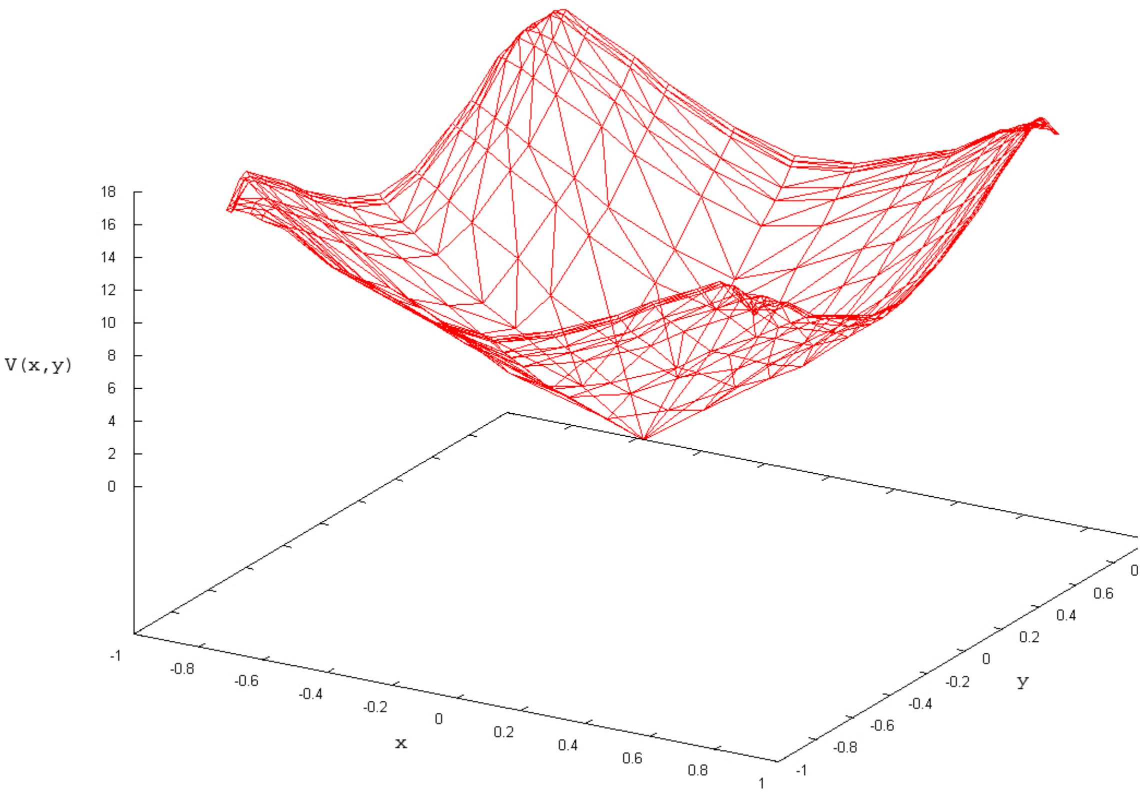
Generated Lyapunov-function

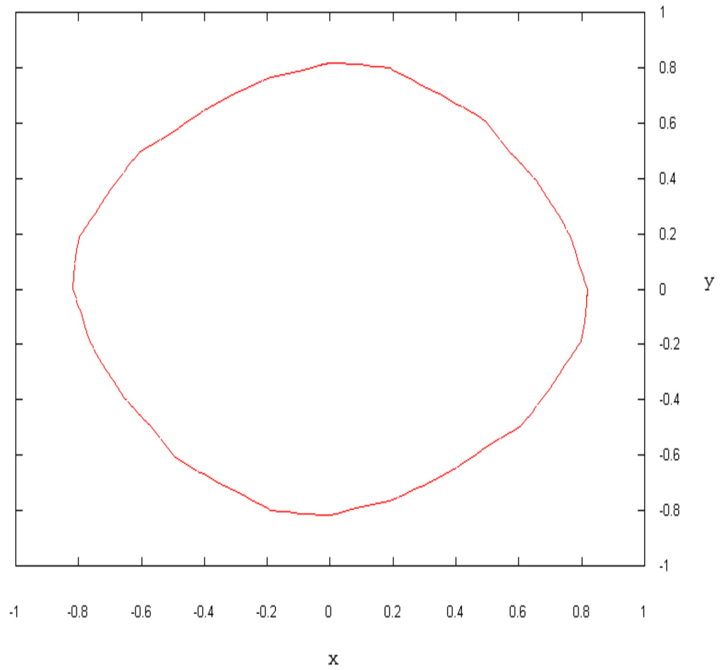




$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}), \quad \text{where} \quad \mathbf{f}_1(x,y) := \begin{pmatrix} -y \\ x - y(1 - x^2 + 0.1x^4) \end{pmatrix}$$

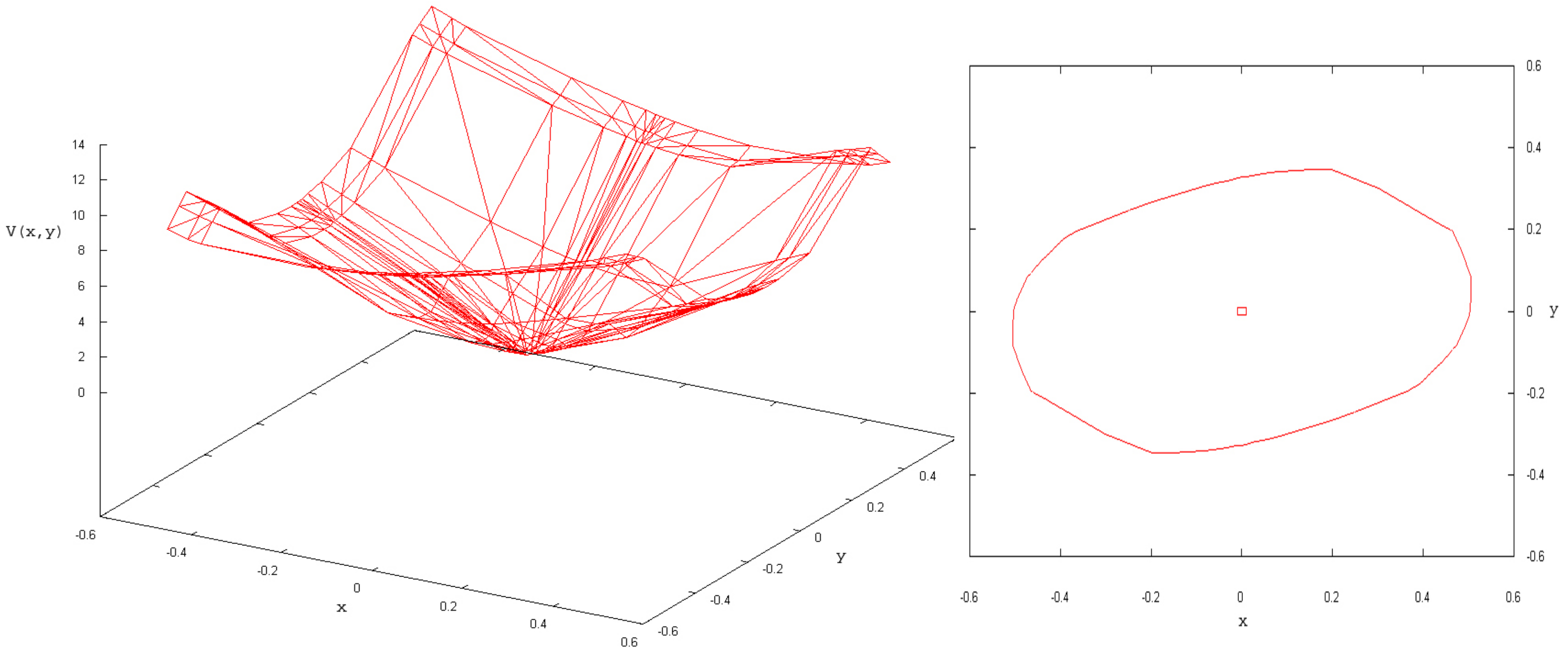
Generated Lyapunov-function





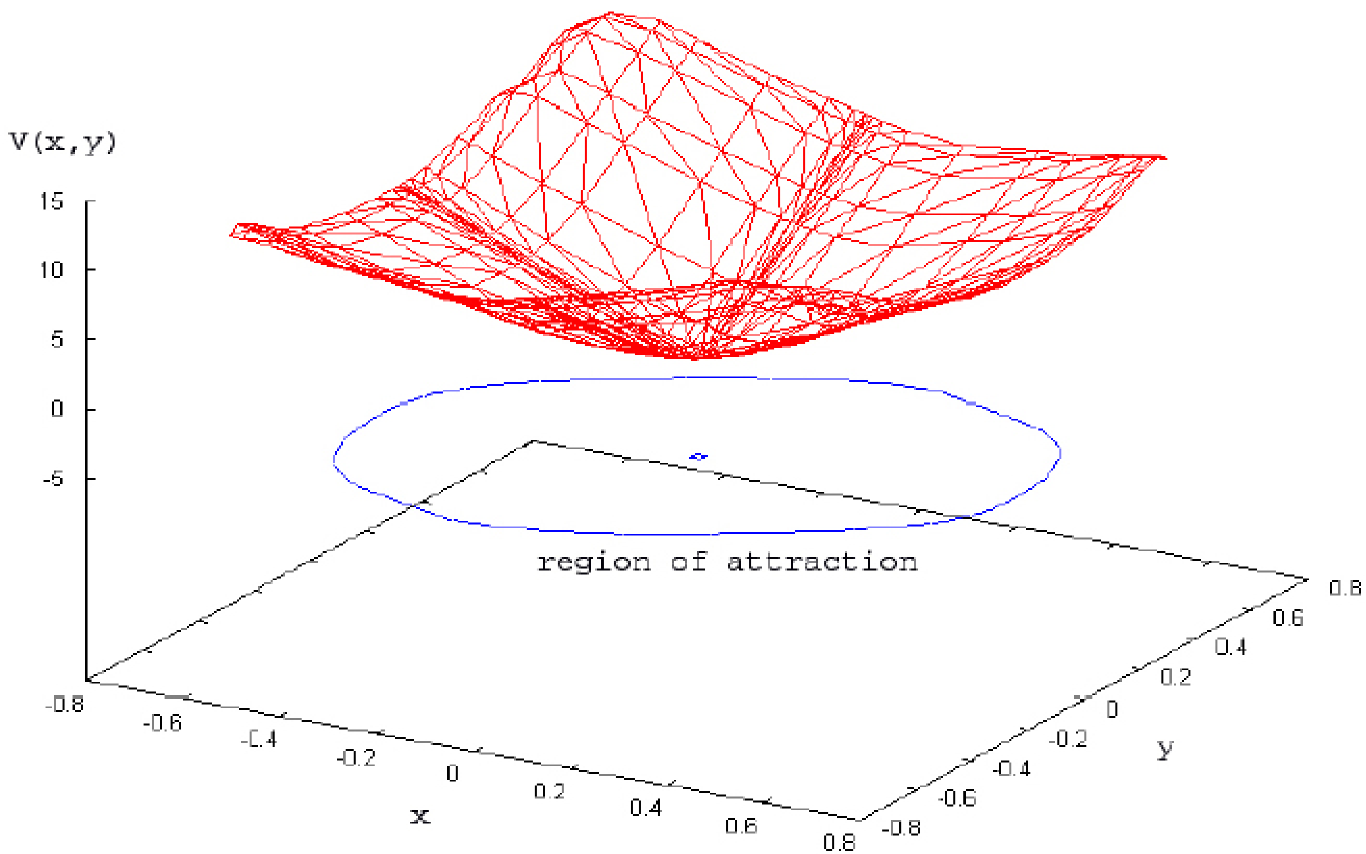
$$\dot{\mathbf{x}} = \mathbf{f}_2(\mathbf{x}), \quad \text{where} \quad \mathbf{f}_2(x, y) := \begin{pmatrix} -y + x(x^2 + y^2 - 1) \\ x + y(x^2 + y^2 - 1) \end{pmatrix}$$

Generated Lyapunov-function



$$\dot{\mathbf{x}} = \mathbf{f}_3(\mathbf{x}), \quad \text{where} \quad \mathbf{f}_3(x,y) := \begin{pmatrix} -1.5y \\ \frac{x}{1.5} + y\left(\left(\frac{x}{1.5}\right)^2 + y^2 - 1\right) \end{pmatrix}$$

Generated common Lyapunov-function



$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}), \quad p \in \{1,2,3\}, \qquad \mathbf{f}_1(x,y) := \begin{pmatrix} -y \\ x - y(1 - x^2 + 0.1x^4) \end{pmatrix},$$

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Arbitrary switched systems

$$\mathbf{f}_1(x,y) := \begin{pmatrix} -y \\ x - y(1 - x^2 + 0.1x^4) \end{pmatrix} \qquad \mathbf{f}_2(x,y) := \begin{pmatrix} -y + x(x^2 + y^2 - 1) \\ x + y(x^2 + y^2 - 1) \end{pmatrix}$$

$$\mathbf{f}_3(x,y) := \begin{pmatrix} -1.5y \\ \frac{x}{1.5} + y\left(\left(\frac{x}{1.5}\right)^2 + y^2 - 1\right) \end{pmatrix}$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}_{\sigma(t)}(\mathbf{x}(t)), \qquad \sigma : \mathbb{R}_+ \longrightarrow \{1, 2, 3\}$$

right-continuous and the discontinuity points form a discrete set

common Lyapunov function

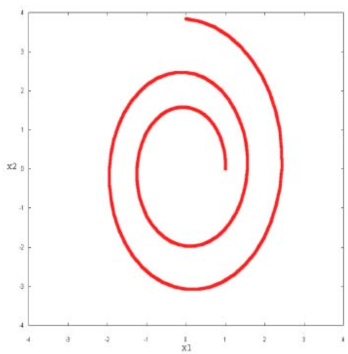
$$\Longleftrightarrow$$

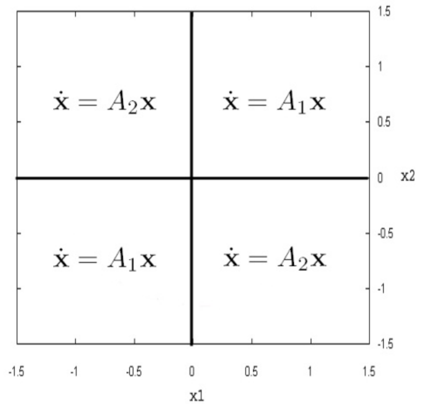
asymptotically stable under arbitrary switching

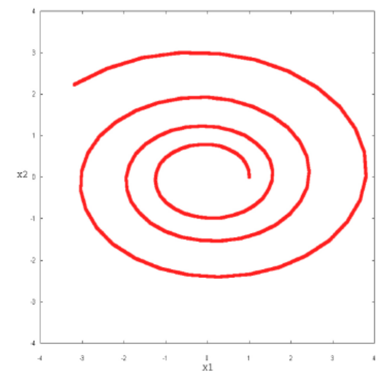
Variable structure system

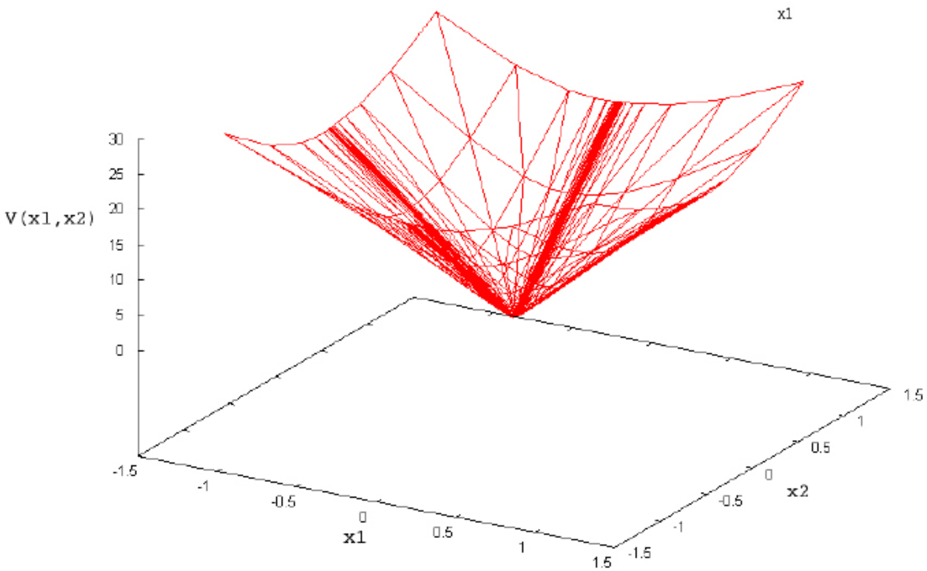
$$\dot{\mathbf{x}} = A_1\mathbf{x}, \quad \text{where} \quad A_1 := \begin{pmatrix} 0.1 & -1 \\ 2 & 0.1 \end{pmatrix}$$

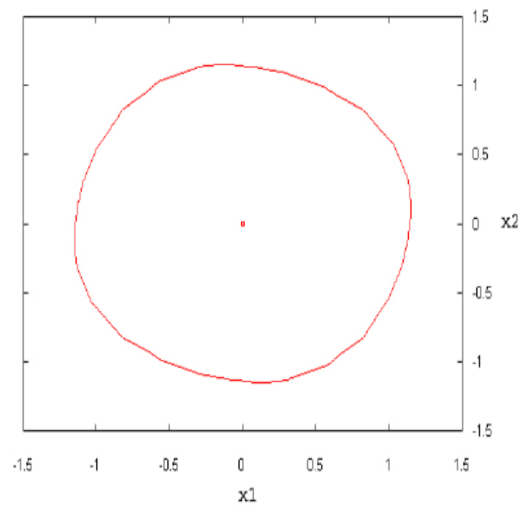
$$\dot{\mathbf{x}} = A_2\mathbf{x}, \quad \text{where} \quad A_2 := \begin{pmatrix} 0.1 & -2 \\ 1 & 0.1 \end{pmatrix}$$



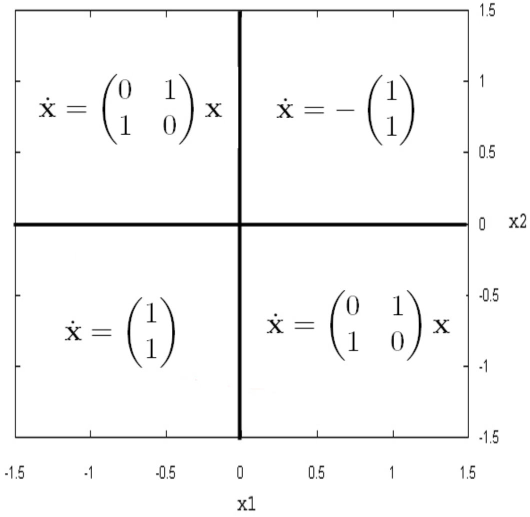


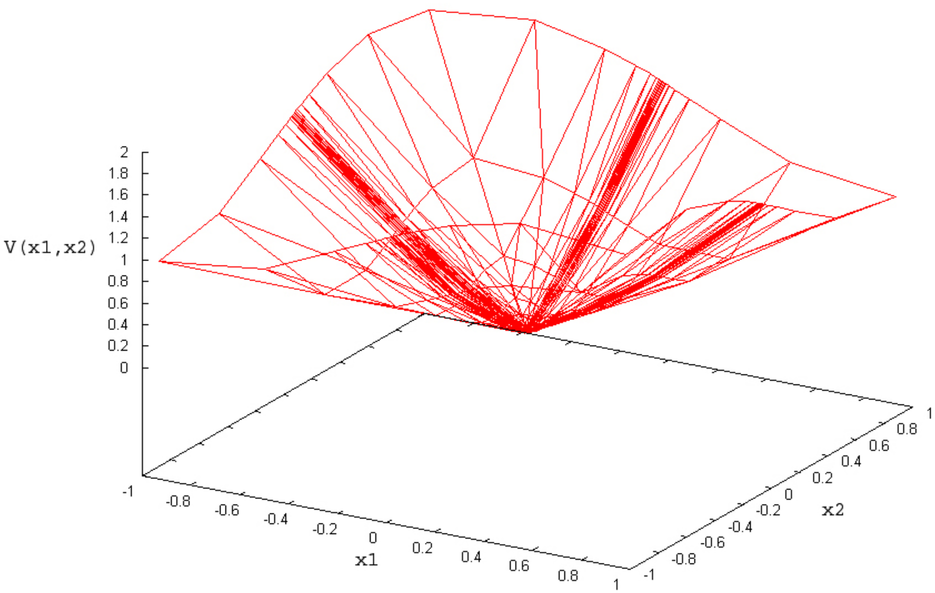


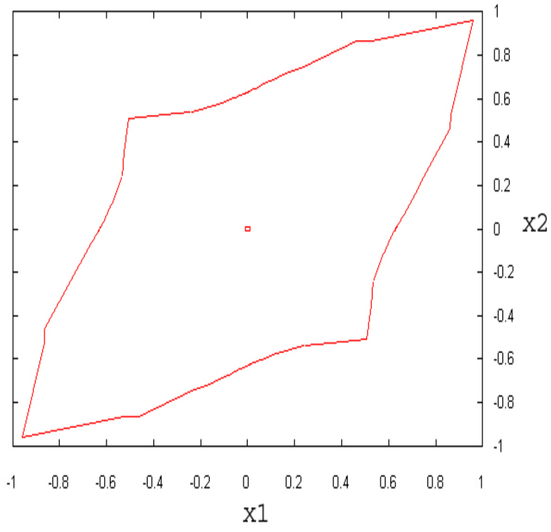




Variable structure system (sliding modes)

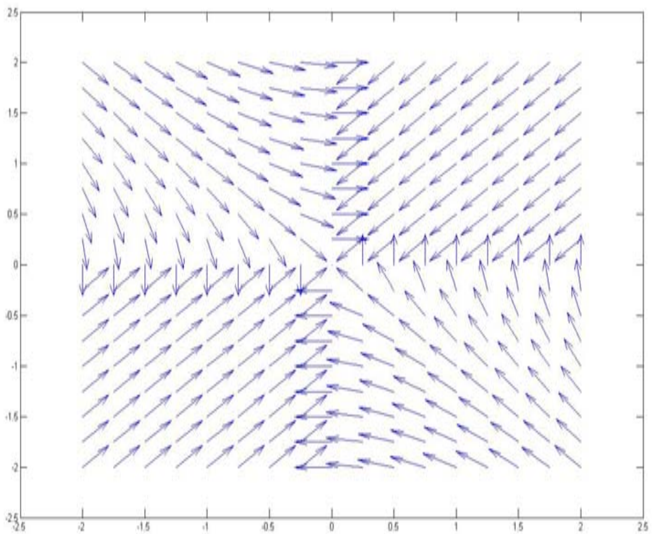


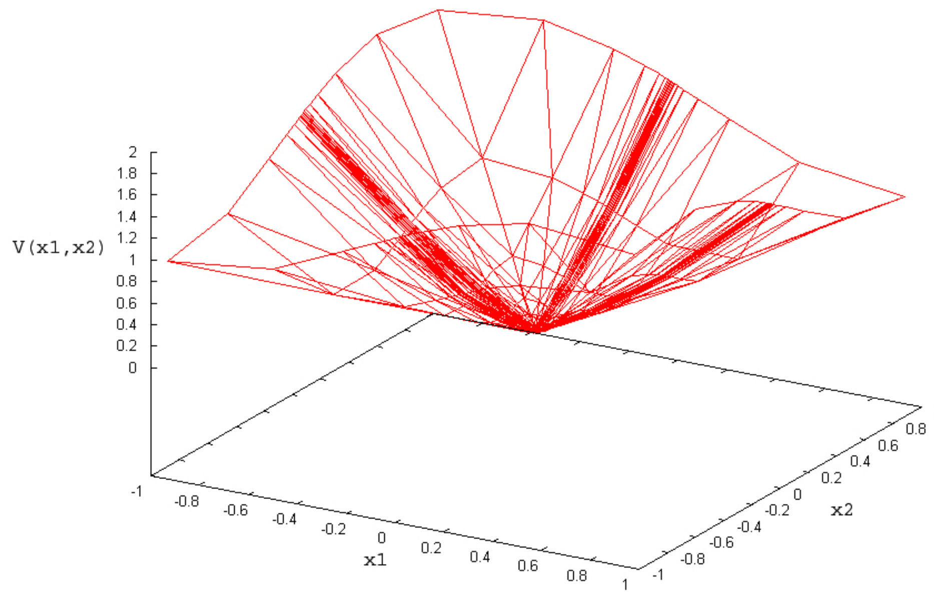




We allow the system to switch arbitiary between the dynamics on a thin strip overlapping the boundaries

Variable structure system (sliding modes)





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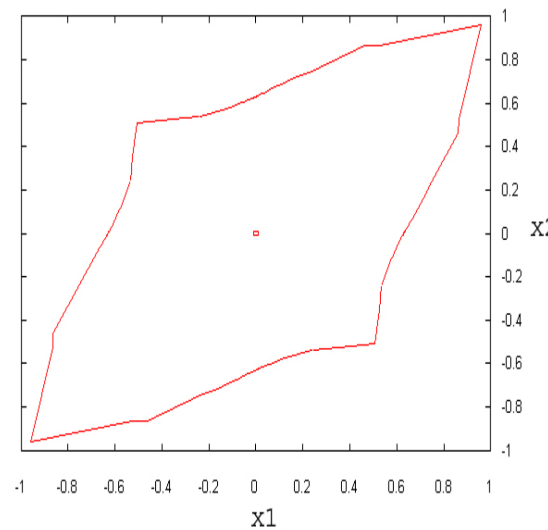
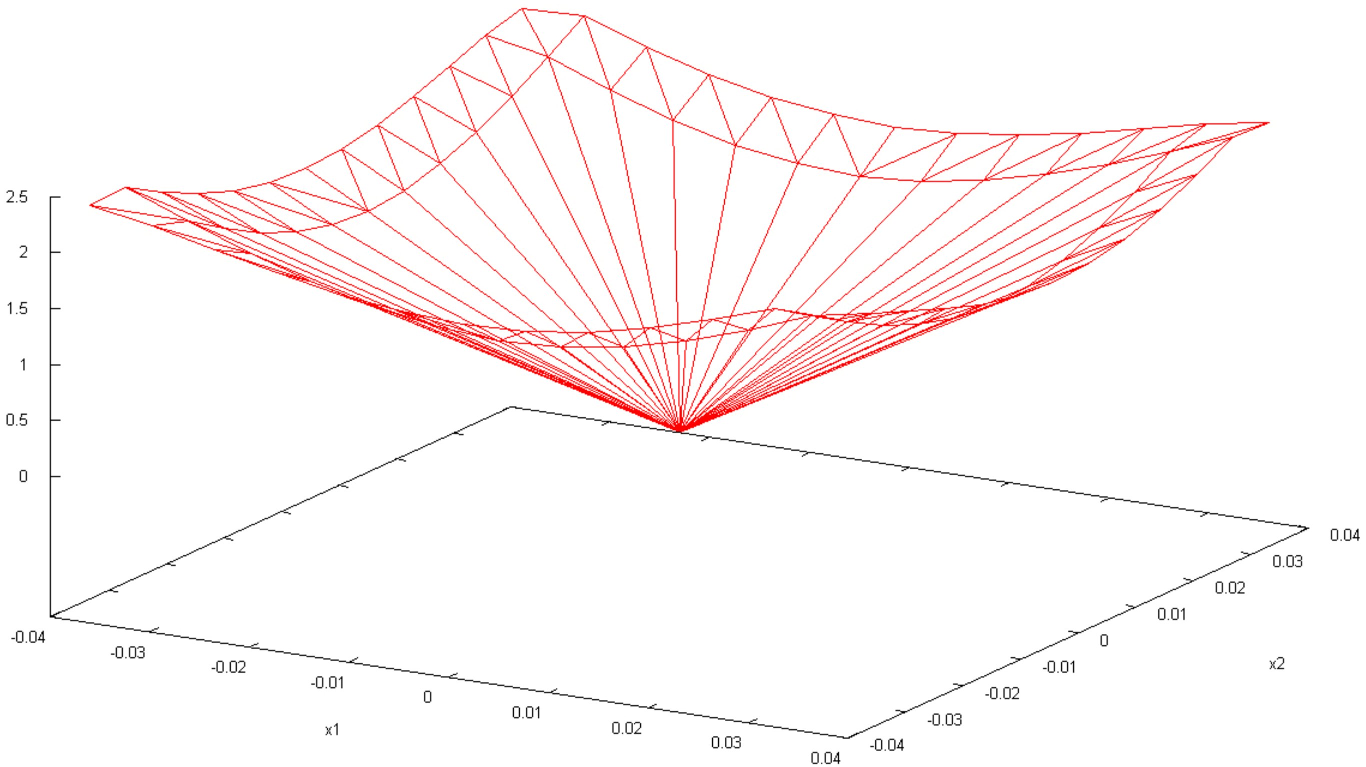
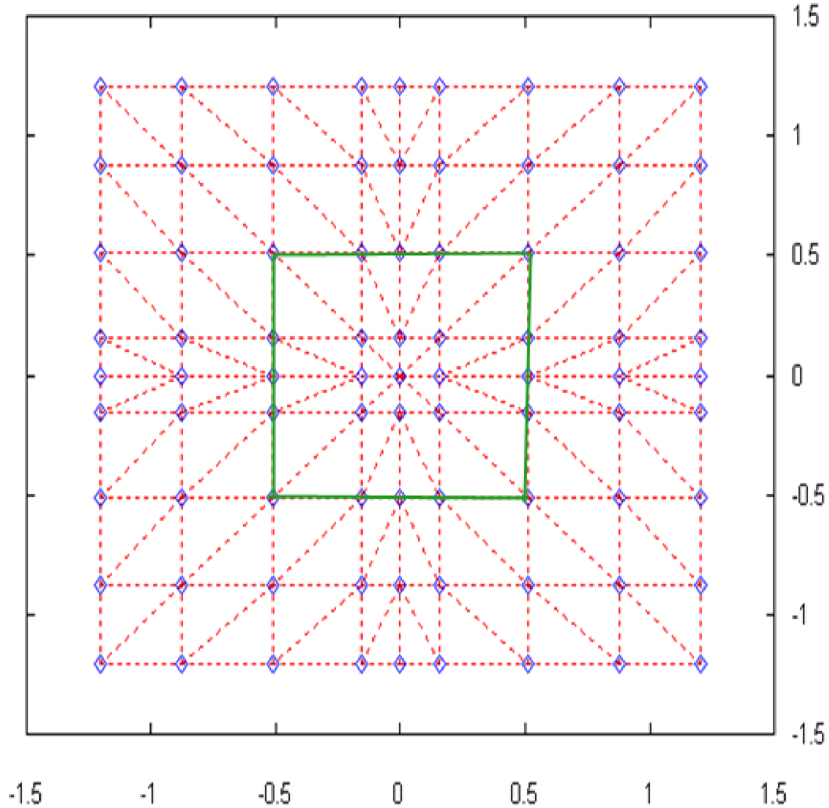


Triangle-Fan Lyapunov function (with Peter Giesl Uni Sussex)



$$\dot{\mathbf{x}} = \begin{pmatrix} -\varepsilon & -1 \\ 1 & -\varepsilon \end{pmatrix} \mathbf{x}, \quad \text{with } \varepsilon = 0.1$$

Extension of the region of attraction



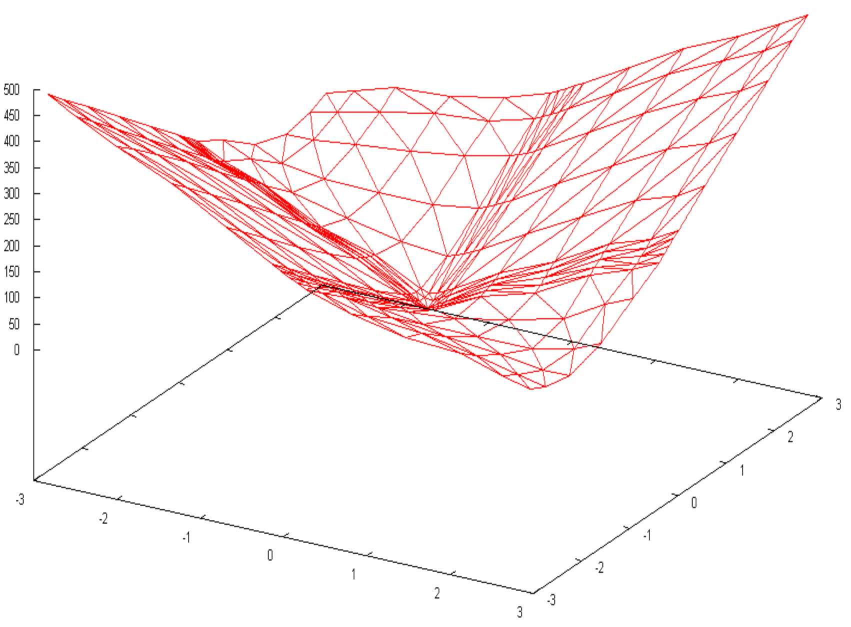
also with Peter

We make additional linear constraints that secure

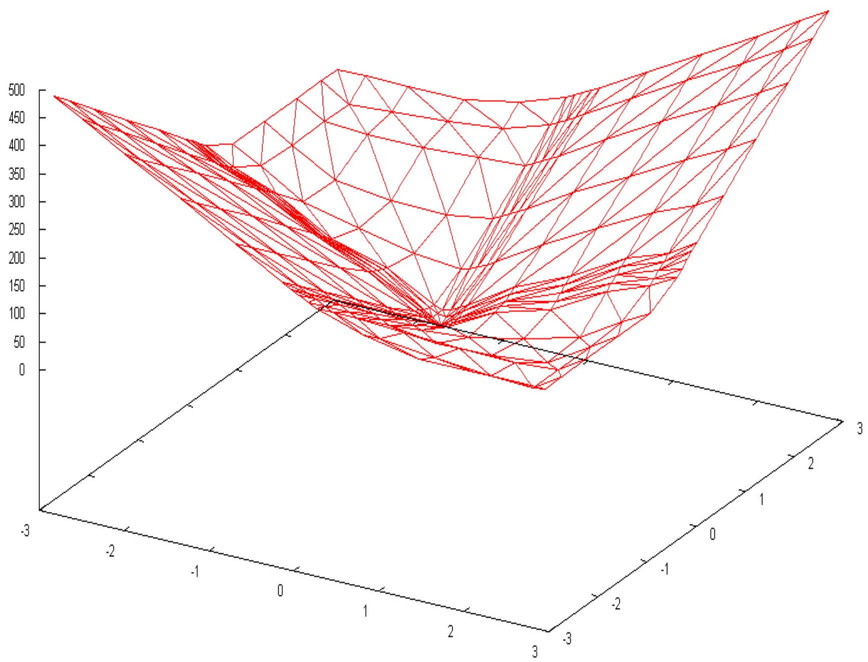
$$\max V(\mathbf{x}_{\text{green}}) < \min V(\mathbf{x}_{\text{boundary}})$$

Then the region of attraction secured by the Lyapunov function must contain the green box

Extension of the region of attraction



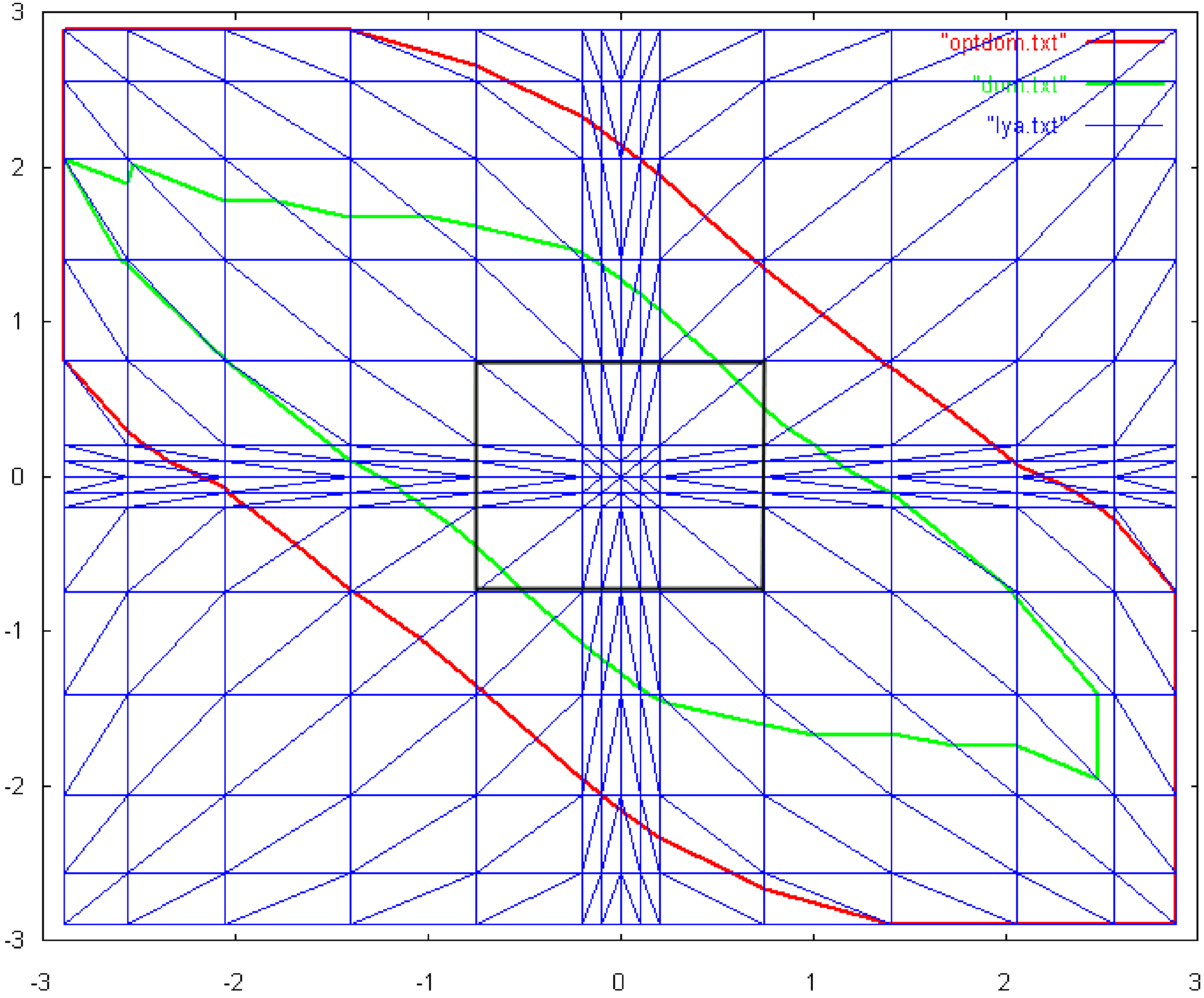
without optimization



with optimization

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \text{with } \mathbf{f}(x, y) := \begin{pmatrix} y \\ -\sin(x) - y \end{pmatrix}$$

Extension of the region of attraction



Differential inclusions and Filippov solutions

(with L. Grüne and R. Baier Uni Bayreuth)

$$\dot{x} \in F(x)$$

$$\mathcal{U} \subseteq \mathbb{R}^n, \quad F : \mathcal{U} \longrightarrow 2^{\mathbb{R}^n}$$

$\forall x \in \mathcal{U}, \quad F(x)$ is convex and compact

$x : J \longrightarrow \mathbb{R}^n$ is a Filippov solution iff $x \in \mathrm{AC}(J)$

and $\dot{x}(t) \in F(x(t))$ a.e.

one allows evil right-hand sides, but demands high regularity of the solutions

Differential inclusions and Filippov solutions

$\forall x \in \mathcal{U}, \quad F(x)$ is convex and compact

$f_\mu : \mathcal{G}_\mu \longrightarrow \mathbb{R}^n, \quad \mathcal{G}_\mu \subset\subset \mathcal{U}$ for $\mu = 1, 2, \ldots, M$

$$F(x) := \mathrm{co}\{f_\mu(x) \big| \mu \in I_\mu(x)\}$$

where $\mu \in I_\mathcal{G}(x) \Leftrightarrow x \in \mathcal{G}_\mu$

$F(x)$ is upper semicontinuous

$$\forall \varepsilon > 0, \ \exists \delta > 0 \ : \ \|x - x'\| < \delta \ \Rightarrow \ F(x') \subseteq F(x) + \mathcal{B}_\varepsilon$$

Differential inclusions and Filippov solutions

Clarke, Ledyaev, Stern 1998

$\dot{x} \in F(x)$ is strongly (every solution) asymptotically stable

$$\Leftrightarrow$$

$\dot{x} \in F(x)$ possesses a smooth Lyapunov function

The algorithm can generate a Lyapunov function for the differential inclusion, if one exists. One just has to demand LC4 for faces of the simplices if necessary

THANKS FOR LISTENING!