Resource bisimilarity and graded bisimilarity coincide☆

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Abstract

Resource bisimilarity has been proposed in the literature on concurrency theory as a notion of bisimilarity over labeled transition systems that takes into account the number of choices that a system has. Independently, g-bisimilarity has been defined over Kripke models as a suitable notion of bisimilarity for graded modal logic. This note shows that these two notions of bisimilarity coincide over image-finite Kripke frames.

Keywords: Concurrency, resource bisimilarity, graded bisimilarity, graded modal logic, Kripke frames, coalgebras

1. Introduction

In the setting of concurrency theory, resource bisimilarity has been defined over labeled transition systems in [3, 4] in order to take into account the number of choices that a system has. Intuitively, unlike ordinary bisimilarity [12], resource bisimilarity relates two processes only if their behaviors unfold to isomorphic synchronization trees. (As remarked in the conclusions of [4], resource bisimilarity is an adaptation of the counting bisimilarity of [6].) In the setting of modal logics, de Rijke defined in [7] the notion of g-bisimilarity ('graded bisimilarity') over Kripke models (which are Kripke frames augmented by a valuation function) as a suitable notion of bisimilarity for graded modal logic.

In light of the results in [4] connecting resource bisimilarity with a version of graded modal logic, it is natural to ask oneself whether there are any relations between resource and graded bisimilarity. This note addresses that question and shows that these two notions of bisimilarity coincide over image-finite Kripke frames. This result provides yet another example of a notion that has been discovered independently in the fields of concurrency theory and modal logics, and may help connect research on graded modal logics in those two fields of research.

In order to show that the above-mentioned notions of bisimilarity coincide, we reconcile some differences in the settings in which they are defined in the original literature. For reasons of notational simplicity, we have chosen to rephrase resource bisimilarity over Kripke frames with a single transition relation. However, the coincidence result we offer in this note also holds over labeled transition systems, over Kripke frames with multiple transition relations as well as over Kripke models, if the notions of resource and graded bisimilarity are rephrased in those settings in the obvious way.

We offer two alternative proofs of our result to the effect that resource and graded bisimilarity coincide over image-finite Kripke frames. The first proof (presented in Section 3) is 'elementary' and 'constructive' in that it only uses the definitions of those relations, and explicitly shows how to transform one type of relation into the other. It also yields the, admittedly easy, observation that resource bisimilarity is included in graded bisimilarity over all Kripke frames, and not just the image-finite ones. The second proof, which is developed in Section 4, relies on a combination of known results

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from the fields of modal logics and coalgebras. This proof highlights the power of the general tools connecting coalgebras and modal logics—see, e.g., the paper [5] for a survey of the coalgebraic approach to modal logics—, and offers yet another application of coalgebraic techniques. However, it is admittedly less direct and 'elementary'. Since we feel that both proofs have their merit, and that different readers will find one more palatable than the other, we decided to present both of them in this paper.

The rest of this note is organized as follows. In Section 2, we will define resource bisimilarity and graded bisimilarity. Those two notions of bisimilarity are shown to coincide over image-finite Kripke frames in Section 3. We present an alternative proof of our main result based on techniques from coalgebra and modal logics in Section 4. We conclude the paper by pointing out in Section 5 the expected fact that the Hennessy-Milner property fails for graded modal logic in the absence of image finiteness.

2. The two bisimulations

We will define both resource bisimilarity and g-bisimilarity over *Kripke frames*, which are pairs (S, R), where S is a set (of 'states') and $R \subseteq S^2$ is a binary relation over S. We define $\mathcal{P}^{<\omega}(S)$ to be the set of all finite subsets of S. Given a state $s \in S$, we define $R(s) = \{s' \mid (s, s') \in R\}$. A Kripke frame (S, R) is *image finite* if $R(s)$ is finite for each $s \in S$. We often use the infix notation for relations, writing $s \; R \; s'$ for $(s, s') \in R$. We write $|X|$ for the cardinality of a finite set X.

2.1. Resource bisimulation

In the setting of Kripke frames, a resource bisimulation, which we hereafter call an r-bisimulation, is defined as follows.

Definition 2.1 (Resource bisimulation). *Given Kripke frames $F_1 = (S_1, R_1)$ and $F_2 = (S_2, R_2)$, a relation $\mathcal{R} \subseteq S_1 \times S_2$ is an* r-bisimulation *if whenever $(s_1, s_2) \in \mathcal{R}$, the following hold:*

1. *There is an injective function $g : R_1(s_1) \to R_2(s_2)$ such that $(s, g(s)) \in \mathcal{R}$ for each $s \in R_1(s_1)$.*
2. *There is an injective function $h : R_2(s_2) \to R_1(s_1)$ such that $(h(s), s) \in \mathcal{R}$ for each $s \in R_2(s_2)$.*

The largest r-bisimulation between a given pair of Kripke frames is called r-bisimilarity*.*

Rather than involving two injective functions, a minor adaptation of the classic Cantor-Schröder-Bernstein theorem can be employed to arrive at a characterization of r-bisimilarity in terms of bijections. We formalize this as follows.

Proposition 2.2. *Given Kripke frames $F_1 = (S_1, R_1)$ and $F_2 = (S_2, R_2)$, a relation $\mathcal{R} \subseteq S_1 \times S_2$ is an r-bisimulation iff whenever $(s_1, s_2) \in \mathcal{R}$, there is a bijective function $i : R_1(s_1) \to R_2(s_2)$ such that $(s, i(s)) \in \mathcal{R}$ for each $s \in R_1(s_1)$.*

Proof This is immediate from the following claim: If

- A and B are sets,
- $\mathcal{R} \subseteq A \times B$,
- $g : A \to B$ is an injection such that $(a, g(a)) \in \mathcal{R}$, for each $a \in A$, and
- $h : B \to A$ is an injection such that $(h(b), b) \in \mathcal{R}$, for each $b \in B$,

then there is a bijection $i : A \to B$ such that $(a, i(a)) \in \mathcal{R}$, for each $a \in A$.

To show the above claim, we adapt a proof of the classic Cantor-Schröder-Bernstein theorem that employs Tarski's fixed-point theorem [16].

Recall that the partially ordered set $(\mathcal{P}(A), \subseteq)$ is a complete lattice. Define the map $f : \mathcal{P}(A) \to \mathcal{P}(A)$ as follows:

$$f(S) = A \setminus h(B \setminus (g(S)).$$

Observe that f is monotone. Indeed, if $S_1 \subseteq S_2 \subseteq A$, then

$$\begin{aligned} a \in f(S_1) &\Leftrightarrow a \in A \setminus h(B \setminus g(S_1)) \\ &\Leftrightarrow a \in A \,\&\, a \notin h(B \setminus g(S_1)) \\ &\Rightarrow a \in A \,\&\, a \notin h(B \setminus g(S_2)) \\ &\Leftrightarrow a \in f(S_2). \end{aligned}$$

Since f is monotone, by Tarski's fixed-point theorem, f has a fixed point C. Thus

$$C = A \setminus h(B \setminus g(C)).$$

This means that

$$A \setminus C = h(B \setminus g(C)).$$

Moreover, $h : B\backslash g(C) \to A\backslash C$ is a bijection because h is injective.

Now define $i : A \to B$ as follows:

$$i(x) = \begin{cases} g(x) & \text{if } x \in C, \\ h^{-1}(x) & \text{if } x \in A \setminus C. \end{cases}$$

Note that i is a bijection by construction. We are therefore left to argue that i is $\mathcal{R}$-preserving. This is clear for each $x \in C$, since $(x, g(x)) \in\mathcal{R}$ holds by our initial assumption about g. For each $x \in A \setminus C$, we have that $i(x) = h^{-1}(x)$. Therefore $x = h(h^{-1}(x))$ and $(x, h^{-1}(x)) \in\mathcal{R}$ by the initial assumption about h. This completes the proof. □

2.2. G-bisimulation

The g-bisimulation in [7] is defined as a tuple of relations. Adapted to our setting, we present it as follows.

Definition 2.3 (G-bisimulation tuple). *Let $F_1 = (S_1, R_1)$, $F_2 = (S_2, R_2)$ be two Kripke frames. A g-*bisimulation tuple *between F_1 and F_2 is a tuple $(\mathcal{Z}_1, \mathcal{Z}_2, \dots)$ of relations satisfying the following requirements:*

1. *$\mathcal{Z}_1$ is non-empty,*
2. *for each i, $\mathcal{Z}_i\subseteq \mathcal{P}^{<\omega}(S_1) \times \mathcal{P}^{<\omega}(S_2)$ (that is, $\mathcal{Z}_i$ only consists of pairs of finite sets),*
3. *for each i, if $X\ \mathcal{Z}_i\ Y$, then $|X| = |Y| = i$ (that is, $\mathcal{Z}_i$ only consists of pairs of sets with cardinality i),*
4. *if $\{x\}\ \mathcal{Z}_1\ \{y\}$ and $X \subseteq R_1(x)$, where $|X| = i \geq 1$, then there exists $Y \in \mathcal{P}^{<\omega}(S_2)$ with $Y \subseteq R_2(y)$ and $X\ \mathcal{Z}_i\ Y$,*
5. *if $\{x\}\ \mathcal{Z}_1\ \{y\}$ and $Y \subseteq R_2(y)$, where $|Y| = i \geq 1$, then there exists $X \in \mathcal{P}^{<\omega}(S_1)$ with $X \subseteq R_1(x)$ and $X\ \mathcal{Z}_i\ Y$, and*
6. *if $X\ \mathcal{Z}_i\ Y$, then*
   (a) *for all $x \in X$ there exists $y \in Y$ with $\{x\}\ \mathcal{Z}_1\ \{y\}$, and*
   (b) *for all $y \in Y$ there exists $x \in X$ with $\{x\}\ \mathcal{Z}_1\ \{y\}$.*

A g-bisimulation tuple is structurally the same as the g-bisimulation defined in [7], except that the notion of g-bisimulation in [7] was presented over Kripke models (that is, Kripke frames augmented with a valuation of proposition letters), and there was a clause requiring that $\mathcal{Z}_1$ relate only pairs of singletons whose elements satisfy the same proposition letters.

Now note that the $\mathcal{Z}_i$ are pairwise disjoint, as each $\mathcal{Z}_i$ only pairs together sets of size i. Thus we can consider a g-bisimulation tuple as being a single relation $\mathcal{Z}= \bigcup \mathcal{Z}_i$.

Definition 2.4 (G-bisimulation relation). *Let $F_1 = (S_1, R_1)$, $F_2 = (S_2, R_2)$ be two Kripke frames. A g-bisimulation relation between F_1 and F_2 is a binary relation $\mathcal{Z}\subseteq \mathcal{P}^{<\omega}(S_1) \times \mathcal{P}^{<\omega}(S_2)$ satisfying the following requirements:*

1. *whenever $X\ \mathcal{Z}\ Y$,*
   (a) *$|X| = |Y|$,*
   (b) *for each $x \in X$, there is a $y \in Y$ such that $\{x\}\ \mathcal{Z}\ \{y\}$, and*
   (c) *for each $y \in Y$, there is an $x \in X$ such that $\{x\}\ \mathcal{Z}\ \{y\}$;*
2. *whenever $\{x\}\ \mathcal{Z}\ \{y\}$*
   (a) *if $X \subseteq R_1(x)$ is finite, then there exists some finite $Y \subseteq R_2(y)$ such that $X\ \mathcal{Z}\ Y$, and*
   (b) *if $Y \subseteq R_2(y)$ is finite, then there exists some finite $X \subseteq R_1(x)$ such that $X\ \mathcal{Z}\ Y$.*

The connection between a g-bisimulation tuple and a g-bisimulation relation can be formalized as follows.

Lemma 2.5. *The following statements hold:*

1. *If $(\mathcal{Z}_1, \mathcal{Z}_2, \dots)$ is a g-bisimulation tuple, then $\mathcal{Z}= \bigcup \mathcal{Z}_i$ is a g-bisimulation relation.*
2. *Let $\mathcal{Z}$ be a g-bisimulation relation and, for each i, let $\mathcal{Z}_i$ be the set of pairs (X, Y) in $\mathcal{Z}$ such that X and Y have size i. Then $(\mathcal{Z}_1, \mathcal{Z}_2, \dots)$ is a g-bisimulation tuple.*

It is easy to see that the union of an arbitrary family of g-bisimulation relations is itself a g-bisimulation relation, and hence the union of all g-bisimulation relations is the largest g-bisimulation relation. If $\mathcal{Z}$ is the largest g-bisimulation between $F_1 = (S_1, R_1)$ and $F_2 = (S_2, R_2)$, we define g-bisimilarity as the relation $\{(s_1, s_2) \mid \{s_1\}\ \mathcal{Z}\ \{s_2\}\}$.

3. Connecting the two

We now proceed to prove the following theorem, to the effect that r- and g-bisimilarity coincide over image-finite Kripke frames.

Theorem 3.1. *Over image-finite Kripke frames, g-bisimilarity coincides with r-bisimilarity.*

In order to show the above result, we first establish that r-bisimilarity is included in g-bisimilarity. We then prove that the converse inclusion also holds. In the proof of the latter result, we employ the assumption that the Kripke frames be image finite.

Definition 3.2. *We say that a g-bisimulation $\mathcal{Z} \subseteq \mathcal{P}^{<\omega}(S_1) \times \mathcal{P}^{<\omega}(S_2)$ is an* extension *of an r-bisimulation $\mathcal{R} \subseteq S_1 \times S_2$ if $\{(\{s_1\},\{s_2\}) \mid (s_1,s_2) \in \mathcal{R}\} \subseteq \mathcal{Z}$.*

Proposition 3.3. *Let $F_1 = (S_1, R_1)$ and $F_2 = (S_2, R_2)$ be Kripke frames. If $\mathcal{R} \subseteq S_1 \times S_2$ is an r-bisimulation between F_1 and F_2, then there is a g-bisimulation $\mathcal{Z} \subseteq \mathcal{P}^{<\omega}(S_1) \times \mathcal{P}^{<\omega}(S_2)$ that is an extension of $\mathcal{R}$.*

Proof Suppose that $\mathcal{R}$ is an r-bisimulation. For each $x \in S_1$ and $y \in S_2$ for which $x \; \mathcal{R} \; y$, by Proposition 2.2 we can select a bijective function $f_{x,y} : R_1(x) \to R_2(y)$ such that $a \; \mathcal{R} \; f_{x,y}(a)$ for each $a \in R_1(x)$. We define

$$\mathcal{Z} = \bigcup_{x\mathcal{R}y, A \in \mathcal{P}^{<\omega}(R_1(x))} \{(\{x\},\{y\}), (A, f_{x,y}(A))\}.$$

One can easily verify that $\mathcal{Z}$ is a g-bisimulation and an extension of $\mathcal{R}$. □

Corollary 3.4. *Over Kripke frames, r-bisimilarity is contained in g-bisimilarity.*

Proof As r-bisimilarity can be extended by Proposition 3.3 to a g-bisimulation relation, the restriction of the g-bisimulation to singletons (after stripping away set-theoretic brackets) is contained in g-bisimilarity. □

In preparation for showing the converse of Corollary 3.4, we prove the following lemma.

Lemma 3.5. *If $\mathcal{Z}$ is the largest g-bisimulation between $F_1 = (S_1, R_1)$ and $F_2 = (S_2, R_2)$, then $\mathcal{Z}$ is difunctional [14], that is, whenever $A_1 \; \mathcal{Z} \; A_2$, $A_1 \; \mathcal{Z} \; B_2$ and $B_1 \; \mathcal{Z} \; A_2$, then also $B_1 \; \mathcal{Z} \; B_2$.*

Proof Let $\mathcal{Z}$ be the largest g-bisimulation between F_1 and F_2. Let $\widehat{\mathcal{Z}}$ be the one-step difunctional closure of $\mathcal{Z}$, that is, $\widehat{\mathcal{Z}} = \mathcal{Z} \cup \mathrm{df}(\mathcal{Z})$, where $\mathrm{df}(\mathcal{Z})$ is the relation

$$\{(B_1, B_2) \mid \exists A_1, A_2 : A_1 \; \mathcal{Z} \; A_2, A_1 \; \mathcal{Z} \; B_2, B_1 \; \mathcal{Z} \; A_2\}.$$

We show that $\widehat{\mathcal{Z}}$ is a g-bisimulation.

First suppose that $B_1 \widehat{\mathcal{Z}} B_2$ because there are A_1, A_2 such that $B_1 \; \mathcal{Z} \; A_2$, $A_1 \; \mathcal{Z} \; A_2$, and $A_1 \; \mathcal{Z} \; B_2$, which we may write as a chain thus:

$$B_1 \; \mathcal{Z} \; A_2 \; \mathcal{Z}^{-1} \; A_1 \; \mathcal{Z} \; B_2.$$

As $\mathcal{Z}$ and $\mathcal{Z}^{-1}$ only relate two sets of the same size, applying this reasoning multiple times reveals $|B_1| = |B_2|$. Next given $b_1 \in B_1$, we have that there is some $a_2 \in A_2$, such that $\{b_1\} \; \mathcal{Z} \; \{a_2\}$. Similarly, there is some $a_1 \in A_1$ such that $\{a_1\} \; \mathcal{Z} \; \{a_2\}$. Finally, there is some $b_2 \in B_2$ such that $\{a_1\} \; \mathcal{Z} \; \{b_2\}$. As $\widehat{\mathcal{Z}}$ is the one-step difunctional closure of $\mathcal{Z}$, we have that $\{b_1\} \widehat{\mathcal{Z}} \{b_2\}$. A nearly identical argument can be used to show that given $b_2 \in B_2$ there is a $b_1 \in B_1$ such that $\{b_1\} \widehat{\mathcal{Z}} \{b_2\}$.

Next suppose that $\{b_1\} \widehat{\mathcal{Z}} \{b_2\}$ because there exist a_1, a_2 such that

$$\{b_1\} \; \mathcal{Z} \; \{a_2\} \; \mathcal{Z}^{-1} \; \{a_1\} \; \mathcal{Z} \; \{b_2\}.$$

Now assume that $B_1 \subseteq R_1(b_1)$. Then there exists some $A_2 \subseteq R_2(a_2)$ such that $B_1 \; \mathcal{Z} \; A_2$, there exists some $A_1 \subseteq R_1(a_1)$ such that $A_1 \; \mathcal{Z} \; A_2$, and finally there exists some $B_2 \subseteq R_2(b_2)$ such that $A_1 \; \mathcal{Z} \; B_2$. As $\widehat{\mathcal{Z}}$ is the one-step difunctional closure of $\mathcal{Z}$, we have that $B_1 \widehat{\mathcal{Z}} B_2$. A nearly identical argument can be used to show that, given $B_2 \subseteq R_2(b_2)$, there exists some $B_1 \subseteq R_1(b_1)$ such that $B_1 \widehat{\mathcal{Z}} B_2$.

This shows that $\widehat{\mathcal{Z}}$ is a g-bisimulation. As $\mathcal{Z} \subseteq \widehat{\mathcal{Z}}$ and $\mathcal{Z}$ is the largest g-bisimulation, $\mathcal{Z} = \widehat{\mathcal{Z}}$. Hence $\mathcal{Z}$ is difunctional, which was to be shown. □

Theorem 3.6. *Over image-finite Kripke frames, g-bisimilarity is an r-bisimulation.*

Proof Let $\mathcal{Z}$ be the largest g-bisimulation relation between frames $F_1 = (S_1, R_1)$ and $F_2 = (S_2, R_2)$. We show that $\mathcal{Z}$ is an r-bisimulation. Suppose that $\{a\} \; \mathcal{Z} \; \{b\}$. By Proposition 2.2, it suffices only to prove that there is a $\mathcal{Z}$-preserving bijection between $R_1(a)$ and $R_2(b)$. To this end, for each $x \in R_1(a)$, let

$$B_x = \{y \in R_2(b) \mid \{x\} \; \mathcal{Z} \; \{y\}\},$$

and for each $y \in R_2(b)$, let

$$A_y = \{x \in R_1(a) \mid \{x\} \; \mathcal{Z} \; \{y\}\}.$$

Now, given A_y and B_x such that $\{x\} \; \mathcal{Z} \; \{y\}$, we claim that $|A_y| = |B_x|$. (Note that A_y and B_x are finite due to the image-finiteness condition in the statement of the theorem.) Indeed, as $\{a\} \; \mathcal{Z} \; \{b\}$

and $A_y \subseteq R_1(a)$, by definition of g-bisimulation we must be able to find a $B \subseteq R_2(b)$ such that $A_y \mathrel{\mathcal{Z}} B$. As $\mathcal{Z}$ is a g-bisimulation, for any $y' \in B$, there is an $x' \in A_y$ such that $\{x'\} \mathrel{\mathcal{Z}} \{y'\}$. By definition of A_y, it holds that $\{x'\} \mathrel{\mathcal{Z}} \{y\}$. Recall that $\{x\} \mathrel{\mathcal{Z}} \{y\}$ by assumption. By Lemma 3.5, $\mathcal{Z}$ is difunctional, and hence we have that $\{x\} \mathrel{\mathcal{Z}} \{y'\}$, which means that $y' \in B_x$. As y' was chosen arbitrarily in B, we have that $B \subseteq B_x$. Hence, $|A_y| = |B| \le |B_x|$. By a symmetric argument, we also have that $|B_x| \le |A_y|$. Thus $|A_y| = |B_x|$ as claimed, and we can then form a bijection from A_y to B_x. The difunctionality of $\mathcal{Z}$ guarantees us that for any $x' \in A_y$ and $y' \in B_x$, we have that $\{x'\} \mathrel{\mathcal{Z}} \{y'\}$, and thus any such bijection will respect the relation $\mathcal{Z}$.

Using a similar reasoning, we also see that if $x_1, x_2 \in A_y$, then $B_{x_1} = B_{x_2}$. Since every $x \in R_1(a)$ is in A_y for some $y \in R_2(b)$, it turns out that $\{A_y\}_{y \in R_2(b)}$ forms a partition of $R_1(a)$. Similarly, $\{B_x\}_{x \in R_1(a)}$ forms a partition of $R_2(b)$.

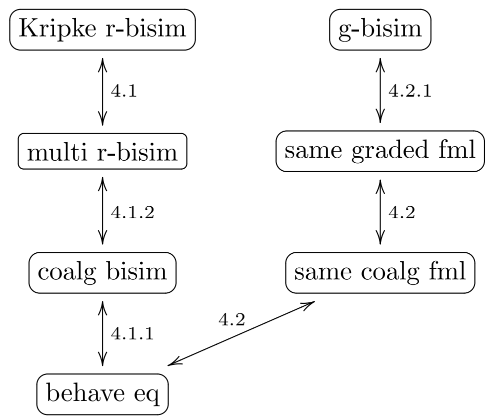
Let $x_1, \ldots, x_\ell$ be representatives of the sets in the partition $\{A_y\}_{y \in R_2(b)}$, and let $y_1, \ldots, y_k$ be representatives of the sets in the partition $\{B_x\}_{x \in R_1(a)}$. We claim that $\ell = k$. Indeed, for each $i \in \{1, \ldots, \ell\}$, there is some $j_i \in \{1, \ldots, k\}$ such that $\{x_i\} \mathrel{\mathcal{Z}} \{y_{j_i}\}$. (Pick the representative of B_{x_i} as y_{j_i}.) Moreover, if $\{x_{i_1}\} \mathrel{\mathcal{Z}} \{y_j\}$ and $\{x_{i_2}\} \mathrel{\mathcal{Z}} \{y_j\}$, for some $i_1, i_2 \in \{1, \ldots, \ell\}$ and $j \in \{1, \ldots, k\}$, then $i_1 = i_2$. Therefore $\ell \le k$. By a symmetric argument, we have that $k \le \ell$ also holds, and therefore $\ell = k$ as claimed. It follows that the function mapping each $i \in \{1, \ldots, \ell\}$ to $j_i \in \{1, \ldots, k\}$ is a bijection. Hence, we can form a $\mathcal{Z}$-respecting bijection from $R_1(a)$ to $R_2(b)$ by 'pasting together' the bijections from $A_{y_{j_i}}$ to B_{x_i}, for $i \in \{1, \ldots, \ell\}$. □

Corollary 3.4 and Theorem 3.6 together show that, over image-finite Kripke frames, r-bisimilarity and g-bisimilarity coincide. This completes the proof of Theorem 3.1.

4. A proof via coalgebras and modal logic

This section proves that graded and resource bisimilarity coincide using a route that involves logic and the Hennessy-Milner Property [2, 10]. The Hennessy-Milner Property essentially says that two states satisfy the same set of formulas in a suitable logic if and only if they are 'bisimilar'. For resource bisimilarity, we will employ coalgebraic modal logic for the finite multiset functor, and for graded bisimilarity we will employ graded modal logic. Although the settings are superficially different, these logics are fundamentally the same.

Coalgebras for the finite multiset functor are identified with multigraphs, which we call *multiframes* in order to draw a parallel with the Kripke frames we have used so far. For this reason, we define in Section 4.1 resource bisimulation in the multiframe setting and show that the resource bisimilarities for Kripke frames and for multiframes coincide. In Section 4.1.1, we show that, for the finite multiset functor, coalgebraic bisimilarity and behavioral equivalence coincide. In Section 4.1.2, we prove that resource bisimilarity on multiframes coincides with coalgebraic bisimilarity for the finite multiset functor. In Section 4.2, we present graded modal logic and coalgebraic modal logic for the finite multiset functor, and we establish the Hennessy-Milner property for the coalgebraic modal logic with respect to resource bisimilarity by way of coalgebraic bisimilarity and behavioral equivalence. Finally in Section 4.2.1, we complete the proof by establishing that the Hennessy-Milner property holds for graded modal logic over the class of image-finite Kripke frames modulo g-bisimilarity. For reader's convenience, we depict these relationships in the following diagram:



4.1. R-bisimulation on multiframes

The coalgebras we will involve in Section 4.1.1 are multigraphs (which we call multiframes), that is, directed graphs with $\mathbb{N}$-weighted edges. A *multiframe* is a structure $M = (S, \Sigma)$, where S is a countable set and Σ is a set of functions $\sigma^s : S \to \mathbb{N}$ for each $s \in S$. Note that Σ can be characterized by a function from $S \times S$ to $\mathbb{N}$, which generalizes a binary relation that can be characterized by a function from $S \times S$ to $\{0, 1\}$. When visualizing a multiframe as a directed graph with weighted edges, we call each s' for which $\sigma^s(s') > 0$ a successor of s. A *pointed multiframe* is a pair (M, s), where $M = (S, \Sigma)$ is a multiframe, and $s \in S$. We now define resource bisimulation for multiframes.

Definition 4.1. *Given multiframes $M_1 = (S_1, \Sigma_1)$ and $M_2 = (S_2, \Sigma_2)$, a relation $\mathcal{R} \subseteq S_1 \times S_2$ is an* r-bisimulation *iff whenever $(s_1, s_2) \in \mathcal{R}$, there is a bijective function f from $\bigcup_{s \in S_1}\{(s, n) \mid 1 \leq n \leq \sigma_1^{s_1}(s)\}$ to $\bigcup_{s \in S_2}\{(s, n) \mid 1 \leq n \leq \sigma_2^{s_2}(s)\}$, such that $s\ \mathcal{R}\ t$ whenever $f(s, n) = (t, m)$.* R-bisimilarity *between M_1 and M_2 is the largest r-bisimulation between M_1 and M_2, which is obtained by taking the union of all r-bisimulations. We say that (M_1, s_1) and (M_2, s_2) are* r-bisimilar *if there exists an r-bisimulation between M_1 and M_2 that relates s_1 with s_2.*

This definition, which will be shown to match the 'matrix property' that will be discussed in Section 4.1.2, says that two states are related by a bisimulation if each successor of one is matched with a successor of the other that is related to the first by the bisimulation. The requirement that the function f be a bijection takes care of handling the weight (or multiplicity) of the edges properly. This will be highlighted by Proposition 4.3 to follow.

We now relate r-bisimilarity on multiframes with r-bisimilarity for Kripke frames. To do so, we must relate multiframes with Kripke frames. A Kripke frame can easily be viewed as a multiframe where $\sigma^s(s') \in \{0, 1\}$. A multiframe can be transformed in a bisimilarity preserving way into a Kripke frame using the following method, which we call the Kripkeization.

Definition 4.2 (Kripkeization). *Given a multiframe $M = (S, \Sigma)$, the* Kripkeization *of M, written $\mathcal{K}(M)$, is the Kripke frame $(\mathcal{K}(S), R)$, where*

- $\mathcal{K}(S) = \bigcup_{s \in S}\{(s, n) \mid n \in \mathbb{N}, 1 \leq n \leq \sup_{a \in S}\{1, \sigma^a(s)\}\}$.
- $R = \{((s, n), (t, m)) \mid (s, n) \in \mathcal{K}(s), 1 \leq m \leq \sigma^s(t)\}$.

Proposition 4.3. *Pointed multiframes (M, s) and (N, t) are r-bisimilar (according to Definition 4.1) if and only if $(\mathcal{K}(M), (s, 1))$ and $(\mathcal{K}(N), (t, 1))$ are r-bisimilar (according to Definition 2.1).*

Proof Suppose $\mathcal{R}$ is r-bisimilarity between $M = (A, \Sigma)$ and $N = (B, T)$, and that $s\ \mathcal{R}\ t$. Let $\widehat{\mathcal{R}}$ be defined by

$$\widehat{\mathcal{R}} = \{((s, m), (t, n)) \in \mathcal{K}(A) \times \mathcal{K}(B) \mid s\ \mathcal{R}\ t\}.$$

We claim that $\widehat{\mathcal{R}}$ is an r-bisimulation. Observe first that $(s, 1)\widehat{\mathcal{R}}(t, 1)$ holds by definition. If $(s, m)\widehat{\mathcal{R}}(t, n)$, then $s\ \mathcal{R}\ t$ and there is a bijective function $f : \bigcup_{a \in A}\{(a, n) \mid 1 \leq n \leq \sigma^s(a)\} \rightarrow \bigcup_{b \in B}\{(b, n) \mid 1 \leq n \leq \tau^t(b)\}$, such that whenever $f(a, n) = (b, m)$, $a\ \mathcal{R}\ b$. But then $(a, n)\widehat{\mathcal{R}}(b, m)$, and f also applies to Definition 2.1. Thus if s and t are r-bisimilar (via $\mathcal{R}$), then so are $(s, 1)$ and $(t, 1)$ (via $\widehat{\mathcal{R}}$).

Suppose conversely that $\mathcal{R}$ is r-bisimilarity between $\mathcal{K}(M) = (\mathcal{K}(A), R)$ and $\mathcal{K}(N) = (\mathcal{K}(B), S)$, and that $(s, 1)\ \mathcal{R}\ (t, 1)$. Define $\overline{\mathcal{R}}$ as

$$\overline{\mathcal{R}} = \{(s, t) \in A \times B \mid \exists m, n.\ (s, m)\ \mathcal{R}\ (t, n)\}.$$

We show that $\overline{\mathcal{R}}$ is an r-bisimulation. If $s\overline{\mathcal{R}}t$, then for some m, n we have $(s, m)\ \mathcal{R}\ (t, n)$. But then there is a bijection $f : R(s, m) \rightarrow S(t, n)$ such that $(a, j)\ \mathcal{R}\ (b, k)$ whenever $f(a, j) = (b, k)$. Then $a\overline{\mathcal{R}}b$. Recall that $R(s, m) = \bigcup_{a \in A}\{(a, n) \mid 1 \leq n \leq \sigma^s(a)\}$ and $S(t, n) = \bigcup_{b \in B}\{(b, n) \mid 1 \leq n \leq \tau^t(b)\}$. Hence the function f applies to Definition 4.1. Thus if $(s, 1)$ and $(t, 1)$ are r-bisimilar, then s and t are also r-bisimilar. □

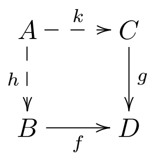
4.1.1. Finite multiset functor, coalgebraic bisimulation, and behavioral equivalence

We now relate r-bisimilarity on multiframes with the notion of behavioral equivalence induced by the finite multiset functor, which we define below. In order to prove that r-bisimilarity coincides with behavioral equivalence, we proceed in two steps. In this section, we prove that the behavioral equivalence and the coalgebraic bisimilarity induced by the finite multiset functor coincide. In Section 4.1.2, we then show that r-bisimilarity coincides with coalgebraic bisimilarity.

Given any structure S with a 0 element, the *support* of a function $f : X \rightarrow S$, written $supp(f)$, is the set $\{x \in X \mid f(x) \neq 0\}$ of all elements that do not map to 0.

Definition 4.4 (Finite Multiset Functor). *The* finite multiset functor *is the functor $\mathcal{B} : \mathsf{Set} \rightarrow \mathsf{Set}$ mapping a set X to the set $\mathcal{B}(X)$ of all functions $\sigma : X \rightarrow \mathbb{N}$ ($\mathbb{N} = \{0, 1, \ldots\}$) with finite support $supp(\sigma)$, and mapping all morphisms $f : X \rightarrow Y$ to the morphism $\mathcal{B}f : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$ for which $\mathcal{B}f\sigma : y \mapsto \sum\{\sigma(x) : f(x) = y\}$ for each $y \in Y$. (Note that if $y \notin f[X]$, then $\mathcal{B}f\sigma(y) = 0$.)*

Definition 4.5 (Weak pullback). *Given functions $f : B \to D$ and $g : C \to D$, a* weak pullback *is a pair of functions $h : A \to B$ and $k : A \to C$, such that $g \circ k = h \circ f$ and whenever $f(b) = g(c)$ for some $b \in B$ and $c \in C$, there exists an $a \in A$ such that $h(a) = b$ and $k(a) = c$. This is depicted by a diagram called a* weak pullback square*:*



Definition 4.6 (Preserving weak pullbacks). *A functor F : Set $\to$ Set* preserves weak pullbacks *if the image of a weak pullback square under F is also a weak pullback square.*

The following proposition can be proved using a minor adaptation of Example 3.5 in [13]. For completeness, we present the proof here.

Proposition 4.7. *The finite multiset functor $\mathcal{B}$ preserves weak pullbacks.*

Proof Suppose we have a weak pullback square as in Definition 4.5. It is easy to see that $\mathcal{B}f \circ \mathcal{B}h = \mathcal{B}g \circ \mathcal{B}k$. Now, let $\sigma \in \mathcal{B}(B)$ and $\tau \in \mathcal{B}(C)$ be such that $\mathcal{B}f\sigma = \mathcal{B}g\tau$. Our aim is to find some $\alpha \in \mathcal{B}(A)$, such that $(\mathcal{B}h)\alpha = \sigma$ and $(\mathcal{B}k)\alpha = \tau$. Pick a $d \in D$. Let $\{b_1, \dots, b_m\} = f^{-1}[d]$, and let $\{c_1, \dots, c_n\} = g^{-1}[d]$. Finally for $1 \le i \le m$ and $1 \le j \le n$, let $\{a^1_{ij}, \dots, a^\ell_{ij}\} = h^{-1}[b_i] \cap k^{-1}[c_j]$. We define α on the a^t_{ij} in any way such that $\sum_{t=1}^{\ell} \alpha(a^t_{ij}) = r_{ij}$, where r_{ij} comes from applying an integer version of the Row/Column Theorem (Theorem 3.6 in [13]), which we present as follows.

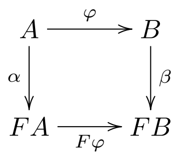
The integer version of the above-mentioned Row/Column Theorem states that if $p_1, \dots, p_m, q_1, \dots, q_n \in \mathbb{N}$ are such that $\sum_{1 \le i \le m} p_i = \sum_{1 \le j \le n} q_j$, then for each $1 \le i \le m$ and $1 \le j \le n$, there exists $r_{ij} \in \mathbb{N}$, such that

$$\begin{aligned} \sum_{1 \le j \le n} r_{ij} &= p_i, \text{ for } 1 \le i \le m, \text{ and} \\ \sum_{1 \le i \le m} r_{ij} &= q_j, \text{ for } 1 \le j \le n. \end{aligned}$$

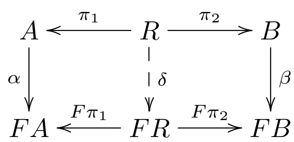
Here the p_i are values $\sigma(b_i)$, and the q_j are values $\tau(c_j)$.

The proof of the integer version of the Row/Column Theorem is similar to the one in [13], but with a variation on the inductive step to avoid non-integer values. The induction is on $m + n$. Let $p_\mu = \min_{1 \le i \le m} p_i$ and $q_\nu = \min_{1 \le j \le n} q_j$. Without loss of generality, suppose $p_\mu \le q_\nu$. Then we set $r_{\mu\nu} = p_\mu$, $r_{\mu j} = 0$ for $j \ne \nu$. To find the rest of the r_{ij}, apply the induction hypothesis to the $p_1, \dots, p_{\mu-1}, p_{\mu+1}, p_m$, and $q'_1, \dots q'_n$, where $q'_\nu = q_\nu - p_\mu$ and $q'_j = q_j$ for $j \ne \nu$. □

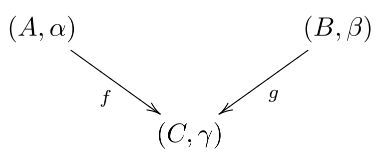
Given an endofunctor F : Set $\to$ Set, an F-coalgebra is a pair $X = (A, \alpha)$ consisting of a set X and a morphism $\alpha : A \to FA$. A coalgebraic homomorphism between F-coalgebras (A, α) and (B, β) is a map φ such that $\beta \circ \varphi = F(\varphi) \circ \alpha$:



Definition 4.8 (Coalgebraic bisimulation). *A* bisimulation *between F-coalgebras (A, α) and (B, β) is a relation $R \subseteq A \times B$, such that there is a morphism $\delta : R \to FR$, such that the projections $\pi_1 : R \to A$ and $\pi_2 : R \to B$ are homomorphisms. States $x \in A$ and $y \in B$ are* bisimilar *if there exists a bisimulation R between (A, α) and (B, β), such that $(x, y) \in R$.*



Definition 4.9 (Behavioral equivalence). *Given F-coalgebras (A, α) and (B, β) and states $x \in A$ and $y \in B$, we say x and y are* behaviorally equivalent *if there exist a coalgebra (C, γ), and morphisms $f : A \to C$ and $g : B \to C$, such that $f(x) = g(y)$.*



Behavioral equivalence implies bisimilarity when the functor preserves weak pullbacks (Proposition 1.2.2 in Kurz [11]). Thus this holds for the multiset functor, as the multiset functor preserves weak pullbacks by Proposition 4.7.

A final coalgebra is a coalgebra (A, α), such that for every coalgebra (B, β) there is a unique coalgebraic homomorphism φ from (A, α) to (B, β). If a final coalgebra exists, then bisimilarity implies behavioral equivalence; indeed, the map from the bisimulation to the final coalgebra factors through the projection maps. It is well known that an ω-accessible set functor has a final coalgebra—see, e.g., [1]. (An ω-accessible functor is a functor that preserves all ω-directed colimits.) It is straightforward to check that the finite multiset functor $\mathcal{B}$ is ω-accessible, and hence has a final coalgebra. Thus, for this functor, behavioral equivalence coincides with coalgebraic bisimilarity.

4.1.2. Matrix Property

The name for the property defined in the following definition (which comes from Lemma 5.5 of [9]) is particular to this paper.

Definition 4.10 (Matrix Property). *Given multiframes (A, Σ) and (B, T), we say that a relation $\mathcal{R} \subseteq A \times B$ satisfies the* matrix property *if for every $(a, b) \in \mathcal{R}$, there exists an $|A| \times |B|$-matrix $(m_{x,y})$ with entries from $\mathbb{N}$ such that*

1. *all but finitely many $m_{x,y}$ are 0,*
2. *$m_{x,y} \neq 0$ implies $(x, y) \in R$,*
3. *for each x, $\sigma^a(x) = \sum\{m_{x,y} \mid y \in B\}$, and*
4. *for each y, $\tau^b(y) = \sum\{m_{x,y} \mid x \in A\}$.*

A relation R satisfies the matrix property if and only if it is an r-bisimulation on multiframes. The existence of a matrix $m_{x,y}$ for the pair (a, b) in Definition 4.10 corresponds to the existence of the function $f : \{\bigcup_{x \in A}\{(x, n) \mid 1 \leq n \leq \sigma^a(x)\} \to \bigcup_{y \in B}\{(y, n) \mid 1 \leq n \leq \tau^b(y)\}$ in Definition 4.1. Each entry $m_{x,y}$ represents the number of integers $n \leq \sigma^a(x)$ for which $f(x, n) = (y, k)$ for some k.

Furthermore, according to Lemma 5.5 in [9], a relation R satisfying the matrix property is an $\mathcal{F}$-coalgebraic bisimulation, where $\mathcal{F}$ is a set functor mapping a set X to a set of functions with finite support from X to a monoid. We recall that a *monoid* is a triple $(S, \star, 0)$, where S is a set and $\star$ is an associative binary operation on S that treats $0 \in S$ as the identity element. For the monoid $\mathcal{N} = (\mathbb{N}, +, 0)$, the functor $\mathcal{F}$ maps each set X to the set of functions $\sigma : X \to \mathcal{N}$ with finite support. Note that $\mathcal{F}$ is the multiset functor $\mathcal{B}$. By way of the matrix property, two states are r-bisimilar on multiframes if and only if they are $\mathcal{B}$-coalgebraically bisimilar, and by the previous section, they are behaviorally equivalent.

4.2. Logic and the Hennessy-Milner property

The language $\mathcal{L}_G$ of graded modal logic is given by the following BNF:

$$\varphi ::= \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond_n \varphi \quad (n \in \mathbb{N}) \ .$$

The semantics of $\mathcal{L}_G$ can be defined on Kripke frames using a satisfaction relation between pointed Kripke frames and formulas, where a pointed Kripke frame is a pair (F, s), with F being a Kripke frame and s a state in F.

- $(F, s) \models \top$ always
- $(F, s) \models \neg\varphi$ if and only if $(F, s) \not\models \varphi$ (where $\not\models$ is the complement of $\models$)
- $(F, s) \models \varphi \wedge \psi$ if and only if $(F, s) \models \varphi$ and $(F, s) \models \psi$
- $(F, s) \models \Diamond_n \varphi$ if and only if there are at least n states t such that sRt and $(F, t) \models \varphi$

We have the following derived formula: $\Box_n \varphi \equiv \neg\Diamond_n \neg\varphi$, which states that fewer than n related states satisfy $\neg\varphi$. For example, $\Box_1 \bot$ is satisfied by (F, s) if state s has no successor. The operators $\Box_1$ and $\Diamond_1$ are the usual modal operators $\Box$ and $\Diamond$.

The semantics can be defined on multiframes using a relation between pointed multiframes and formulas:

- $(M, s) \models \Diamond_n \varphi$ if and only if $\sum\{\sigma^s(t) \mid (M, t) \models \varphi\} \geq n$.

Here, the derived formula $\Box_n \varphi$ is true at (M, s) iff $\sum\{\sigma^s(t) \mid (M, t) \models \neg\varphi\} < n$. In what follows, we simply say that a formula is true at a state s of a (multi)frame when the (multi)frame is understood from the context.

Define a sequence of predicate liftings $\lambda^n_X : 2 \to 2 \circ \mathcal{B}$ (with 2 being the contravariant powerset functor), that maps each set $A \subseteq X$ according to

$$A \mapsto \{\sigma \in \mathcal{B}(X) \mid \sum_{x \notin A} \sigma(x) < n\}.$$

We see that this is a natural transformation, for given $f : X \to Y$, and $A \subseteq Y$,

$$
\begin{aligned}
&(\mathcal{B}f)^{-1}(\lambda^n_Y(A)) = \\
&\quad (\mathcal{B}f)^{-1}[\{\sigma \in \mathcal{B}(Y) \mid \sum_{y \notin A} \sigma(y) < n\}] = \\
&\quad \{\sigma \in \mathcal{B}(X) \mid \sum_{y \notin A} \mathcal{B}f\sigma(y) < n\} = \\
&\quad \{\sigma \in \mathcal{B}(X) \mid \sum_{x \notin f^{-1}[A]} \sigma(x) < n\} = \\
&\quad \lambda^n_X(f^{-1}[A]).
\end{aligned}
$$

The first equality is from the definition of λ^n_Y. The second is from the definition of $\mathcal{B}f^{-1}$. The third is from the definition of $\mathcal{B}$ and the fact that $X \setminus f^{-1}[A] = f^{-1}[Y \setminus A]$. The fourth is from the definition of λ^n_X.

We now define a coalgebraic semantics for $\mathcal{L}_G$ over a $\mathcal{B}$-coalgebra $X = (A, \alpha)$ by

- $(X, s) \models \Box_n \varphi$ if and only $\alpha(s) \in \lambda^n_X [\![\varphi]\!]$, where $[\![\varphi]\!] = \{t \mid (X, t) \models \varphi\}$.

By identifying $\mathcal{B}$-coalgebras with multiframes, one can observe that the coalgebraic semantics for $\Box_n$ matches the multiframe semantics: $\Box_n \varphi$ is true at s if the number of connections from s to points t not in $[\![\varphi]\!]$ is less than n (where the connections are given by either $\sigma^s(t)$ in the multiframe or values $\alpha(s)(t)$ in the case of the coalgebra).

By Definition 7 in [15], the set $\{\lambda^n_X\}_{n \in \mathbb{N}}$ of predicate liftings is *separating* if, for every set X, any multiset $\sigma \in \mathcal{B}(X)$ can be uniquely determined by the set

$$\{(\lambda^n_X, A) \mid n \in \mathbb{N}, A \subseteq X, \sigma \in \lambda^n_X(A)\}.$$

We also see that the set $\{\lambda^n_X\}_{n \in \mathbb{N}}$ is separating, since given a multiset $\sigma : X \to \mathbb{N}$, we can check for each $x \in X$ that $\{(\lambda^n, A) \mid A = X - \{x\}, n \in \mathbb{N}, \lambda^n_X(A)\} = \{(\lambda^n_X, X - \{x\}) \mid n > \sigma(x)\}$.

Thus applying Theorem 14[1] and Example 37(1) in [15], we see that the coalgebraic modal logic for $\{\lambda^n_X\}_{n \in \mathbb{N}}$ is *expressive*. According to [15], a $\mathcal{B}$-coalgebraic language is expressive if 'logical indistinguishability under $\mathcal{L}_G$ implies behavioral equivalence.'

[1] This theorem requires the functor to be ω-accessible. As the finite multiset functor $\mathcal{B}$ is ω-accessible, the theorem can be applied.

The converse of this statement is immediate from the fact that the homomorphisms preserve the semantics, meaning that if f is a homomorphism from (A, α) to (B, β), then for every formula φ and $x \in A$, $(A, x) \models \varphi$ if and only if $(B, f(x)) \models \varphi$. Written another way, $[\![\varphi]\!]_A = f^{-1}[[\![\varphi]\!]_B]$, where the subscript indicates the set the semantic function maps to; this is the form that will be used in the induction hypothesis. To see that indeed the language preserves the semantics, we argue by induction on the structure of the formula. The base case and the boolean connective cases are simple. For the inductive step $\Box_n \varphi$, we proceed as follows:

$$
\begin{aligned}
s \in [\![\Box_n \varphi]\!]_A &\Leftrightarrow \alpha(s) \in \lambda_n [\![\varphi]\!]_A \\
&\Leftrightarrow \sum_{x \notin [\![\varphi]\!]_A} \alpha(s)(x) < n \\
&\Leftrightarrow_1 \sum_{y \notin [\![\varphi]\!]_B} (\mathcal{B}f \circ \alpha(s))(y) < n \\
&\Leftrightarrow_2 \sum_{y \notin [\![\varphi]\!]_B} \beta(f(s))(y) < n \\
&\Leftrightarrow \beta(f(s)) \in \lambda_n [\![\varphi]\!]_B \\
&\Leftrightarrow f(s) \in [\![\Box_n \varphi]\!]_B.
\end{aligned}
$$

The equivalence $\Leftrightarrow_1$ makes use of the induction hypothesis ($[\![\varphi]\!]_A = f^{-1}[[\![\varphi]\!]_B]$) and the definition of $\mathcal{B}\sigma$. The equivalence $\Leftrightarrow_2$ makes use of the definition of homomorphism.

Summarizing, the behavioral equivalence associated with the finite multiset functor, and therefore r-bisimilarity, coincides with logical indistinguishability under $\mathcal{L}_G$.

4.2.1. G-bisimilarity and the Hennessy-Milner property

In light of the development so far, to complete the alternative proof of Theorem 3.1, it suffices to show that the Hennessy-Milner property holds for graded modal logic over the class of image-finite Kripke frames modulo g-bisimilarity. The Hennessy-Milner property is proved in [7] for ω-saturated models. We adapt this proof for image-finite Kripke frames. For a state w in a Kripke frame, let $L(w)$ be the set of formulas true at w.

Proposition 4.11. *Let M_1 and M_2 be image-finite Kripke frames, and let $w_1 \in X_1$ and $w_2 \in X_2$. Then $L(w_1) = L(w_2)$ if and only if w_1 and w_2 are g-bisimilar.*

Proof The proof from right to left is a simple induction on the formulas (Proposition 3.3 in [7]).

From left to right, assume that $L(w_1) = L(w_2)$. We define a g-bisimulation relation $\mathcal{Z}$ as follows: $X_1 \mathcal{Z} X_2$ if and only if each of the following holds:

- $|X_1| = |X_2|$,
- for each $x_1 \in X_1$, there is $x_2 \in X_2$ such that $L(x_1) = L(x_2)$, and
- for each $x_2 \in X_2$, there is $x_1 \in X_1$ such that $L(x_1) = L(x_2)$.

It remains to check that $\mathcal{Z}$ is indeed a g-bisimulation relation. First $\mathcal{Z}$ is non-empty, since $L(w_1) = L(w_2)$, and hence $\{w_1\} \mathcal{Z} \{w_2\}$.

Condition 1(a) of Definition 2.4 is immediate from the definitions. Conditions 1(b) holds due the the following argument. Suppose $X_1 \mathcal{Z} X_2$, then for every $x_1 \in X_1$, there is an $x_2 \in X_2$, such that $L(x_1) = L(x_2)$. But then $\{x_1\} \mathcal{Z} \{x_2\}$, thus satisfying Condition 1(b). The proof for condition 1(c) is similar.

For Condition 2(a), suppose $\{x_1\} \mathcal{Z} \{x_2\}$. Let $X_1 \subseteq R_1(x_1)$ be chosen, and suppose toward a contradiction that there is no $X_2 \subseteq R_2(x_2)$, such that $X_1 \mathcal{Z} X_2$. Suppose $X_1 = \{a_1, \ldots, a_n\}$, and for a suitable permutation π of $\{1, \ldots, n\}$, let $\{a_{\pi_1}, \ldots, a_{\pi_k}\}$ be a maximal set for which $L(a_{\pi_i}) \neq L(a_{\pi_j})$, but for each a_i, there is some j, such that $L(a_i) = L(a_{\pi_j})$. Let $\sigma : \{a_{\pi_1}, \ldots, a_{\pi_k}\} \to \mathbb{N}$ be the multiset for which $\sigma(a_{\pi_i}) = |\{j \mid L(a_{\pi_i}) = L(a_j)\}|$. By our assumption, there must be a j, such that there is no subset $Y \subseteq R_2(x_2)$ of size $\sigma(a_{\pi_j})$, where for each $y \in Y$, $L(y) = L(a_{\pi_j})$. For each of the finitely many $Y \subseteq R_2(x_2)$ of size $\sigma(a_{\pi_j})$, let φ_Y be a formula that is true at a_{π_j}, but not at every $y \in Y$. Then $\Diamond_{\sigma(a_{\pi_j})} \bigwedge\{\varphi_Y : Y \subseteq R_2(x_2), |Y| = \sigma(a_{\pi_j})\}$ is true at x_1, but not at x_2, contradicting our initial assumption that $\{x_1\} \mathcal{Z} \{x_2\}$. Thus there is an $X_2 \subseteq R_2(x_2)$, such that $X_1 \mathcal{Z} X_2$. A similar argument shows that for every $X_2 \subseteq R_2(x_2)$, there is an $X_1 \subseteq R_1(x_1)$, such that $X_1 \mathcal{Z} X_2$, thus giving us Condition 2(b). □

This completes the alternative proof of Theorem 3.1.

5. The Hennessy-Milner property fails without image-finiteness

Consider the following two frames:

1. $F_1 = (S_1, R_1)$, with $S_1 = \mathbb{N} \cup \{0\}$ and with $R_1 = \{(n+1, n) \mid n \geq 1\} \cup \{(0, n) \mid n \geq 1\}$, and
2. $F_2 = (S_2, R_2)$, with $S_2 = S_1 \cup \{\omega\}$ and $R_2 = R_1 \cup \{(0, \omega), (\omega, \omega)\}$.

We observe that $(F_1, 0)$ and $(F_2, 0)$ are neither r-bisimilar nor g-bisimilar according to a natural extension of g-bisimulation to infinite sets. Indeed, suppose that $\mathcal{R}$ is an r-bisimulation in which $0 \mathcal{R} 0$. Then using Proposition 2.2, there is a bijection $i : R_1(0) \to R_2(0)$ in which for each $a \in R_1(0)$, $(a, i(a)) \in \mathcal{R}$. Then some $a \in R_1(0)$ maps under i to ω. But $a \models \Diamond_1^a \Box_1 \bot$, while $\omega \not\models \Diamond_1^a \Box_1 \bot$, where $\Diamond_1^0 \varphi = \varphi$ and $\Diamond_1^{a+1} \varphi = \Diamond_1 \Diamond_1^a \varphi$. As the submodels generated from a and ω are image-finite, the Hennessy-Milner Property holds in those submodels. Thus a and ω cannot be bisimilar, contradicting the definition of $\mathcal{R}$.

Suppose now that we were to relax the constraint on g-bisimulation in Definition 2.4 that only finite sets can be paired, and suppose for a contradiction that $\mathcal{Z}$ were such a bisimulation between F_1 and F_2 in which $\{0\} \mathcal{Z} \{0\}$. Then $R_1(0) \mathcal{Z} R_2(0)$, in which case given ω, there must be an $a \in S_1$, such that $\{a\} \mathcal{Z} \{\omega\}$, which cannot be true for the same reason as given for the r-bisimulation.

We next observe that $(F_1, 0)$ and $(F_2, 0)$ satisfy the same graded model logic formulas, and hence cannot have the Hennessy-Milner Property. This is argued by induction on the structure of the formula. The key step is $\varphi = \Diamond_n \psi$. Clearly any n states in S_1 can be matched by the identical states in S_2. But conversely, were ω among the n states chosen in S_2, we can safely select any state m in S_1 greater than the modal (nesting) depth of ψ. We appeal to the fact that the submodels generated from (S_1, m) and from (S_2, ω) are non-branching, on which setting graded modal logic and ordinary modal logic coincide. Furthermore, modal logic formulas with modal depth k cannot distinguish states that are no fewer than k relational steps from a terminating state (Theorem 32 in [8]).

By Corollary 3.4, r-bisimilarity is included in g-bisimilarity over arbitrary Kripke frames. We have moreover shown that the converse inclusion holds over image-finite Kripke frames. We do not know whether g-bisimilarity is included in r-bisimilarity over arbitrary Kripke frames.

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