Approximation Algorithms for the Weighted Indepndent Set Problem in Sparse Graphs ⋆

Akihisa Kako [a,1], Takao Ono [a,*], Tomio Hirata [a], Magnús M. Halldórsson [b]

[a] *School of Information Science, Nagoya University, Nagoya, Japan*

[b] *School of Computer Science, Reykjavík University, Reykjavík, Iceland.*

Abstract

The approximability of the unweighted independent set problem has been analyzed in terms of sparseness parameters such as the average degree and inductiveness. In the weighted case, no corresponding results are possible for average degree, since adding vertices of small weight can decrease the average degree arbitrarily without significantly changing the approximation ratio. In this paper, we introduce two weighted measures, namely *weighted average degree* and *weighted inductiveness*, and analyze algorithms for the weighted independent set problem in terms of these parameters.

Key words: weighted independent set problem, approximation algorithm, weighted degree

1991 MSC: 68W25, 05C69

1 Introduction

An independent set in a graph is a set of vertices in which no two vertices are adjacent. The (weighted) independent set problem is that of finding a maximum (weight) independent set. Numerous approximation algorithms have been proposed and analyzed for this problem. In the unweighted case, an algorithm with approximation ratio $(\Delta+3)/5$ was given by Berman and Fujito [2] for graphs of maximum degree Δ. Vishwanathan proposed an SDP-based algorithm with approximation ratio $O(\Delta \log\log \Delta / \log \Delta)$, which first appeared in [5]. For graphs of average degree $\bar{d}$, Hochbaum [10] proved that an LP-based algorithm has approximation ratio $(\bar{d}+1)/2$. Halldórsson and Radhakrishnan [9] improved this approximation ratio to $(2\bar{d}+3)/5$. Moreover, an algorithm with approximation ratio $O(\bar{d} \log\log \bar{d} / \log \bar{d})$ was proposed by Halldórsson [6]. In the weighted case, Halldórsson and Lau [7] gave an algorithm with approximation ratio $(\Delta+2)/3$. For δ-inductive graphs approximation ratio $(\delta+1)/2$ is known due to Hochbaum [10], and Halldórsson [6] proposed an algorithm with approximation ratio $O(\delta \log\log \delta / \log \delta)$. Note that $\delta \leq \Delta$ for any graph.

In this paper, we extend the approximation algorithms of [6,10] to the weighted case. In the weighted independent set problem, by inserting vertices of small weight we can arbitrarily reduce the average degree $\bar{d}$ of the input graph

⋆ A preliminary version appeared in the Proceedings of the 31st Workshop on Graph-Theoretic Concepts in Computer Science, WG2005, LNCS 3787.
* Corresponding author.
Email addresses: ono@is.nagoya-u.ac.jp (Takao Ono), hirata@is.nagoya-u.ac.jp (Tomio Hirata), mmh@ru.is (Magnús M. Halldórsson).
[1] Currently with Denso Create Inc.

without significantly changing the approximation ratio. Under the assumption P $\neq$ NP, we will show that no approximation algorithms for this problem can have an approximation ratio depending only on $\bar{d}$. Thus we introduce the *weighted average degree* measure $\bar{d}_w$ and analyze the approximation of several algorithms in terms of it. For weighted graphs, there exist approximation algorithms whose approximation ratio is analyzed in terms of inductiveness. We extend inductiveness to weighted version and introduce the *weighted inductiveness* δ_w.

We note that the definition of the weighted average degree and Theorem 6 of this paper have already appeared in the paper of Demange and Paschos [3]. We will give the proof of the theorem in order to make this paper self-contained. We also note that some arguments in this paper ~~follows ones in~~ [3,5].

The rest of this paper is organized as follows. In Section 2 we define the weighted average degree and the weighted inductiveness. We also show the relationship between the various parameters. In Section 3 we propose a greedy algorithm for finding an independent set with weight at least $\max(W/(\bar{d}_w + 1), W/(\delta_w + 1))$, where W is the total weight of the graph. We also prove that this algorithm has approximation ratio $\max(\delta_w, 1)$. In Section 4 we prove that the approximation ratio of $\min((\bar{d}_w + 1)/2, (\delta_w + 1)/2)$ can be achieved by an LP-based algorithm. Finally we will prove that the approximation ratios of $O(\bar{d}_w \log\log \bar{d}_w / \log \bar{d}_w)$ and $O(\delta_w \log\log \delta_w / \log \delta_w)$ can be achieved by an SDP-based algorithm in Section 5.

2 Preliminaries

2.1 Definitions

Let G be an undirected graph where each vertex v has positive weight w_v. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively, as usual. Without loss of generality, we will assume that G is connected. Let $W(G)$ be the sum of the weights of all vertices. The number of vertices in G is denoted by $n(G)$. Let $\Delta(G)$ and $\bar{d}(G)$ denote the maximum and the average degree of G, respectively. Let $d(v, G)$ be the degree of vertex v in G. The inductiveness $\delta(G)$ of a graph G is given by

$$\delta(G) = \max_{H \subseteq G} \min_{v \in V(H)} d(v, H), \tag{1}$$

where $H \subseteq G$ denotes that H is a subgraph of G. Let π be an ordering of the vertices in V, that is, a one to one map $V \rightarrow \{1, 2, \ldots, n\}$ $(n = |V|)$. We define the *right degree* of a vertex v in G with respect to π by:

$$d^{\pi}(v, G) = |\{u \in V | (u, v) \in E, \pi(u) > \pi(v)\}|. \tag{2}$$

The right degree of a vertex v is the number of adjacent vertices to the right when we arrange vertices from left to right according to π. If there exists π such that $m \geq \max_v d^{\pi}(v, G)$, we call G an m-inductive graph.

For a vertex set X, let $w(X)$ denote the sum of the weights of the vertices in X. Let $N_G(v)$ denote the set of vertices adjacent to vertex v in G. For a vertex v, we define the *weighted degree* $d_w(v, G)$ in G by:

$$d_w(v,G) = \frac{w(N_G(v))}{w_v}. \tag{3}$$

Let $\Delta_w(G) = \max_v d_w(v,G)$ be the maximum weighted degree of G. We will omit G if clear from the context. We define the *weighted average degree* $\bar{d}_w(G)$ of graph G as follows:

$$\bar{d}_w(G) = \frac{\sum_{v\in V} w_v d_w(v,G)}{W}. \tag{4}$$

In fact, we can represent the weighted average degree in the following alternative forms:

$$\begin{aligned}\bar{d}_w(G) &= \frac{\sum_{v\in V} w(N(v))}{W} &(5)\\ &= \frac{\sum_{v\in V} w_v d(v)}{W}. &(6)\end{aligned}$$

The *weighted inductiveness* $\delta_w(G)$ of a graph G is given by

$$\delta_w(G) = \max_{H\subseteq G} \min_{v\in V(H)} d_w(v,H). \tag{7}$$

We define the *right weighted degree* of a vertex v for an ordering π in G by:

$$d_w^\pi(v,G) = \frac{w(\{u\in V | (u,v)\in E, \pi(u) > \pi(v)\})}{w_v}.$$

If there exists π such that $m \geq \max_v d_w^\pi(v,G)$, we call G a weighted m-inductive graph.

We note that the weighted degree has the following "scaling property" that it is not affected when we uniformly multiply all the weights by a constant. This means that both the weighted average degree and the weighted inductiveness satisfy the scaling property. We also note that the weighted degree is monotone in the sense that if G' is a subgraph of G, then $d_w(v, G') \leq d_w(v, G)$ for any vertex $v \in V(G)$. The weighted inductiveness is also monotone, that is, $\delta_w(G') \leq \delta_w(G)$ if G' is a subgraph of G.

We denote by $\alpha_w(G)$ the maximum weight of an independent set in G. For an algorithm A, $A(G)$ denotes the weight of the independent set obtained by A on G. Then the approximation ratio of A is defined by

$$\sup_G \frac{\alpha_w(G)}{A(G)}.$$

We will consider unweighted graphs as weighted ones where each vertex has unit weight. We use $\alpha(G)$ for the size of a maximum cardinality independent set on G.

2.2 Properties of the degrees

Let π be an ordering of the vertices of G and v_i a vertex with $\pi(v_i) = i$. We define $V_i^\pi = \{v_j | j \geq i\}$ as the suffix of the vertex set starting with i in the ordering π. Let G_i^π be the subgraph of G induced by V_i^π. Smallest-first ordering π is an ordering such that the weighted degree of v_i is minimum in G_i^π for all i ($1 \leq i \leq n$). We can find a smallest-first ordering in polynomial time by greedily choosing vertices of minimum weighted degree. We can prove the following theorem in the same manner as in the case of unweighted inductiveness [12].

Theorem 1 *For any ordering π, the inequality*

$$\delta_w(G) \le \max_v d_w^{\pi}(v, G)$$

holds. Moreover, equality holds when π is a smallest-first ordering.

For unweighted graphs, the relationships $\delta \le \Delta$ and $\bar{d} \le \Delta$ are obvious. Their counterpart for the weighted case, $\delta_w \le \Delta_w$ and $\bar{d}_w \le \Delta_w$ are also obvious. We can further show that both Δ and Δ_w dominate all the measures δ, δ_w, $\bar{d}$, and $\bar{d}_w$:

Theorem 2 *The following relationships hold for all graphs G:*

$$\delta \le \Delta_w \tag{8}$$
$$\delta_w \le \Delta \tag{9}$$
$$\bar{d} \le \Delta_w \tag{10}$$
$$\bar{d}_w \le \Delta. \tag{11}$$

PROOF. Let π_1 be the vertex ordering such that $\pi_1(u) < \pi_1(v)$ if $w_u < w_v$. Theorem 1 and the definition of the maximum weighted degree Δ_w ensure the inequalities

$$\delta \le \max_{v \in V} d^{\pi_1}(v, G), \qquad \max_{v \in V} d_w^{\pi_1}(v, G) \le \Delta_w.$$

Observe that the right-neighbors of a vertex v under π_1 (i.e, those neighbors u of v with $\pi_1(u) > \pi_1(v)$) are all of weight at least that of v. That implies that $d^{\pi_1}(v, G) \le d_w^{\pi_1}(v, G)$. Thus we have the inequality (8). We can prove (9) in a similiar way by considering the ordering that is the reverse of π_1.

In order to prove inequality (10), observe that we can bound the sum of the weighted degree in the graph from below by twice the degree sum:

$$\sum_{v\in V} d_w(v) = \sum_{v\in V}\sum_{u:(u,v)\in E}\frac{w_u}{w_v} = \sum_{(u,v)\in E}\left(\frac{w_u}{w_v}+\frac{w_v}{w_u}\right) \geq 2|E| = n\bar{d}.$$

Thus,

$$\Delta_w = \max_{v\in V} d_w(v) \geq \frac{1}{n}\sum_{v\in V} d_w(v) \geq \bar{d}.$$

Finally, inequality (11) follows immediately from Equation (6). □

Thus, we have the following partial order on the degree measures

$$\{\delta, \delta_w, \bar{d}, \bar{d}_w\} \leq \{\Delta, \Delta_w\}.$$

There exist graphs where δ_w and $\bar{d}_w$ are arbitrarily smaller than δ: Consider the complete bipartite graph $G = K_{n/2,n/2}$, where vertices have weight 1 on one side and w on the other side. Then, $\delta(G) = n/2$, while $\delta_w(G) = (n/2)/w$. For $\bar{d}_w$, we consider an n-clique of $\{v_0, v_1, \ldots, v_{n-1}\}$ plus v_n connected to only v_{n-1}. The weight w_i of v_i is given by $w_i = 1$ for $0 \leq i \leq n-1$ and $w_n = w$. In the graph, $\delta = n-1$ and

$$\bar{d}_w = \frac{w + (n-1)^2 + n}{w+n} = 1 + O\left(\frac{n^2}{w}\right).$$

2.3 Motivation for the weighted average degree

As mentioned already, there are no approximation results with the parameter $\bar{d}$ for the weighted case, whereas Δ and δ have such results. The main difference is that Δ and δ are monotone while $\bar{d}$ is not. That is, for a subgraph G' of G, it is clear that $\Delta(G') \leq \Delta(G)$ and $\delta(G') \leq \delta(G)$ but $\bar{d}(G')$ can be larger than $\bar{d}(G)$.

Because of this lack of monotonicity, we can construct a weighted graph of constant average degree by adding some vertices, without affecting much the size of the maximum weighted independent set. Combining with the fact that we cannot approximate the unweighted independent set within constant factor unless P = NP [1], the following theorem holds:

Theorem 3 *Let f be any real-valued function. If there exists an $f(\bar{d})$-approximation algorithm for the weighted independent set problem on graphs with average degree $\bar{d}$, then* P = NP.

PROOF. We assume that A_w is an $f(\bar{d})$-approximation algorithm for the weighted maximum independent set problem. We will show that we can then construct a constant-ratio approximation algorithm A for the (unweighted) independent set problem using A_w.

We are given a connected graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. We assume that $n \geq 7$, because otherwise we can find a maximum independent set in G in polynomial time. We then construct a supergraph $G' = (V', E')$ of G as follows: $V' = V + U$ where $U = \{u_1, u_2, \ldots, u_m\}$ is a set of dummy vertices. $E' = E + E_1 + E_2$, where E_1 consists of the m edges of the form (v_1, u_i), making G' connected, and E_2 is a set of edges connecting $2n$ arbitrary pairs in U. (This construction always works because $m \geq n - 1$ and $n \geq 7$.) The vertex weights of G' are defined by:

$$w(v) = \begin{cases} 1 & v \in V, \\ 1/(2f(4)m) & v \in U. \end{cases}$$

We note that the average degree of G' is 4, because it has $(m+n)$ vertices and $m + m + 2n = (2m + 2n)$ edges.

Algorithm A uses A_w on the graph G' with weights w, and removes vertices outside of V from the solution to return an independent set of G.

Let $A(G)$ be the size of the independent set found by A on G. Similary we use $A_w(G')$ for the weight of the independent set found by A_w for G'.

Our construction of G' ensures that any independent set of G is also an independent set of G'. This immediately implies that $\alpha(G) \leq \alpha_w(G')$. Thus $A_w(G') \geq \alpha_w(G')/f(4) \geq \alpha(G)/f(4)$. The size of the independent set found by A is bounded from below by $A_w(G') - |U|/(2f(4)m) \geq \alpha(G)/f(4) - 1/(2f(4))$. Moreover, any singleton vertex is an independent set, or $\alpha(G) \geq 1$. This means that $A(G) \geq \alpha(G)/f(4) - \alpha(G)/(2f(4)) = \alpha(G)/(2f(4))$. Thus the algorithm A has approximation ratio $2f(4)$, which is constant. $\square$

Theorem 3 states that the unweighted average degree $\bar{d}$ is not a valid para~~tem~~er for the approximation ratio for the weighted independent set problem. It is natural to ask whether the unweighted inductiveness δ is valid or not. In fact, as we will see in Section 3.1, δ can be used as the parameter ~~for the approximation ratio for the weighted independent set problem~~.

2.4 Reduction from weighted graph to unweighted one

The weighted independent set problem can easily be reduced to the unweighted version as follows: Assume that we are given a graph G with weight w. We construct an unweighted graph $G' = (V', E')$ by $V' = \{(v, i) | v \in V,\, 1 \leq i \leq w_v\}$ and $E' = \{((u, i), (v, j)) | (u, v) \in E,\, 1 \leq i \leq w_u,\, 1 \leq j \leq w_v\}$, where we assume that the weights w_v are positive integers. This reduction preserves the independent set, that is, any independent set S of G induces the independent set $S' = \{(v, i) | v \in S,\, 1 \leq i \leq w_v\}$ of G' of size $|S'| = w(S)$. Conversely, for any independent set S' of G', the set $S = \{v | (v, i) \in S' \text{ for some } i\}$ is the independent set of G of weight $w(S) \geq |S'|$.

We note that this translation increases the degree of the vertex: a vertex (v, i) of G' has degree $d((v, i), G') = w(N(v)) = d_w(v) \cdot w_v$. Thus the maximum degree, the average degree, and the inductiveness of G' must be at least the weighted counterparts of G. This means that with this translation no interesting results for approximation ratios using Δ_w, $\bar{d}_w$, and δ_w can be achieved.

Demange and Paschos [3] have introduced the notion of FA-reduction and proposed a general FA-reduction between the maximization problem and the weighted maximization problem on graphs. An FA-reduction from problem P to problem Q is a triple (f, g, h), where f is a polynomial function which converts an instance p of P to the instance $f(p)$ of Q, h is a function taking an instance p of P and a feasible solution x of $f(p)$ to produce the feasible solution $h(x)$ of p in polynomial time, and g is a function such that for any approximation algorithm A for Q with approximation ratio ρ the sequential application of f, A, and h is an $g(\rho)$-approximation algorithm for P. They have shown a generic FA-reduction from the weighted problems to unweighted ones transforming any approximation ratio ρ for latter into an approximation ratio $\Omega(\rho/\log n)$ for former. Moreover they have improved this FA-reduction for the maximum independent set problem. However the improved reduction still introduces extra $\log\log n$ factor to the approximation ratio.

3 Greedy algorithm

3.1 Previous results

For unweighted graphs, the greedy algorithm can be described as follows. We select a minimum degree vertex, add it to an independent set solution I, and delete this vertex and all of its neighbors from the graph. We repeat this process for the remaining subgraph until the subgraph becomes empty, and then output I. This algorithm attains the Turán bound [9,10]:

$$|I| \geq \frac{n}{\bar{d}+1}. \tag{12}$$

For weighted graphs, the lower bound

$$w(I) \geq \frac{W}{\delta+1}. \tag{13}$$

can be achieved by the smallest-last coloring procedure [12], as it produces a vertex coloring using at most $(\delta+1)$ colors.

The greedy algorithm WG for weighted graph $G = (V, E)$ is as follows:

(1) Let $i \leftarrow 1$, $G_1 \leftarrow G$, and $I \leftarrow \emptyset$

(2) Repeat (2)–(6) until G_i becomes empty:

(3) Select a vertex v_i of minimum weighted degree.

(4) Add v_i to I.

(5) Remove v_i and its neighbors from G_i. The remaining graph is G_{i+1}.

(6) Increment i by 1.

(7) Return I as an independent set.

Let $R = |I|$ be the number of iterations of the loop of WG.

On unweighted graphs, WG is equivalent to the classical minimum-degree greedy algorithm, since in this case the weighted degree is identical to the (unweighted) degree.

Sakai, Togasaki, and Yamazaki proposed an algorithm which is essentially the same as WG and proved the following theorem [14].

Theorem 4 ([14]) WG *finds an independent set satisfying:*

$$\mathsf{WG}(G) \geq \sum_{v \in V} \frac{w_v^2}{w(N(v)) + w_v}.$$

3.2 Lower bound

We use the following proposition.

Proposition 5 *Assume that* $a_i > 0$, $b_i > 0$ *for all* $1 \leq i \leq n$. *Then the inequality*

$$\sum_{i=1}^{n} \frac{b_i^2}{a_i} \geq \frac{\left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n} a_i}$$

holds.

PROOF. The inequality is equivalent to

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} \frac{b_i^2}{a_i} \geq \left(\sum_{i=1}^{n} b_i\right)^2.$$

This inequality comes from the Cauchy-Schwarz inequality $\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \geq \left(\sum_{i=1}^{n} x_i y_i\right)^2$, by assigning $x_i = \sqrt{a_i}$ and $y_i = b_i/\sqrt{a_i}$. □

Theorem 6 (Theorem 5 of [3]) WG *produces an independent set satisfying:*

$$\mathsf{WG}(G) \geq \frac{W}{\bar{d}_w + 1}.$$

PROOF. We obtain a lower bound of $\bar{d}_w W$:

$$\begin{aligned}
\bar{d}_w W &= \sum_{v \in V(G)} w_v d_w(v, G) && \text{(from (5))} \\
&\geq \sum_{i=1}^{R} \sum_{v \in N_{G_i}(v_i) \cup \{v_i\}} w_v d_w(v, G_i) && \text{(by monotonicity, as } G_i \subseteq G\text{)} \\
&\geq \sum_{i=1}^{R} \sum_{v \in N_{G_i}(v_i) \cup \{v_i\}} w_v d_w(v_i, G_i) && (v_i \text{ has the least weighted degree}) \\
&= \sum_{i=1}^{R} \left[w(N_{G_i}(v_i)) + w_{v_i}\right] d_w(v_i, G_i). && (d_w(v_i, G_i) \text{ is fixed in the inner sum})
\end{aligned}$$

Adding $W = \sum_{i=1}^{R} \left[w(N_{G_i}(v_i)) + w_{v_i}\right]$, we can deduce, using (3), the inequality

$$\left(\bar{d}_w + 1\right) W \geq \sum_{i=1}^{R} \frac{\left[w(N_{G_i}(v_i)) + w_{v_i}\right]^2}{w_{v_i}}.$$

Finally we apply Proposition 5 with $a_i = w_{v_i}$, $b_i = w(N_{G_i}(v_i)) + w_{v_i}$, giving

$$\left(\bar{d}_w + 1\right) W \geq \frac{W^2}{\mathsf{WG}(G)}.$$

This implies the theorem. □

One may observe that Theorem 4 also leads to Theorem 6.

We note that this analysis depends on our definition of the weighted degree. In fact, our definition is a natural extension of the (unweighted) degree in the following sense. (1) the weighted degree satisfies the scaling property, and (2) our definition captures the relation between the gain and the possible loss when adding a vertex v to be in an independent set: we gain its weight $w(v)$ while possibly losing the weights of its neighbors $w(N(v)) = w(v)d_w(v)$, just as we gain one vertex while losing $d(v)$ vertices in the unweighted case.

Theorem 6 is a natural extension of (12) to the weighted independent set problem. Similarly, for the weighted inductiveness δ_w we can prove the theorem corresponding to (13) for the unweighted inductiveness δ.

Theorem 7 WG *produces an independent set satisfying:*

$$\mathsf{WG}(G) \geq \frac{W}{\delta_w + 1}.$$

PROOF. Because $W = \sum_{i=1}^{R} \left[w(N_{G_i}(v_i)) + w_{v_i}\right]$ and $\delta_w \geq d_w(v_i, G_i)$ for $i = 1, \ldots, R$, the inequality

$$\delta_w W \geq \sum_{i=1}^{R} [w(N_{G_i}(v_i)) + w_{v_i}]\, d_w(v_i, G_i)$$

holds. With this inequality, we can prove this theorem in the same way as Theorem 6. □

The following example shows that the lower bounds given by Theorems 6 and 7 are both tight. Let G be a star with n vertices. We assign weight 1 to the center vertex and $1/\sqrt{n-1}$ to the other vertices. In this graph, all vertices have the same weighted degree of $\sqrt{n-1}$, so WG may output the center vertex alone for $\mathsf{WG}(G) = 1$. We have $\bar{d}_w = \delta_w = \sqrt{n-1}$, and $W = \sqrt{n-1} + 1$. Therefore, the inequalities in Theorems 6 and 7 hold here with equality.

It is clear that the maximum weighted independent set consists of the non-center vertices, giving $\alpha_w(G) = \sqrt{n-1}$. Thus the approximation ratios of WG on this instance are $\bar{d}_w$ and δ_w. This gives lower bounds on the approximation ratios of WG.

3.3 Approximation ratio

From Theorems 6 and 7, the approximation ratios $\bar{d}_w + 1$ and $\delta_w + 1$ are immediate. The latter ratio can be slightly improved.

Theorem 8 WG *attains approximation ratio* $\max(\delta_w, 1)$.

PROOF. Let $V_i = N_{G_i}(v_i) \cup \{v_i\}$, and H_i be the subgraph of G induced by V_i. If $\delta_w \leq 1$, it is easy to see that $\alpha_w(H_i) = w_{v_i}$ and thus $\alpha_w(G) \leq \sum_{i=1}^{R} \alpha_w(H_i) = \sum_{i=1}^{R} w_{v_i} = \mathsf{WG}(G)$. Otherwise, by the property of WG and the definition of inductiveness, $\alpha_w(H_i) \leq \max(w_{v_i}, w(N_{H_i}(v_i))) = w_{v_i} \cdot \max(1, d_w(v_i, H_i)) \leq w_{v_i} \cdot \max(1, \delta_w(G)) = w_{v_i} \cdot \delta_w(G)$. The inequalities

$$\alpha_w(G) \leq \sum_{i=1}^{R} \alpha_w(H_i) \leq \sum_{i=1}^{R} w_{v_i} \cdot \delta_w(G) = \mathsf{WG}(G) \cdot \delta_w(G)$$

are immediate. □

This theorem immediately implies that this problem is polynomial time solvable for the graphs with $\delta_w \leq 1$; we will ignore this case hereafter.

4 LP-based algorithms

We will consider the combination of linear programming and the greedy algorithm. With the lower bound (12), Hochbaum [10] proved that this combination achieves the approximation ratio $(\bar{d}+1)/2$. Similarly the approximation ratio $(\delta+1)/2$ can be shown. In this section we extend Hochbaum's analysis to the weighted case and prove that the proposed algorithm has corresponding approximation ratios $(\bar{d}_w+1)/2$ and $(\delta_w+1)/2$.

4.1 LP relaxation for the weighted independent set problem

The weighted independent set problem has the following integer programming formulation:

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in V} w_i x_i, & (14)\\ \text{subject to} \quad & x_i + x_j \leq 1 \quad \text{for all } (i,j) \in E, \\ & x_i \in \{0,1\} \quad \text{for all } i \in V. \end{aligned}$$

Relaxing the integral constraint, we deduce the following linear program:

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in V} w_i x_i, \\ \text{subject to} \quad & x_i + x_j \leq 1 \ \ \text{for all } (i,j) \in E, \\ & 0 \leq x_i \leq 1 \ \ \text{for all } i \in V. \end{aligned} \tag{15}$$

We can obtain an optimal solution to this LP each of whose elements is 0, 1/2, or 1 [16]. Note that this LP can be solved with a combinatorial algorithm [13,15]. We classify the vertices into three sets according to the value of x_i, that is, $S_1 = \{i \in V | x_i = 1\}$, $S_{1/2} = \{i \in V | x_i = 1/2\}$, $S_0 = \{i \in V | x_i = 0\}$. Note that S_1 is an independent set of G and no vertex in $S_{1/2}$ has a neighbor in S_1. We also note that $S_{1/2}$ induces a subgraph with no isolated vertices.

4.2 Algorithm

We first solve the LP relaxation to divide the vertex set V into three subsets S_1, $S_{1/2}$, and S_0 as above. We then apply WG to the subgraph H induced by $S_{1/2}$ to obtain an independent set I_H of H. Finally, we output the independent set $I = S_1 \cup I_H$. We call this algorithm WGL.

4.3 Approximation ratio

From Theorem 6, we can prove the following theorem in the same manner as the proof of Hochbaum [10] of the approximation ratio $(\bar{d}+1)/2$ for unweighted graphs.

Theorem 9 *Approximation ratio of* WGL *is* $(\bar{d}_w + 1)/2$.

PROOF. We prove the following chain of inequalities:

$$
\begin{aligned}
\frac{\alpha_w(G)}{\mathsf{WGL}(G)} &\le \frac{w(S_1)+w(S_{1/2})/2}{w(S_1)+w(S_{1/2})/(\bar{d}_w(H)+1)} && (16)\\
&\le \frac{1}{2}\left[\frac{w(S_{1/2})\bar{d}_w(H)+w(S_1)+w(S_0)}{w(S_{1/2})+w(S_1)+w(S_0)}+1\right] && (17)\\
&\le \frac{\bar{d}_w+1}{2}. && (18)
\end{aligned}
$$

We have used the optimal solution to LP (15) to partition V into S_0, $S_{1/2}$, S_1. This guarantees that $w(S_1) \ge w(S_0)$. Moreover, we mentioned that H has no isolated vertices. This means that $d(v, H) \ge 1$ for each vertex $v \in S_{1/2}$, which in combination with Equation (6) ensures that $\bar{d}_w(H) \ge 1$. Thus we can show Inequality (17) as follows, in which we use $D = \bar{d}_w(H) + 1$ for readability:

$$
\begin{aligned}
&\frac{w(S_1)+w(S_{1/2})/2}{w(S_1)+w(S_{1/2})/(\bar{d}_w(H)+1)}\\
&= \frac{Dw(S_1)+Dw(S_{1/2})/2}{Dw(S_1)+w(S_{1/2})}\\
&= 1+\frac{(D/2-1)w(S_{1/2})}{Dw(S_1)+w(S_{1/2})}\\
&\le 1+\frac{(D/2-1)w(S_{1/2})}{w(S_1)+w(S_0)+w(S_{1/2})}\\
&= \frac{Dw(S_{1/2})/2+w(S_1)+w(S_0)}{w(S_{1/2})+w(S_1)+w(S_0)}\\
&= \frac{[\bar{d}_w(H)+1]w(S_{1/2})/2+w(S_1)+w(S_0)}{w(S_{1/2})+w(S_1)+w(S_0)}\\
&= \frac{1}{2}\left[\frac{w(S_{1/2})\bar{d}_w(H)+w(S_1)+w(S_0)}{w(S_{1/2})+w(S_1)+w(S_0)}+1\right].
\end{aligned}
$$

We argue Inequality (18) as follows. Since we have assumed that the input graph G is connected, each vertex is of positive degree, or, $d(v, G) \ge 1$ for each vertex $v \in V$. Moreover, because H is a subgraph of G induced by $S_{1/2}$, for each vertex $v \in S_{1/2}$ the degree in H is at most that in G, that is, $d(v,H) \leq d(v,G)$. Hence,

$$\begin{aligned}\sum_{v\in V} d(v,G)w(v) &= \sum_{v\in S_{1/2}} d(v,G)w(v) + \sum_{v\in S_1\cup S_0} d(v,G)w(v)\\ &\geq \sum_{v\in S_{1/2}} d(v,H)w(v) + \sum_{v\in S_0\cup S_1} 1\cdot w(v)\\ &= \bar{d}_w(H)w(S_{1/2}) + [w(S_0)+w(S_1)].\end{aligned}$$

Using Equation (6), this is equivalent to

$$\bar{d}_w(G)W(G) \geq \bar{d}_w(H)w(S_{1/2}) + w(S_1) + w(S_0), \tag{19}$$

which in turn is equivalent to Inequality (18). □

We also prove an approximation ratio in terms of the weighted inductiveness.

Theorem 10 *Approximation ratio of* WGL *is* $(\delta_w + 1)/2$.

PROOF. From Theorem 7 and our assumption that $\delta_w \geq 1$,

$$\begin{aligned}\frac{\alpha_w(G)}{\mathsf{WGL}(G)} &\leq \frac{w(S_1)+w(S_{1/2})/2}{w(S_1)+w(S_{1/2})/(\delta_w(H)+1)}\\ &\leq \max\left(1, \frac{\delta_w(H)+1}{2}\right)\\ &\leq \frac{\delta_w+1}{2}. \qquad \square\end{aligned}$$

Proposition 11 *The approximation ratios of Theorems 9 and 10 are tight.*

PROOF. Let t be a number. We consider the split graph $G = (V, E)$, where $V = \{u_1, u_2, \ldots, u_t,\ v_1, v_2, \ldots, v_{2t-1}\}$ and $E = \{(u_i, v_j) | 1 \le i \le t, 1 \le j \le 2t-1\} \cup \{(u_i, u_j) | 1 \le i < j \le t\}$. The subgraph induced by $\{u_i | 1 \le i \le t\}$ is a clique and the vertex set $\{v_j | 1 \le j \le 2t-1\}$ is an independent set. We give weight $1/t + 1/t^3$ to each u_i and weight $1/(2t-1)$ to each v_j. The weighted degree of each vertex is

$$d_w(u_i) = 2t - 1 - \frac{t}{t^2+1}, \qquad d_w(v_j) = 2t - 1 + \frac{2t-1}{t^2}.$$

The weighted average degree and weighted inductiveness of G are:

$$\bar{d}_w = 2t - 1 + \frac{t-1}{2t^2+1}, \qquad \delta_w = 2t - 1 - \frac{t}{t^2+1}.$$

In the optimal solution to LP (15), each x_i has value 1/2. Thus, $S_{1/2} = V(G)$. Because $d_w(u_i) < d_w(v_j)$ for each i and j, WGL returns some singleton set $\{u_i\}$ as an independent set in G. Thus $\mathsf{WGL}(G) = 1/t + 1/t^3$ while it is clear that $\alpha_w = 1$, which is achieved by the independent set $\{v_j | 1 \le j \le 2t-1\}$. So, the approximation ratio is

$$\frac{\alpha_w(G)}{\mathsf{WGL}(G)} = \frac{1}{1/t + 1/t^3} = t - \frac{t}{t^2+1}.$$

This ratio can be evaluated, with $\bar{d}_w$ and δ_w, as follows:

$$
\begin{aligned}
\frac{\alpha_w(G)}{\mathsf{WGL}(G)} &= \frac{\bar{d}_w + 1}{2} - \frac{t-1}{2(2t^2+1)} - \frac{t}{t^2+1} = \frac{\bar{d}_w + 1}{2} - O\left(\frac{1}{t}\right), \\
\frac{\alpha_w(G)}{\mathsf{WGL}(G)} &= \frac{\delta_w + 1}{2} + \frac{t}{2(t^2+1)} - \frac{t}{t^2+1} = \frac{\delta_w + 1}{2} - O\left(\frac{1}{t}\right).
\end{aligned}
$$

As we can set t arbitrarily large, we have that Theorems 9 and 10 are tight. □

5 SDP-based algorithms

5.1 Previous result

The following theorem was proved in [6], based on an unweighted version of Karger, Motwani and Sudan [11]:

Theorem 12 ([6]) *For any fixed real k such that $\vartheta_w(G) \geq 2W/k$, we can construct an independent set in G whose weight is $\Omega(W/(k\delta^{1-1/(2k)}))$.*

The function $\vartheta_w(G)$, defined in [4], is the weighted version of Lovász's ϑ-function. This function can be computed using semi-definite programming (SDP) in polynomial time, and has the property that $\alpha_w(G) \leq \vartheta_w(G)$.

For unweighted graphs, the combination of this theorem and the greedy algorithm yields the approximation ratios $O(\bar{d} \log\log \bar{d}/\log \bar{d})$ and $O(\delta \log\log \delta/\log \delta)$. We show the approximation ratios using the weighted degrees, namely $O(\bar{d}_w \times \log\log \bar{d}_w/\log \bar{d}_w)$ and $O(\delta_w \log\log \delta_w/\log \delta_w)$, are achived by the combination of the greedy algorithm and SDP.

5.2 Approximation ratio for weighted graphs

We will prove the following result for the weighted version of the algorithm with the approximation ratio $O(\bar{d}\log\log\bar{d}/\log\bar{d})$.

Theorem 13 *For any fixed real t such that $t \geq W(G)/\alpha_w(G)$, we can approximate the weighted independent set problem within $O(t^2\bar{d}_w^{1-1/(8t)})$ factor.*

PROOF. Assume that $t \geq W(G)/\alpha_w(G)$ is fixed. Let V' be the subset of vertices with degree less than $2t\bar{d}_w$. Then we can estimate the value $\bar{d}_w W(G)$ as follows:

$$\bar{d}_w W(G) = \sum_{v\in V(G)} w_v d(v) \geq \sum_{v\in V(G)\setminus V'} w_v d(v) \geq 2t\bar{d}_w \sum_{v\in V(G)\setminus V'} w_v.$$

Thus, the inequality $\sum_{v\in V(G)\setminus V'} w_v \leq W(G)/(2t) \leq \alpha_w(G)/2$ holds. We now consider the subgraph G' of G induced by V'. It is obvious that $\alpha_w(G') \geq \alpha_w(G) - \sum_{v\in V(G)\setminus V'} w_v \geq \alpha_w(G)/2$ and that $w(V') \leq w(V(G)) = W$. Thus the value of the weighted ϑ-function for G' satisfies

$$\vartheta_w(G') \geq \alpha_w(G') \geq \alpha_w(G)/2 \geq W/(2t) = 2W/(4t).$$

We apply Theorem 12 with $k = 4t$. The result is that, there exists an algorithm which finds an independent set I of G' with weight $\Omega(w(V')/(t\delta(G')^{1-1/(8t)}))$. Our selection of V' ensures that $\delta(G') \leq 2t\bar{d}_w$. With the inequality $w(V') \geq \alpha_w(G') \geq \alpha_w(G)/2$, weight of I is estimated as follows:

$$w(I) = \Omega(w(V')/(t\delta(G')^{1-1/(8t)})) = \Omega(\alpha_w(G)/(t^2\bar{d}_w^{1-1/(8t)})).$$

This inequality implies the following:

$$\frac{\alpha_w(G)}{w(I)} = O(t^2 \bar{d}_w^{1-1/(8t)}). \qquad \square$$

Theorem 14 *For any fixed real t such that $t \geq W(G)/\alpha_w(G)$, we can approximate the weighted independent set problem within $O(t^2\delta_w^{1-1/(8t)})$ factor.*

PROOF. Let π be an ordering of vertices in G for which the value of $\max_v d_w^\pi(v)$ is equal to δ_w. Let π' be the reverse ordering of π. Assume that $t \geq W(G)/\alpha_w(G)$ is fixed. Let V' be the subset of vertices with right degree less than $2t\delta_w$. Because V' induces a $2t\delta_w$-inductive subgraph of G, the following inequalities hold:

$$\begin{aligned}\delta_w W &\geq \sum_{v\in V(G)} w_v d_w^\pi(v) = \sum_{v\in V(G)} w_v d^{\pi'}(v) \\ &\geq \sum_{v\in V(G)\setminus V'} w_v d^{\pi'}(v) \geq 2t\delta_w \sum_{v\in V(G)\setminus V'} w_v.\end{aligned}$$

The rest of the proof is nearly identical to the one of Theorem 13. $\square$

5.3 Algorithm

In this section we propose two algorithms: WGSA, whose approximation ratio is a function of $\bar{d}_w$, and WGSI, whose approximation ratio is a function of δ_w.

WGSA is the following algorithm: Obtain an independent set by applying WG, independently apply the algorithm of Theorem 13 to obtain another set, and output the one with larger weight.

Theorem 15 WGSA *achieves approximation ratio* $O(\bar{d}_w \log\log \bar{d}_w / \log \bar{d}_w)$ *for the weighted independent set problem.*

PROOF. Let t be a fixed constant. If $t \geq W(G)/\alpha_w(G)$, then the independent set I in the proof of Theorem 13 satisfies the inequality

$$\frac{\alpha_w(G)}{w(I)} = O(t^2 \bar{d}_w^{1-1/(8t)}). \tag{20}$$

Otherwise, WG finds an independent set I' satisfying

$$w(I') \geq \frac{W}{\bar{d}_w + 1} \geq \frac{t\alpha_w(G)}{\bar{d}_w + 1},$$

that is,

$$\frac{\alpha_w(G)}{w(I')} = O(\bar{d}_w / t). \tag{21}$$

Equations (20) and (21) approximately coincide when $t = \log \bar{d}_w / \log\log \bar{d}_w$, giving the theorem. □

WGSI is identical to WGSA, except we replace the algorithm of Theorem 13 with the one of Theorem 14. The analysis is also identical, by simply substituting δ_w for $\bar{d}_w$.

Theorem 16 WGSI *achieves approximation ratio* $O(\delta_w \log\log \delta_w / \log \delta_w)$ *for the weighted independent set problem.*

6 Conclusion

In this paper, we defined the weighted average degree $\bar{d}_w$ and the weighted inductiveness δ_w, and proved lower bounds on the weight of the independent set obtained by the weighted greedy algorithm. We also proved that this algorithm has approximation ratio δ_w. Combining with LP, we obtained the approximation ratio $\min((\bar{d}_w+1)/2, (\delta_w+1)/2)$. Also combining with SDP, we proved that approximation ratios of $O(\bar{d}_w \log\log \bar{d}_w / \log \bar{d}_w)$ and $O(\delta_w \log\log \delta_w / \log \delta_w)$ can be attained.

Here we briefly discuss whether our weighted parameters are applicable to the weighted clique problem or not. In unweighted case, [3] showed the following reduction from the maximum clique problem to the maximum independent set problem. For a vertex v, let G_v be the subgraph of G induced by the vertex v and its neighbors. It is clear to see that any maximum clique of G is a maximum clique of at least one subgraph G_v. This means that finding a maximum clique of G is identical to finding maximum cliques of all of G_v, which is the same as finding maximum independent sets of $\bar{G}_v$. Moreover, G_v and its complement $\bar{G}_v$ have at most $\Delta(G) + 1$ vertices and both $\Delta(G_v)$ and $\Delta(\bar{G}_v)$ are at most $\Delta(G)$. Thus, any $f(\Delta)$-approximation algorithm for the maximum independent set problem can be converted to $f(\Delta)$-approximation algorithm for the maximum clique problem. However, in weighted case, our definition of weighted degree does not allow similar property. Specifically, $\Delta_w(\bar{G}_v)$ can be larger than $\Delta_w(G)$. Thus our weighted degree is not applicable to the maximum weighted clique problem.

Acknowledgments

We thank Toshihiro Fujito for his fruitful comments. We also thank anonymous referees, for their helpful comments and the reference to a closely related paper.

References

[1] S. Arora, C. Lund, R. Motwani, M. Sudan, M. Szegedy, Proof verification and hardness of approximation problems, *J. ACM* 45(3) (1998) 501–555,

[2] P. Berman, T. Fujito, On approximation properties of the independent set problem for low degree graphs, *Theory Comput. Syst.* 32(2) (1999) 115–132.

[3] M. Demange, V. Paschos, Improved approximations for weighted and unweighted graph problems, *Theory Comput. Syst.* 38(6) (2005), 763–787.

[4] M. Grötschel, L. Lovász, A. Schrijver, Geometric algorithms and combinatorial optimization, 2nd ed., Springer-Verlag, 1993.

[5] M. Halldórsson, Approximations of independent sets in graphs, in: The First International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX), 1998.

[6] M. Halldórsson, Approximations of weighted independent set and hereditary subset problems, Journal of Graph Algorithms and Applications 4(1) (2000) 1–16.

[7] M. Halldórsson, H. Lau, Low-degree graph partitioning via local search with applications to constraint satisfaction, max cut, and 3-coloring, Journal of Graph Algorithms and Applications 1(3) (1997) 1–13.

[8] M. Halldórsson, J. Radhakrishnan, Improved approximations of independent sets in bounded-degree graphs via subgraph removal, Nordic Journal of Computing 1(4) (1994) 475–482.

[9] M. Halldórsson, J. Radhakrishnan, Greed is good: Approximating independent sets in sparse and bounded-degree graphs, Algorithmica 18 (1997) 45–163.

[10] D. Hochbaum, Efficient bounds for the stable set, vertex cover and set packing problems, Discrete Applied Mathematics 6 (1983) 243–254.

[11] D. Karger, R. Motwani, M. Sudan, Approximate graph coloring by semidefinite programming, *J. ACM* 45(2) (1998) 246–265.

[12] D. Matula, L. Beck, Smallest-last ordering and clustering and graph coloring algorithms, *J. ACM* 30(2) (1983) 417–427.

[13] G. Nemhauser, L. Trotter, Vertex packing: Structural properties and algorithms, Mathematical Programming 8 (1975) 232–248.

[14] S. Sakai, M. Togasaki, K. Yamazaki, A note on greedy algorithms for maximum weighted independent set problem, Discrete Applied Mathematics 126 (2003) 313–322.

[15] A. Schrijver, Combinatorial optimization: Polyhedra and efficiency, Vol. A., Springer-Verlag, 2003.

[16] V. Vazirani, Approximation algorithms, Springer-Verlag, 2001.