Lyapunov function construction for ordinary differential equations with linear programming

Sigurður F. Marinósson

Department of Mathematics, Gerhard-Mercator-University Duisburg, Lotharstr. 65, D-47048 Duisburg, Germany

Tel: (0049) 203 379 3150, Fax: (0049) 203 379 6564, Email: marino@math.uni-duisburg.de

Abstract

An algorithm that derives a linear program for an ordinary differential equation is presented, of which a feasible solution defines a continuous piecewise affine linear Lyapunov function for the differential equation. The linear program can be generated for an arbitrary region containing an equilibrium of the differential equation. The domain of the Lyapunov function is the region used in the generation of the linear program. The Lyapunov function secures the asymptotic stability of the equilibrium and gives a lower bound on its region of attraction.

1. Introduction

The Lyapunov theory of dynamical systems delivers some powerful tools for the stability analysis of dynamical systems. For several stability concepts, the stability of an equilibrium of a system, can be shown to be equivalent to the existence of a real valued energy-like function with some additional properties. An energy-like function that can be used to prove the stability of an equilibrium of a system is called a Lyapunov function for the system. In this paper we will derive a linear program for continuous autonomous dynamical systems with an equilibrium at zero, whose dynamics are governed by an ordinary differential equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$. The function $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is not assumed to have some specific algebraic structure like linear or piecewise affine linear. The linear program is constructed by the use of function values of $\mathbf{f}$ on a grid and bounds on the second order derivatives of its components in a compact subset containing the grid. A feasible solution of the linear program, i.e. a solution that does not violate the constraints of the linear program, defines a continuous piecewise affine linear Lyapunov function for the system. The domain of the Lyapunov function is the convex hull of the grid used to generate the linear program.

In the literature there have been several approaches to construct Lyapunov functions for nonlinear systems. In (Johansson 1999) and (Johansson and Rantzer 1997, 1998) piecewise quadratic Lyapunov functions are constructed for piecewise affine linear systems. The construction is based on continuity matrices for the partition of the space. Computing them is not a closed problem. In (Brayton and Tong 1979, 1980), (Michel, Sarabudla, and Miller 1982), and (Michel, Nam, and Vittal 1984) the Lyapunov function construction for a set of linear systems is reduced to the design of a balanced polytope fulfilling some invariance properties. In (Boyd, El Ghaoui, Feron, and Balakrishnan 1994) a convex optimization problem is used to compute a quadratic Lyapunov function for a system linear in a band containing the origin, in (Johansson 1999) there is an illustrating example of its use. In (Ohta, Imanishi, Gong, and Haneda 1993) the stability of an uncertain nonlinear system is analyzed through a set of piecewise affine linear systems. In (Kiendl and Ruger 1995) piecewise affine linear Lyapunov function candidates are used to prove the stability of linear systems with piecewise affine linear control. In (Karweina 1989), (Kiendl 1999), (Knicker 1999), and (Knicker and Krause 1999) a convex partition is used to prove $G_{H,N}$-stability of time discrete piecewise affine linear systems. There is no Lyapunov function involved, but the region of attraction can be calculated. In (Scheel and Kiendl 1995) and (Scheel 1997) a converse Lyapunov theorem is used to construct integral Lyapunov functions for time discrete systems. None of the methods above can be used for general continuous nonlinear systems.

In (Julian, Guivant, and Desages 1999) and (Julian 1999) linear programming is used to calculate Lyapunov-like functions for continuous autonomous nonlinear systems. When such a function can be found the system trajectories are proven to be ultimately bounded, but the equilibrium is not necessarily stable. The linear program presented here was to a large extent inspired by the work of Julian *et al.* and we will compare the linear programs in detail in the next section, where we explain how both methods work. The main advantage of the linear program introduced in this paper over the approach of Julian *et al.*, is that a feasible solution of it defines a true Lyapunov function for the system, which implies asymptotic stability of the equilibrium at zero.

2. How the method works

Julian *et al.* proposed a linear program, that can prove the solutions of a continuous autonomous system to be ultimately bounded. An exact definition of the stability concept they used is:

> The solution $\boldsymbol{\phi}$ of a dynamical system
>
> $$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), \quad \mathbf{x}(0) = \boldsymbol{\xi}$$
>
> is said to be *uniformly ultimately bounded*, with the *ultimate bound* $b > 0$, if and only if there exists a positive constant C, such that
>
> $$\limsup_{t \to +\infty} \|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| < b$$
>
> for all initial values $\boldsymbol{\xi}$ with $\|\boldsymbol{\xi}\| < C$.

Their method works in essence in the following way:

i) Partition the area around the equilibrium under consideration in a family $\mathfrak{S}$ of simplices.

ii) Limit the search for a Lyapunov-like function V^{Lya} to the class of continuous functions, whose restriction to any $S^{(i)} \in \mathfrak{S}$ is affine linear.

The essential inequality needed for V^{Lya} to be a Lyapunov function of the system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ is

$$-\gamma(\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|) \geq D_t^+[V^{Lya}(\boldsymbol{\phi}(t, \boldsymbol{\xi}))], \tag{1}$$

where D_t^+ is a Dini derivative with respect to t,

$$D_t^+[V^{Lya}(\boldsymbol{\phi}(t, \boldsymbol{\xi}))] := \limsup_{h \to 0+} \frac{V^{Lya}(\boldsymbol{\phi}(t+h, \boldsymbol{\xi})) - V^{Lya}(\boldsymbol{\phi}(t, \boldsymbol{\xi}))}{h},$$

$\boldsymbol{\phi}$ is the solution of the system, γ is a strictly monotone increasing continuous function with $\gamma(0) = 0$, and $\| \cdot \|$ is an arbitrary norm on $\mathbb{R}^n$.

Because V^{Lya} is assumed to be affine linear on any simplex $S^{(i)} \in \mathfrak{S}$, there is for any such simplex a vector $\mathbf{w}^{(i)}$ and a real number a_i, such that

$$V^{Lya}(\mathbf{x}) = \mathbf{w}^{(i)} \cdot \mathbf{x} + a_i$$

for all $\mathbf{x} \in S^{(i)}$. Further, is it possible to show that if

$$-\gamma(\|\mathbf{x}\|) \geq \mathbf{w}^{(i)} \cdot \mathbf{f}(\mathbf{x}) \tag{2}$$

for all $\mathbf{x} \in S^{(i)}$ and for all $S^{(i)} \in \mathfrak{S}$, then the inequality (1) it fulfilled. Because it is impossible to handle inequality (2) for all $\mathbf{x}$ numerically, one must limit this check to a finite number of $\mathbf{x}$. By limiting oneself to the vertices of the simplices in $\mathfrak{S}$, one gets by irrelevant assumptions about γ:

> If $\mathbf{f}$ is a continuous piecewise affine linear function, defined through its values on the vertices, then inequality (1) holds for all $\mathbf{x}$, if it holds for all vertices of all simplices in $\mathfrak{S}$.

This makes the following straight forward. Let $\mathbf{f_p}$ be the continuous piecewise affine linear function, defined through $\mathbf{f_p}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all vertices $\mathbf{x}$ of the simplices in $\mathfrak{S}$ and $\mathbf{f_p}$ is affine linear on any simplex in $\mathfrak{S}$. Solve, in consideration of the continuity of V^{Lya}, the linear constraints

$$-\gamma(\|\mathbf{x}\|) \geq \mathbf{w}^{(i)} \cdot \mathbf{f_p}(\mathbf{x}) \tag{3}$$

with respect to $\mathbf{w}^{(i)}$ for all vertices $\mathbf{x}$ of the simplex $S^{(i)}$, and do this for all simplices $S^{(i)}$ in $\mathfrak{S}$. Note that

$$\begin{aligned}\mathbf{w}^{(i)} \cdot \mathbf{f}(\mathbf{x}) &= \mathbf{w}^{(i)} \cdot \mathbf{f_p}(\mathbf{x}) + \mathbf{w}^{(i)} \cdot [\mathbf{f}(\mathbf{x}) - \mathbf{f_p}(\mathbf{x})] \\ &\leq \mathbf{w}^{(i)} \cdot \mathbf{f_p}(\mathbf{x}) + \|\mathbf{w}^{(i)}\|_1 \|\mathbf{f}(\mathbf{x}) - \mathbf{f_p}(\mathbf{x})\|_\infty\end{aligned}$$

for all $\mathbf{x}$. Define $W := \max_{S^{(i)} \in \mathfrak{S}} \|\mathbf{w}^{(i)}\|_1$ and $\varepsilon := \sup_{\mathbf{x}} \|\mathbf{f}(\mathbf{x}) - \mathbf{f_p}(\mathbf{x})\|_\infty$ and consider, that for any i such that $\mathbf{x} \in S^{(i)}$, we have

$$-\gamma(\|\mathbf{x}\|) + W\varepsilon \geq \mathbf{w}^{(i)} \cdot \mathbf{f_p}(\mathbf{x}) + W\varepsilon \geq \mathbf{w}^{(i)} \cdot \mathbf{f}(\mathbf{x}).$$

By choosing a constant $\eta \in]0, 1[$ and write the last inequality as

$$-(1-\eta)\gamma(\|\mathbf{x}\|) - \eta\gamma(\|\mathbf{x}\|) + W\varepsilon \geq \mathbf{w}^{(i)} \cdot \mathbf{f}(\mathbf{x}),$$

we see that

$$-(1-\eta)\gamma(\|\mathbf{x}\|) \geq \mathbf{w}^{(i)} \cdot \mathbf{f}(\mathbf{x})$$

if

$$-\eta\gamma(\|\mathbf{x}\|) + W\varepsilon \leq 0,$$

i.e.

$$\|\mathbf{x}\| \geq \gamma^{-1}(\frac{W\varepsilon}{\eta}).$$

This means that the system is ultimately bounded with the ultimate bound

$$b := \alpha_1^{-1}(\alpha_2(\gamma^{-1}(\frac{W\varepsilon}{\eta}))),$$

where the α_1 and α_2 are strictly increasing continuous functions $\mathbb{R} \longrightarrow \mathbb{R}$, such that $\alpha_1(0) = \alpha_2(0) = 0$ and

$$\alpha_1(\|\mathbf{x}\|) \leq V^{Lya}(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|).$$

Consequently, the function V^{Lya}, defined through the vectors $\mathbf{w}^{(i)}$ and $V^{Lya}(\mathbf{0}) = 0$, is almost a Lyapunov function. The only difference to a true Lyapunov function is, that

$$-\gamma(\|\mathbf{x}\|) \geq D_t^+[V^{Lya}(\boldsymbol{\phi}(t,\mathbf{x}))]\Big|_{t=0}$$

for all relevant $\mathbf{x}$, which additionally satisfy $\|\mathbf{x}\| \geq \gamma^{-1}(\frac{W\varepsilon}{\eta})$, instead of for all relevant $\mathbf{x}$. Note that the ultimate bound is not a priori known.

What we do in this paper, is first for every $S^{(i)}$ in $\mathfrak{S}$ to find a set $\{D_1^{(i)}, D_2^{(i)}, .., D_{n+1}^{(i)}\}$ of real numbers, so that if

$$\mathbf{x} = \sum_{j=1}^{n+1} \lambda_j \mathbf{x}_j, \quad \lambda_j \in [0,1], \quad \sum_{j=1}^{n+1} \lambda_j = 1,$$

where the vectors $\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_{n+1}$ are the vertices of the simplex $S^{(i)}$, then

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}_\mathbf{p}(\mathbf{x})\|_\infty \leq \sum_{j=1}^{n+1} \lambda_j D_j^{(i)}.$$

Then we solve the linear constraints

$$|w_j^{(i)}| \leq C_j^{(i)} \quad \text{for } j = 1,2,..,n$$

and

$$-\gamma(\|\mathbf{x}_j\|) \geq \mathbf{w}^{(i)} \cdot \mathbf{f}(\mathbf{x}_j) + D_j^{(i)} \sum_{k=1}^{n} C_k^{(i)} \quad \text{for } j = 1,2,..,n+1,$$

where $\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_{n+1}$ are the vertices of the simplex $S^{(i)}$ in $\mathfrak{S}$, with respect to $C_1^{(i)}$, $C_2^{(i)}$,..,$C_n^{(i)}$, and $\mathbf{w}^{(i)}$, for all simplexes $S^{(i)} \in \mathfrak{S}$. This will, as shown in the Appendix, deliver a true Lyapunov function and thus secures asymptotic stability.

Note that when $\mathbf{x}_j = \mathbf{0}$, we must have $D_j^{(i)} = 0$, for else there would be no feasible solution to the linear constraints. This is a difficulty that must be solved, if the method is to be of any use. The solution lies in a proper choice of the simplicial partition combined with an appropriate bound on $\|\mathbf{f}(\mathbf{x})-\mathbf{f_p}(\mathbf{x})\|_\infty$, proved in the Appendix for $\mathbf{f}$ with bounded second order derivatives in the interior of every simplex $S^{(i)}$ in $\mathfrak{S}$.

3. Parameterization of the Lyapunov function

To be able to search for a Lyapunov function $V^{Lya} : \mathbb{R}^n \longrightarrow \mathbb{R}$ with linear programing one clearly needs to limit the search to some parameterized set of functions. The set of the continuous piecewise affine linear functions $\mathbb{R}^n \longrightarrow \mathbb{R}$, with a predefined boundary configuration, is probably the most simple of such sets. It further has the nice property, that with a suitable choice of the boundary configuration, one can approximate any continuous function with arbitrary precision.

In order to explain the boundary configuration used in the generation of the linear program we need some definitions. The boundary configuration is basically the same as the of Julian *et al.*, but the notations differ a lot. In the rest of this paper $\mathbf{e}_i$ is the i-th unit vector, $\mathfrak{G}_n$ is the set of the permutations of $\{1,2,..,n\}$, $\lfloor x \rfloor$ is the largest integer smaller than or equal to x, δ_{ij} is the Kronecker delta, and $\chi_{\mathcal{N}}$ is the characteristic function of the set $\mathcal{N}$.

Definition 1: *The set $\mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+} \subset \mathbb{Z}^n$ is defined as*

$$\mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+} := \{(z_1,z_2,..,z_n) \in \mathbb{Z}^n \big| \; N_i^- \leq z_i \leq N_i^+, \; i=1,2,..,n\},$$

where the N_1^-, N_2^-,.., N_n^- are strictly negative integers and the N_1^+, N_2^+,.., N_n^+ are strictly positive integers. The set $\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+}$ is defined through

$$\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+} := \{(x_1,x_2,..,x_n) \in \mathbb{R}^n \big| \; N_i^- \leq x_i \leq N_i^+, \; i=1,2,..,n\}.$$

To generate the linear program we want to use a closer meshed grid than $\mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$. To achieve this we define a piecewise scaling function for $\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+}$.

Definition 2: *The piecewise scaling function* $\mathrm{PS} : \mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+} \longrightarrow \mathbb{R}^n$ *is defined through*

$$\mathrm{PS}(\mathbf{x}) := \sum_{i=1}^n \left(-\sum_{k=N_i^-}^{-1} h_{i,k} + \sum_{k=N_i^-}^{\lfloor x_i \rfloor -1} h_{i,k} + h_{i,\lfloor x_i \rfloor}(x_i - \lfloor x_i \rfloor) \right) \mathbf{e}_i$$

for all $\mathbf{x} \in \mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+}$, for some strictly positive real constants $h_{i,k}$, for $i=1,2,..,n$, and $k = N_i^-, N_i^- + 1, .., N_i^+ - 1$.

By choosing the constants $h_{i,k}$ in the last definition properly, the image of $\mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$ under PS is a grid as closely meshed as one wishes. We are going to use the function values of $\mathbf{f}$ on the grid $\mathrm{PS}(\mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+})$, i.e. the image of $\mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$ under the function PS, as constants in the linear program. Further, we are going to use the linear program to calculate the values of the Lyapunov function V^{Lya} on the same grid. What we need to construct a properly defined continuous piecewise affine linear function from these values, is an appropriate boundary configuration. The atoms of the partition used, leading to an advantageous boundary configuration, are the simplices S_σ, where σ is a permutation.

Definition 3: *The simplex S_σ, where $\sigma \in \mathfrak{S}_n$, is defined as the set*

$$S_\sigma := \{\mathbf{x} \in \mathbb{R}^n \,\big|\, 0 \leq x_{\sigma(1)} \leq x_{\sigma(2)} \leq .. \leq x_{\sigma(n)} \leq 1\},$$

where $x_{\sigma(i)}$ is the $\sigma(i)$-th component of the vector $\mathbf{x}$.

An equivalent definition of the set S_σ is

$$S_\sigma := \{\mathbf{x} \in \mathbb{R}^n \,\big|\, \mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)},\ \sum_{k=1}^{n+1} \lambda_k = 1\},$$

i.e. the set S_σ is a simplex in $\mathbb{R}^n$. We need one more definition before we can use the values of V^{Lya} on the grid $\mathrm{PS}(\mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+})$ to define a continuous piecewise affine linear function $\mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+}) \longrightarrow \mathbb{R}$.

Definition 4: *The functions $R^\mathcal{N} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, where $\mathcal{N} \subset \{1, 2, .., n\}$, are defined through*

$$R^\mathcal{N}(\mathbf{x}) := [(-1)^{\chi_\mathcal{N}(1)} x_1, (-1)^{\chi_\mathcal{N}(2)} x_2, .., (-1)^{\chi_\mathcal{N}(n)} x_n] \ \ \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

The next theorem is proved in Chapter 4 in (Marinosson 2002). The proof is rather long and will not be given here.

Theorem 5: *For every $\mathbf{z} \in \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$ let $V[\mathbf{z}]$ be a real number. Then a continuous piecewise affine linear function $V^{Lya} : \mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+}) \longrightarrow \mathbb{R}$ is properly defined through:*

i) $V^{Lya}(\mathrm{PS}(\mathbf{z})) := V[\mathbf{z}]$ *for all* $\mathbf{z} \in \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$.

ii) The restriction of V^{Lya} on any set of the form $\mathrm{PS}(R^\mathcal{N}(\mathbf{y} + S_\sigma))$ contained in $\mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+})$, where $\mathcal{N} \subset \{1, 2, .., n\}$, $\sigma \in \mathfrak{S}_n$, and $\mathbf{y} \in \mathbb{Z}^n_{\geq 0}$, is affine linear.

4. The linear program

From the last section we have everything we need to state the linear program.

Linear Program 6: *Let the sets* $\mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+} \subset \mathbb{Z}^n$ *and* $\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+} \subset \mathbb{R}^n$ *be as in Definition 1, and let the functions* $\mathrm{PS} : \mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+} \longrightarrow \mathbb{R}^n$ *and* $R^{\mathcal{N}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be as in Definition 2 and Definition 4. We define the sets* $Y^{\mathcal{N}}$, Z, *and* $X^{\|\cdot\|}$.

i) For every $\mathcal{N} \subset \{1,2,..,n\}$ *we define the set* $Y^{\mathcal{N}} \subset \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$ *through*

$$Y^{\mathcal{N}} := \{(y_1,y_2,..,y_n) \in \mathbb{Z}^n \big|\ y_i = \begin{cases} 0,1,..,N_i^+ - 1, & \text{if } i \notin \mathcal{N} \\ N_i^- + 1, N_i^- + 2, .., 0, & \text{if } i \in \mathcal{N} \end{cases} \}.$$

ii) We define the set Z *through*

$$Z := \{\ \{\mathbf{x},\mathbf{y}\} \subset \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+} \big|\ \mathbf{x} - \mathbf{y} = (\delta_{i1},\delta_{i2},..,\delta_{in}) \text{ for some } i \in \{1,2,..,n\}\}.$$

The set Z *is the set of neighboring pairs in* $\mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$.

iii) Let $\|\cdot\|$ *be an arbitrary norm on* $\mathbb{R}^n$. *The set* $X^{\|\cdot\|}$ *is defined through*

$$X^{\|\cdot\|} := \{\|\mathrm{PS}(\mathbf{z})\| \ \big|\ \mathbf{z} \in \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}\}.$$

Let $\mathcal{U} \subset \mathbb{R}^n$ *be an open set containing the set* $\mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+})$ *and* $\mathbf{f} : \mathcal{U} \longrightarrow \mathbb{R}^n$ *be a function with the property, that the second order partial derivatives* $\partial_l\partial_k f_i$, $i,l,k = 1,2,..,n$ *exist and are bounded on every open set* $\mathcal{M}^{\mathcal{N}}_{\mathbf{y}^{\mathcal{N}}}$ *of the form*

$$\mathcal{M}^{\mathcal{N}}_{\mathbf{y}^{\mathcal{N}}} := \mathrm{PS}(\mathbf{y}^{\mathcal{N}} + R^{\mathcal{N}}(]0,1[^n))$$

for every $\mathcal{N} \subset \{1,2,..,n\}$ *and every* $\mathbf{y}^{\mathcal{N}} \in Y^{\mathcal{N}}$. *Further assume that* $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

We define the constants $F_i[\mathbf{z}]$, $i = 1,2,..,n$ *and* $\mathbf{z} \in \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$, *and* $B^{\mathcal{N}}_{kl}[\mathbf{y}^{\mathcal{N}}]$ *for all* $k,l = 1,2,..,n$, *all* $\mathcal{N} \subset \{1,2,..,n\}$, *and all* $\mathbf{y}^{\mathcal{N}} \in Y^{\mathcal{N}}$, *through*

$$F_i[\mathbf{z}] := f_i(PS(\mathbf{z}))$$

and

$$B^{\mathcal{N}}_{lk}[\mathbf{y}^{\mathcal{N}}] \geq \max_{i=1,2,..,n} \sup_{\mathbf{x} \in \mathcal{M}^{\mathcal{N}}_{\mathbf{y}^{\mathcal{N}}}} |\partial_k\partial_l f_i(\mathbf{x})|.$$

Further, we define the variables $V[\mathbf{z}]$ *for all* $\mathbf{z} \in \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$, *the variables* $C[\{\mathbf{x},\mathbf{y}\}]$ *for all* $\{\mathbf{x},\mathbf{y}\} \in Z$, *and the variables* $\Psi[x]$ *and* $\Gamma[x]$ *for all* $x \in X^{\|\cdot\|}$.

To shorten writings we define for every $\mathcal{N} \subset \{1,2,..,n\}$, *every* $\sigma \in \mathfrak{S}_n$, *and every* $i = 1,2,..,n+1$, *the vectors* $\mathbf{x}^{\mathcal{N},\sigma}_i$ *through*

$$\mathbf{x}^{\mathcal{N},\sigma}_i := \sum_{j=i}^{n} (-1)^{\chi_{\mathcal{N}}(\sigma(j))} \mathbf{e}_{\sigma(j)}.$$

Further, we define for every $\mathcal{N} \subset \{1,2,..,n\}$, every $\sigma \in \mathfrak{G}_n$, every $\mathbf{y}^{\mathcal{N}} \in Y^{\mathcal{N}}$, every $k = 1,2,..,n$, and every $i = 1,2,..,n+1$, the real value $A_{k,i}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}$ through

$$A_{k,i}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}} := |\mathbf{e}_k \cdot [\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_i^{\mathcal{N},\sigma}) - \mathrm{PS}(\mathbf{y}^{\mathcal{N}})]|.$$

The constraints of the linear program are:

LC1) *Let $\varepsilon > 0$ be an arbitrary number and let $x_1, x_2, .., x_K$ be the elements of $X^{\|\cdot\|}$ in an increasing order. Then*

$$\Psi[x_1] = \Gamma[x_1] = 0,$$

$$\varepsilon x_2 \leq \Psi[x_2],$$

$$\varepsilon x_2 \leq \Gamma[x_2],$$

and for every $i = 2,3,..K-1$:

$$\frac{\Psi[x_i] - \Psi[x_{i-1}]}{x_i - x_{i-1}} \leq \frac{\Psi[x_{i+1}] - \Psi[x_i]}{x_{i+1} - x_i}$$

and

$$\frac{\Gamma[x_i] - \Gamma[x_{i-1}]}{x_i - x_{i-1}} \leq \frac{\Gamma[x_{i+1}] - \Gamma[x_i]}{x_{i+1} - x_i}.$$

LC2)

$$V[\mathbf{0}] = 0$$

and for every $\mathbf{z} \in \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$:

$$\Psi[\|\mathrm{PS}(\mathbf{z})\|] \leq V[\mathbf{z}].$$

LC3) *For every $\{\mathbf{x},\mathbf{y}\} \in Z$:*

$$-C[\{\mathbf{x},\mathbf{y}\}]\cdot\|\mathrm{PS}(\mathbf{x})-\mathrm{PS}(\mathbf{y})\|_\infty \leq V[\mathbf{x}]-V[\mathbf{y}] \leq C[\{\mathbf{x},\mathbf{y}\}]\cdot\|\mathrm{PS}(\mathbf{x})-\mathrm{PS}(\mathbf{y})\|_\infty.$$

LC4) *For every $\mathcal{N} \subset \{1,2,..,n\}$, every $\sigma \in \mathfrak{G}_n$, every $\mathbf{y}^{\mathcal{N}} \in Y^{\mathcal{N}}$, and $i = 1,2,..,n+1$:*

$$\begin{aligned}
&-\Gamma[\|\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_i^{\mathcal{N},\sigma})\|] \geq \\
&\quad \sum_{j=1}^{n} (-1)^{\chi_{\mathcal{N}}(\sigma(j))} \frac{V[\mathbf{y}^{\mathcal{N}} + \mathbf{x}_j^{\mathcal{N},\sigma}] - V[\mathbf{y}^{\mathcal{N}} + \mathbf{x}_{j+1}^{\mathcal{N},\sigma}]}{\|\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_j^{\mathcal{N},\sigma}) - \mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_{j+1}^{\mathcal{N},\sigma})\|_\infty} F_{\sigma(j)}[\mathbf{y}^{\mathcal{N}} + \mathbf{x}_i^{\mathcal{N},\sigma}] \\
&\quad + \frac{1}{2} \sum_{r,s=1}^{n} B_{rs}^{\mathcal{N}}[\mathbf{y}^{\mathcal{N}}] A_{r,i}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}} (A_{s,i}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}} + A_{s,1}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}) \sum_{j=1}^{n} C[\{\mathbf{y}^{\mathcal{N}} + \mathbf{x}_j^{\mathcal{N},\sigma}, \mathbf{y}^{\mathcal{N}} + \mathbf{x}_{j+1}^{\mathcal{N},\sigma}\}].
\end{aligned}$$

Note that the value of the constant ε in LC1 is not important. If there is a solution for one $\varepsilon_{sol} > 0$, then there is a solution for all $\varepsilon > 0$. Just multiply all variables with $\varepsilon/\varepsilon_{sol}$. This comes as no surprise for if V^{Lya} is a Lyapunov function for a system, then so is aV^{Lya} for any $a > 0$. The next theorem contains the essential results of this paper. A proof of it is given in the Appendix.

Theorem 7: *Assume that the linear program above has a feasible solution. Then the function* $V^{Lya} : \mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+}) \longrightarrow \mathbb{R}$*, defined as in Theorem 5, with the real numbers* $V[\mathbf{z}]$, $\mathbf{z} \in \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$*, from a feasible solution of the Linear Program 6, is a (true) Lyapunov function for the system* $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$.

If there is a feasible solution of the Linear Program 6 then one can use the Lyapunov function from Theorem 7 to extract information about the stability behavior of the equilibrium at zero in a very easy way. It is asymptotically stable and a lower bound on its region of attraction is given by the largest preimage $\{\mathbf{x} \in \mathbb{R}^n \big| V^{Lya}(\mathbf{x}) \leq c\}$ entirely contained in the interior of $\mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+})$.

In the next section we will see an example of the use of Linear Program 6 to generate a Lyapunov functions for a nonlinear system. A linear solver generally minimizes an objective, i.e. a function linear in the variables of the linear program, in consideration of the linear constraints. In order to simply generate a Lyapunov function by the Linear Program 6 the objective is not needed. This makes it possible to use the objective to optimize the Lyapunov function in some way. In the example given the objective

$$\sum_{\mathbf{z} \in \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}} (V[\mathbf{z}] - \Psi[\|\mathrm{PS}(\mathbf{z})\|_\infty])$$

is minimized. This should not be considered to deliver the optimal Lyapunov function, but it leads to a reasonably looking one. In the example the constant ε was set equal to one and the norm used was the maximum norm $\|\cdot\| := \|\cdot\|_\infty$. The linear solver CPLEX® [1] 6.5 was used to solve the linear program. In order to be able to visualize the Lyapunov function properly we consider a two-dimensional system.

5. Example

Consider the dynamical system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$, where

$$\mathbf{f}(\mathbf{x}) := \begin{pmatrix} x_2 \\ -\sin(x_1) - x_2 \end{pmatrix}.$$

[1]CPLEX is a registered trademark of ILOG®

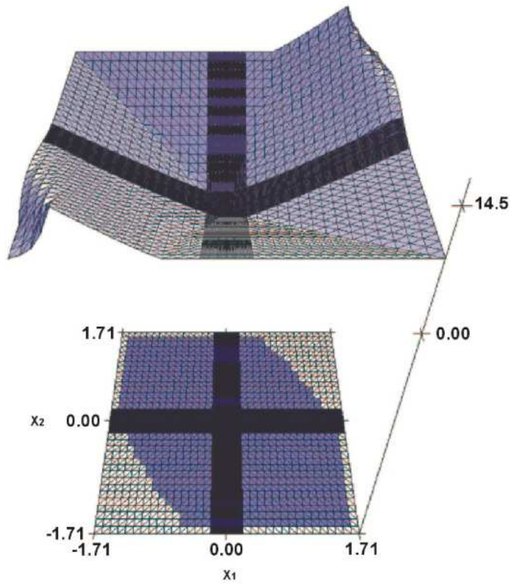

Figure 1: Lyapunov function for $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$

It is the state equation of a pendulum with friction, where

$$\frac{g}{l} = \frac{k}{m} = 1,$$
$$g := \text{acceleration due to gravity},$$
$$l := \text{the length of the pendulum},$$
$$k := \text{the coefficient of friction},$$
$$m := \text{the mass of the pendulum}.$$

For a further discussion on this system see, for example, Subsection 1.1.1 and Example 1.3 in (Khalil 1992). One easily verifies that $\mathbf{f}$ has an equilibrium at zero and that the second order derivatives of its components are bounded. The Linear Program 6 was used to generate linear constraints for a continuous piecewise affine linear Lyapunov function $[-1.71, 1.71]^2 \longrightarrow \mathbb{R}$ for the system, which the linear solver was able to satisfy. In figure 1 a Lyapunov function for a 91×91 grid ($= 8281$ grid points) is drawn. The grid steps are identical for the x_1- and x_2-axes, 0.001 in the interval $[-0.01, 0.01]$, 0.01 in the intervals $[-0.21, -0.01]$ and $[0.01, 0.21]$, and 0.1 in the intervals $[-1.71, -0.21]$ and $[0.21, 1.71]$. The boundary configuration is shown at the base of the figure and the shaded area is a lower bound on the region of attraction of the equilibrium at zero.

6. Conclusions

In this work we derived an algorithm that generates linear programs for ordinary differential equations. A feasible solution of the linear program for a particular system defines a continuous piecewise affine linear Lyapunov function for it. An examples of its use was shown, where a Lyapunov function for a nonlinear system was generated. The procedure presented here can be seen as a major improvement of the procedure presented in (Julian, Guivant, and Desages 1999) and (Julian 1999), where a solution of a similar linear program results in an almost Lyapunov function. The advantage of a true Lyapunov function over the Lyapunov-like function of Julian *et al.*, is that it secures asymptotic stability of the equilibrium instead of the weaker "the trajectories of the system are ultimately bounded". A very interesting open problem related to the results of this work is the following. Sometimes there is a feasible solution to a linear program generated for an asymptotically stable system and sometimes not. Just using a more closely meshed grid does not always seem to help if there is no solution. The function PS has more degrees of freedom because of the variable grid steps sizes allowed, but using them is currently a trial-and-error process. What is needed is a converse theorem for the existence of continuous piecewise affine linear Lyapunov functions. Preferably such a theorem should be based on some easily checked properties of the dynamical system in question and should at least give an idea of how to construct the grid. If this problem can be solved in a satisfactory way one would have a general method to generate Lyapunov functions for a very large class of continuous dynamical systems.

Appendix

In this appendix we are going to prove Theorem 7. To keep it at a reasonable length not all details can be thoroughly worked out. In (Marinosson 2002) they are, so it can be taken as a general reference for the proof. What we are going to show, is that if the functions ψ, γ, and V^{Lya} are defined as piecewise affine linear interpolation of the values of the variables Ψ, Γ, and V respectively from a feasible solution of the Linear Program 6, then ψ and γ are strictly monotone increasing continuous functions with $\psi(0) = \gamma(0) = 0$,

$$\psi(\|\mathbf{x}\|) \leq V^{Lya}(\mathbf{x}) \tag{4}$$

for all $\mathbf{x} \in \mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+})$, and

$$D^+_t[V^{Lya}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))] \leq -\gamma(\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|), \tag{5}$$

for all $\boldsymbol{\phi}(t,\boldsymbol{\xi})$ in the interior of $\mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+})$, where $\boldsymbol{\phi}(\cdot,\boldsymbol{\xi})$ is the solution of the equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ with the initial value $\boldsymbol{\xi}$. Because $V^{Lya}(\mathbf{0}) = 0$ this implies that V^{Lya} is a Lyapunov function of the system. The property (5) is usually formulated with the classical derivative, but it is well known that the Dini derivative can be used just as well.

Let $x_1, x_2, .., x_K$ be the elements of $X^{\|\cdot\|}$ in an increasing order. We define the piecewise affine linear functions $\psi, \gamma : [x_1, +\infty[\longrightarrow \mathbb{R}$ through

$$\psi(y) := \Psi[x_i] + \frac{\Psi[x_{i+1}] - \Psi[x_i]}{x_{i+1} - x_i}(y - x_i)$$

and

$$\gamma(y) := \Gamma[x_i] + \frac{\Gamma[x_{i+1}] - \Gamma[x_i]}{x_{i+1} - x_i}(y - x_i),$$

for all $y \in [x_i, x_{i+1}]$ and all $i = 1, 2, .., K-1$. The values of ψ and γ on $]x_K, +\infty[$ do not really matter, but to have everything properly defined, we set

$$\psi(y) := \Psi[x_{K-1}] + \frac{\Psi[x_K] - \Psi[x_{K-1}]}{x_K - x_{K-1}}(y - x_{K-1})$$

and

$$\gamma(y) := \Gamma[x_{K-1}] + \frac{\Gamma[x_K] - \Gamma[x_{K-1}]}{x_K - x_{K-1}}(y - x_{K-1})$$

for all $y > x_K$. From this definition, it is clear that the functions ψ and γ are continuous. The function V^{Lya} is defined as in Theorem 5 with the constants $V[\mathbf{z}]$, $\mathbf{z} \in \mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}$, from the Linear Program 6. We will now prove that the linear constraints LC1, LC2, LC3, and LC4 imply the properties (4) and (5).

The Constraints LC1:
It is easy to see that the constraints LC1 imply, that the functions ψ and γ are convex strictly monotone increasing functions and that $\psi(0) = \gamma(0) = 0$. Usually, it is not demanded that these comparison functions are convex, but we will need that later in the proof.

The Constraints LC2:
We are going to show that the constraints LC2 imply the inequality (4). To do this let $\mathbf{x} \in \mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+})$. Then there is an $\mathcal{N} \subset \{1, 2, .., n\}$, a $\sigma \in \mathfrak{S}_n$, and an $\mathbf{y}^{\mathcal{N}} \in Y^{\mathcal{N}}$, such that

$$\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_i^{\mathcal{N},\sigma}), \ \lambda_j \in [0,1] \text{ for } j = 1, 2, .., n+1 \text{ and } \sum_{i=1}^{n+1} \lambda_i = 1.$$

By using the constraints LC2, the convexity of ψ, the affine linearity of V^{Lya} on the simplex defined through the vertices $\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+\mathbf{x}_1^{\mathcal{N},\sigma})$, $\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+\mathbf{x}_2^{\mathcal{N},\sigma})$,..,$\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+$

$\mathbf{x}_{n+1}^{\mathcal{N},\sigma})$, that $\mathbf{y}^{\mathcal{N}}+\mathbf{x}_i^{\mathcal{N},\sigma}\in\mathcal{G}_{\mathbf{N}^+,\mathbf{N}^-}$ for all $i=1,2,..,n+1$, and that ψ is monotone increasing, we get

$$\begin{aligned}\psi(\|\mathbf{x}\|) &\leq \sum_{i=1}^{n+1}\lambda_i\psi(\|\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+\mathbf{x}_i^{\mathcal{N},\sigma})\|)\\ &= \sum_{i=1}^{n+1}\lambda_i\Psi[\|\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+\mathbf{x}_i^{\mathcal{N},\sigma})\|]\\ &\leq \sum_{i=1}^{n+1}\lambda_i V[\mathbf{y}^{\mathcal{N}}+\mathbf{x}_i^{\mathcal{N},\sigma}]\\ &= \sum_{i=1}^{n+1}\lambda_i V^{Lya}(\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+\mathbf{x}_i^{\mathcal{N},\sigma}))\\ &= V^{Lya}(\sum_{i=1}^{n+1}\lambda_i\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+\mathbf{x}_i^{\mathcal{N},\sigma}))\\ &= V^{Lya}(\mathbf{x}).\end{aligned}$$

Because $\mathbf{x}$ was arbitrary, this inequality is valid for all $\mathbf{x}\in\mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+})$.

The Constraints LC3:
The constrains LC3 imply that

$$\left|\frac{V[\mathbf{x}]-V[\mathbf{y}]}{\|\mathrm{PS}(\mathbf{x})-\mathrm{PS}(\mathbf{y})\|_\infty}\right|\leq C[\{\mathbf{x},\mathbf{y}\}]$$

for every $\{\mathbf{x},\mathbf{y}\}\in Z$. This can be used to give a bound on the gradient of V^{Lya} in LC4. To see this verify that for every $\mathcal{N}\subset\{1,2,..,n\}$, every $\mathbf{y}^{\mathcal{N}}\in Y^{\mathcal{N}}$, and every $\sigma\in\mathfrak{S}_n$, the function V^{Lya} has the algebraic form

$$V^{Lya}(\mathbf{z})=[\mathbf{z}-\mathrm{PS}(\mathbf{y}^{\mathcal{N}})]\cdot\sum_{j=1}^{n}(-1)^{\chi_{\mathcal{N}}(\sigma(j))}\frac{V[\mathbf{y}^{\mathcal{N}}+\mathbf{x}_j^{\mathcal{N},\sigma}]-V[\mathbf{y}^{\mathcal{N}}+\mathbf{x}_{j+1}^{\mathcal{N},\sigma}]}{\|\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+\mathbf{x}_j^{\mathcal{N},\sigma})-\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+\mathbf{x}_{j+1}^{\mathcal{N},\sigma})\|_\infty}\mathbf{e}_{\sigma(j)}+V[\mathbf{y}^{\mathcal{N}}]$$

on the simplex $\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+R^{\mathcal{N}}(S_\sigma))$. The gradient $\mathbf{w}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}$ of V^{Lya} on $\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+R^{\mathcal{N}}(S_\sigma))$ is therefore

$$\mathbf{w}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}=\sum_{j=1}^{n}(-1)^{\chi_{\mathcal{N}}(\sigma(j))}\frac{V[\mathbf{y}^{\mathcal{N}}+\mathbf{x}_j^{\mathcal{N},\sigma}]-V[\mathbf{y}^{\mathcal{N}}+\mathbf{x}_{j+1}^{\mathcal{N},\sigma}]}{\|\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+\mathbf{x}_j^{\mathcal{N},\sigma})-\mathrm{PS}(\mathbf{y}^{\mathcal{N}}+\mathbf{x}_{j+1}^{\mathcal{N},\sigma})\|_\infty}\mathbf{e}_{\sigma(j)},\qquad(6)$$

from which

$$\|\mathbf{w}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}\|_1\leq\sum_{j=1}^{n}C[\{\mathbf{y}^{\mathcal{N}}+\mathbf{x}_j^{\mathcal{N},\sigma},\mathbf{y}^{\mathcal{N}}+\mathbf{x}_{j+1}^{\mathcal{N},\sigma}\}]\qquad(7)$$

immediately follows.

The Constraints LC4:
We are going to prove that the constraints LC4 imply, that

$$-\gamma(\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|) \geq D_t^+[V^{Lya}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))]$$

for all $\boldsymbol{\phi}(t,\boldsymbol{\xi})$ in the interior of $\mathrm{PS}(\mathcal{G}^F_{\mathbf{N}^-,\mathbf{N}^+})$. To do this let $\boldsymbol{\phi}(t,\boldsymbol{\xi})$ be arbitrary and set $\mathbf{x} := \boldsymbol{\phi}(t,\boldsymbol{\xi})$. Then there is an $\mathcal{N} \subset \{1,2,..,n\}$, a $\sigma \in \mathfrak{S}_n$, and an $\mathbf{y}^{\mathcal{N}} \in Y^{\mathcal{N}}$, such that

$$\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_i^{\mathcal{N},\sigma}),\ \ \lambda_j \in [0,1] \text{ for } j = 1,2,..,n+1 \text{ and } \sum_{i=1}^{n+1} \lambda_i = 1,$$

and such that

$$\mathbf{x} + h\mathbf{f}(\mathbf{x}) \in \mathrm{PS}(\mathbf{y}^{\mathcal{N}} + R^{\mathcal{N}}(S_\sigma))$$

for all $h \geq 0$ small enough. With $\mathbf{w}$ defined as the gradient of V^{Lya} on $\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + R^{\mathcal{N}}(S_\sigma))$, it is not difficult to show that

$$\mathbf{w} \cdot \mathbf{f}(\mathbf{x}) = D_t^+[V^{Lya}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))]\Big|_{t=0}.$$

In the same way as in the considerations about LC2, we get

$$-\gamma(\|\mathbf{x}\|) \geq -\sum_{i=1}^{n+1} \lambda_i \Gamma[\|\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_i^{\mathcal{N},\sigma})\|],$$

so if we can show that

$$-\sum_{i=1}^{n+1} \lambda_i \Gamma[\|\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_i^{\mathcal{N},\sigma})\|] \geq \mathbf{w} \cdot \mathbf{f}(\mathbf{x})$$

we have proved the theorem. Because of

$$\begin{aligned}
\mathbf{w} \cdot \mathbf{f}(\mathbf{x}) &= \mathbf{w} \cdot \sum_{i=1}^{n+1} \lambda_i \mathbf{f}(\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_i^{\mathcal{N},\sigma})) \\
&\quad + \mathbf{w} \cdot \left(\mathbf{f}(\mathbf{x}) - \sum_{k=1}^{n+1} \lambda_k \mathbf{f}(\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_k^{\mathcal{N},\sigma})) \right) \\
&\leq \sum_{i=1}^{n+1} \lambda_i \mathbf{w} \cdot \mathbf{f}(\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_i^{\mathcal{N},\sigma})) \\
&\quad + \|\mathbf{w}\|_1 \|\mathbf{f}(\mathbf{x}) - \sum_{k=1}^{n+1} \lambda_k \mathbf{f}(\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_k^{\mathcal{N},\sigma}))\|_\infty,
\end{aligned}$$

the formula (6) for $\mathbf{w}$, the definition of the constants $F_i[\mathbf{z}]$, and the inequality (7), the proof is further reduced to showing that

$$\begin{aligned}\|\mathbf{f}(\mathbf{x}) - &\sum_{k=1}^{n+1} \lambda_k \mathbf{f}(\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_k^{\mathcal{N},\sigma}))\|_\infty \\ &\leq \frac{1}{2}\sum_{i=1}^{n+1}\lambda_i \left(\sum_{r,s=1}^{n} B_{rs}^{\mathcal{N}}[\mathbf{y}^{\mathcal{N}}]\left[A_{r,i}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}[A_{s,1}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}} + A_{s,i}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}]\right]\right).\end{aligned}$$

To proof this last inequality it is convenient to define

$$\mathbf{y} := \mathrm{PS}(\mathbf{y}^{\mathcal{N}})$$

and

$$\mathbf{z}_i := \mathrm{PS}(\mathbf{y}^{\mathcal{N}} + \mathbf{x}_i^{\mathcal{N},\sigma}) - \mathrm{PS}(\mathbf{y}^{\mathcal{N}})$$

for $i = 1, 2, .., n+1$. By Taylor's theorem there are vectors $\mathbf{r}$, $\mathbf{r}_1$,..,$\mathbf{r}_{n+1}$ in $\mathrm{PS}(\mathbf{y}^{\mathcal{N}} + R^{\mathcal{N}}(S_\sigma))$, such that for all $i = 1, 2, .., n$ we have

$$\begin{aligned}f_i(\mathbf{y} + &\sum_{k=1}^{n+1}\lambda_k \mathbf{z}_k) - \sum_{k=1}^{n+1}\lambda_k f_i(\mathbf{y} + \mathbf{z}_k) \\ &= f_i(\mathbf{y}) + \nabla f_i(\mathbf{y}) \cdot \sum_{k=1}^{n+1}\lambda_k \mathbf{z}_k + \frac{1}{2}\sum_{r,s=1}^{n}[\mathbf{e}_r \cdot \sum_{k=1}^{n+1}\lambda_k \mathbf{z}_k][\mathbf{e}_s \cdot \sum_{l=1}^{n+1}\lambda_l \mathbf{z}_l]\frac{\partial^2 f_i}{\partial x_r \partial x_s}(\mathbf{r}) \\ &- \sum_{k=1}^{n+1}\lambda_k\left(f_i(\mathbf{y}) + \nabla f_i(\mathbf{y}) \cdot \mathbf{z}_k + \frac{1}{2}\sum_{r,s=1}^{n}[\mathbf{e}_r \cdot \mathbf{z}_k][\mathbf{e}_s \cdot \mathbf{z}_l]\frac{\partial^2 f_i}{\partial x_r \partial x_s}(\mathbf{r}_k)\right),\end{aligned}$$

i.e.

$$\begin{aligned}\|\mathbf{f}(\mathbf{x}) - \sum_{k=1}^{n+1}\lambda_k \mathbf{f}(\mathbf{y} + \mathbf{z}_k)\|_\infty &= \max_{i=1,2,..,n}|f_i(\mathbf{y} + \sum_{k=1}^{n+1}\lambda_k \mathbf{z}_k) - \sum_{k=1}^{n+1}\lambda_k f_i(\mathbf{y} + \mathbf{z}_k)| \\ &\leq \frac{1}{2}\left(\sum_{r,s=1}^{n} B_{rs}^{\mathcal{N}}[\mathbf{y}^{\mathcal{N}}]\left[\sum_{k,l=1}^{n+1}\lambda_k\lambda_l A_{r,k}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}\, A_{s,l}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}} + \sum_{k=1}^{n+1}\lambda_k A_{r,k}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}\, A_{s,k}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}\right]\right) \\ &\leq \frac{1}{2}\sum_{k=1}^{n+1}\lambda_k\left(\sum_{r,s=1}^{n} B_{rs}^{\mathcal{N}}[\mathbf{y}^{\mathcal{N}}]\left[A_{r,k}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}[A_{s,1}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}} + A_{s,k}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}]\right]\right),\end{aligned}$$

where we used the easy to see

$$\sum_{l=1}^{n+1}\lambda_l A_{s,l}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}} \leq A_{s,1}^{\mathcal{N},\sigma,\mathbf{y}^{\mathcal{N}}}.$$

We have proved Theorem 7.

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