Lyapunov Functions for Interconnected Systems

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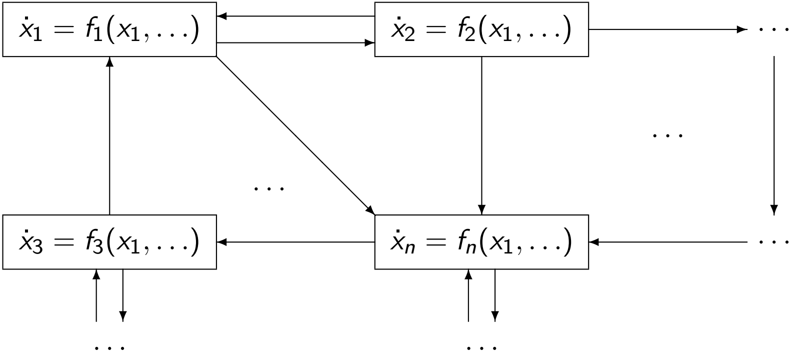
Stability of networks of ISS subsystems



What are conditions for input-to-state stability of such a network?

How can we construct Lyapunov functions ?

Input-to-state stability (ISS) — Lyapunov version

$$\xrightarrow{u} \boxed{\dot{x} = f(x,u)} \xrightarrow{x} \qquad \Sigma : \dot{x} = f(x,u)$$

Definition

A locally Lipschitz continuous function $V : \mathbb{R}^N \to \mathbb{R}_+$ is an **ISS Lyapunov function** for Σ if there exist $\psi_1, \psi_2, \gamma \in \mathcal{K}_\infty$ and a positive definite function α such that

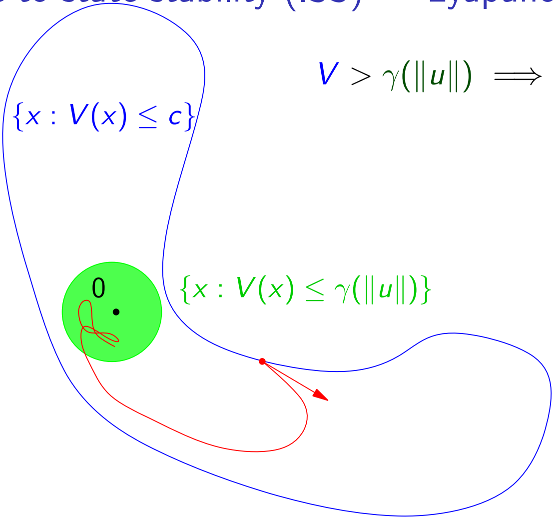
$$\psi_1(\|x\|) \leq \; V(x) \leq \psi_2(\|x\|)$$

and for a.a. $x \in \mathbb{R}^N$

$$V(x) \geq \gamma(\|u\|) \implies \nabla V(x) \cdot f(x,u) \leq -\alpha(V(x)).$$

Input-to-state stability (ISS) — Lyapunov version

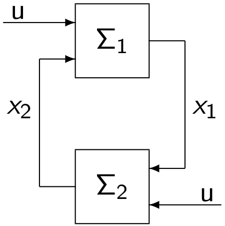
$$V > \gamma(\|u\|) \implies \dot{V} < 0$$



Interconnections: First steps

Consider a simple feedback loop

$$\begin{aligned}\Sigma_1 : \dot{x}_1 &= f_1(x_1, x_2, u)\\ \Sigma_2 : \dot{x}_2 &= f_2(x_1, x_2, u)\end{aligned}$$



$$f_i : \mathbb{R}^{N_1+N_2+N_u} \to \mathbb{R}^{N_i}$$

with

$$V_1(x_1) > \max\{\gamma_{12}(V_2(x_2)), \gamma_u(\|u\|)\} \quad \Rightarrow \dot{V}_1 < -\alpha_1(\|x_1\|)$$

$$V_2(x_2) > \max\{\gamma_{21}(V_1(x_1)), \gamma_u(\|u\|)\} \quad \Rightarrow \dot{V}_2 < -\alpha_2(\|x_2\|)$$

Two systems in feedback interconnection

Theorem [Jiang, Mareels, Wang 1996]
If there exist $\mathcal{K}_\infty$-functions α_1, α_2 such that

$$\gamma_{12} \circ \gamma_{21} \leq \mathrm{id},$$

then

$$\dot{x} = f(x, u)$$

with

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad f(x, u) = \begin{pmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{pmatrix}$$

is input-to-state stable from u to (x_1, x_2).
See also [Jiang, Teel, Praly (1994)] [Grüne 2002].

Two systems in feedback interconnection

The small gain condition is

$$\gamma_{12} \circ \gamma_{21} \le \mathrm{id},$$

Introduce the matrix

$$\Gamma := \begin{pmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{pmatrix}.$$

Γ may be interpreted as an operator

$$\Gamma : \mathbb{R}^2_+ \to \mathbb{R}^2_+ \quad \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \mapsto \begin{pmatrix} \gamma_{12}(s_2) \\ \gamma_{21}(s_1) \end{pmatrix}.$$

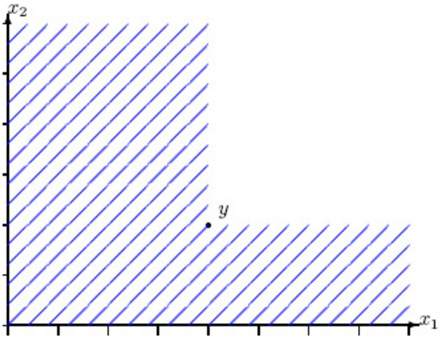
With this interpretation, the small gain condition is equivalent to

$$\Gamma(s) \not\geq s \quad \text{for all } s \in \mathbb{R}^2_+, s \neq 0.$$

The small gain condition

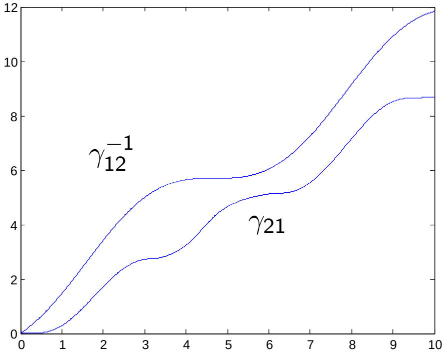
The condition for the existence of Lyapunov functions will turn out to be

$$\Gamma(s) \not\geq s\,, \quad \forall s \in \mathbb{R}^2_+ \setminus \{0\}$$



In two dimensions

$$\Gamma(s) \ngeq s \quad \text{means} \quad \Gamma(s) = \begin{bmatrix} \gamma_{12}(s_2) \\ \gamma_{21}(s_1) \end{bmatrix} \ngeq \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$



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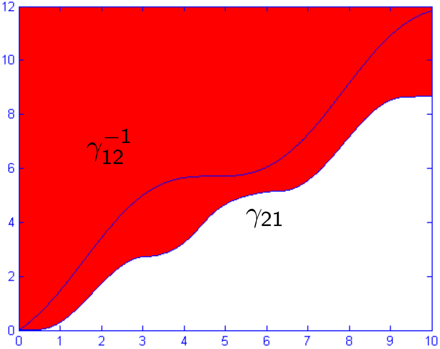

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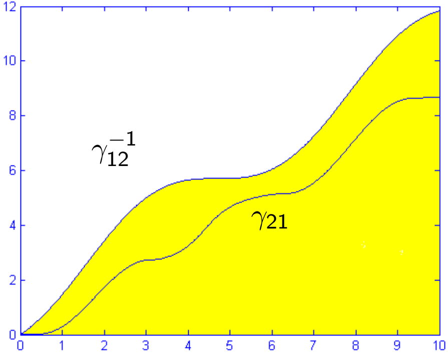
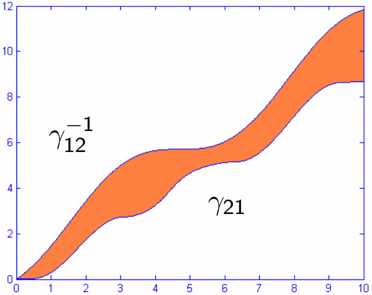


By the small gain conditions for some $s \in \mathbb{R}^2+$

$$\Gamma(s) = \begin{bmatrix} \gamma_{12}(s_2) \\ \gamma_{21}(s_1) \end{bmatrix} \ll \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$



By the small gain conditions for some $s \in \mathbb{R}^2_+$

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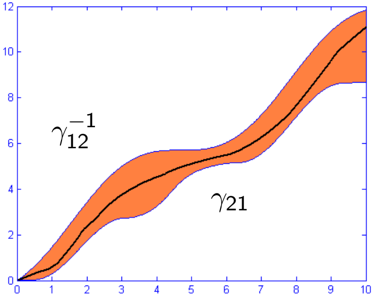


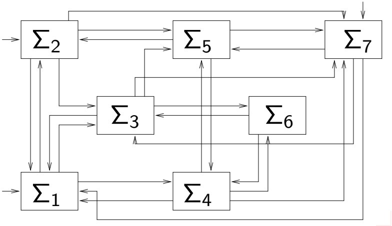
Construction of the Lyapunov function: Find a strictly increasing, unbounded path σ in the orange region. Set

$$V(x) = \max\{\sigma_1^{-1}(V_1(x_1)), \sigma_2^{-1}(V_2(x_2))$$

Large-scale interconnections

$$\Sigma_i : \dot{x}_i = f_i(x_1, \dots, x_n, u),$$

with $f_i : \mathbb{R}^{\sum N_j + M} \to \mathbb{R}^{N_i}$, such that each system satisfies



$$V_i(x_i) \geq \mu\{ \gamma_{ij}(V_j(x_j)), \gamma_i(\|u\|) \} \implies \nabla V_i(x_i) \cdot f_i(x, u) \leq -\alpha_i(V_i(x_i))$$

where $\gamma_{ij}, \gamma_{ij} \in \mathcal{K}_\infty$ or constantly zero.

Monotone Aggregation Functions

A continuous function $\mu : \mathbb{R}^n_+ \to \mathbb{R}_+$ is called a monotone aggregation function if the following two properties hold

(M1) positive definiteness: $\mu(0) = 0$, $\mu(s) > 0$ for all $s \in \mathbb{R}^n_+, s \neq 0$;

(M2) monotonicity: if $x < y$, then $\mu(x) < \mu(y)$.

The induced monotone operator

The interconnection structure given by the gain functions γ_{ij} defines a matrix

$$\Gamma = (\gamma_{ij}) \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}.$$

Such a matrix together with monotone aggregation functions $\mu_1, \ldots, \mu_n$ define a monotone operator $\Gamma_\mu : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

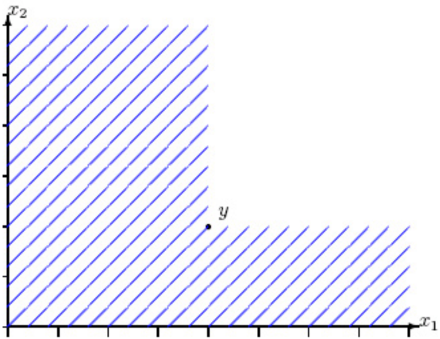
$$\Gamma_\mu(s) = \big(\mu_1(\gamma_{1j}(s_j)) \quad \ldots \quad \mu_n(\gamma_{nj}(s_j))\big)^T \,, \quad s \in \mathbb{R}^n_+ \,.$$

Here and in the following we assume $\gamma_{ii} \equiv 0$ for all i.

The small gain condition

The condition for the existence of Lyapunov functions is the existence of a $D = \operatorname{diag}(\operatorname{id} + \alpha_i), \alpha_i \in \mathcal{K}_\infty$ such that

$$D \circ \Gamma_\mu(s) \not\geq s\,, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}$$



The small gain condition

In the rest of the talk D will be ignored.
The condition for the existence of Lyapunov functions is

$$\Gamma_\mu(s) \not\geq s\,, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}$$

The small gain condition

The condition for the existence of Lyapunov functions is

$$\Gamma_\mu(s) \not\geq s\,, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}$$

Why is this condition interesting ?
We want again a path in the set $\{\Gamma_\mu(s) \ll s\}$.
Consider the sets

$$\Omega_i := \{s \in \mathbb{R}^n_+ \mid \Gamma(s)_i < s_i\} \quad i = 1, \ldots, n$$

If $\Gamma_\mu(s) \not\geq s$ then $\{0\} \cup \bigcup_i \Omega_i = \mathbb{R}^n_+$ and

$$\Omega := \bigcap_i \Omega_i \neq \emptyset$$

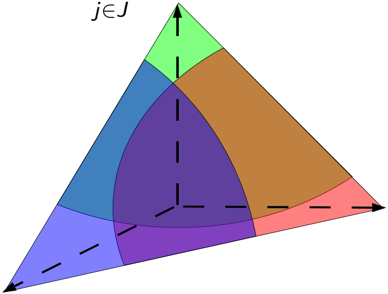
Theorem

(Knaster-Kuratowski-Mazurkiewicz 1929)
Let S be an $n-1$-dimensional simplex with extremal points $e_1, \ldots, e_n$. Let $A_1, \ldots, A_n$ be open sets in S such that for all subsets $\emptyset \neq J \subset \{1, \ldots, n\}$ we have

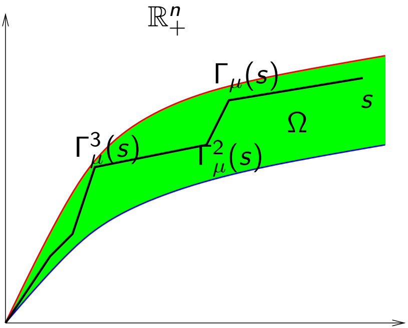
$$\operatorname{conv}\{e_j \mid j \in J\} \subset \bigcup_{j \in J} A_j \,,$$

then

$$\bigcap_{i=1}^{n} A_i \neq \emptyset \,.$$



Properties of Ω



$$\Omega = \{s \in \mathbb{R}^n_+ : \left(\Gamma_\mu(s)\right)_i < s_i \, \forall i\}$$

- Knaster-Kuratowski-Masurkiewicz-Theorem ensures that decay set intersected with S_r is nonempty for any $r > 0$.
- Γ_μ is strictly decreasing on Ω.
- The decay set contains a backward invariant set that is unbounded in every component and pathwise connected to the origin.

The final construction

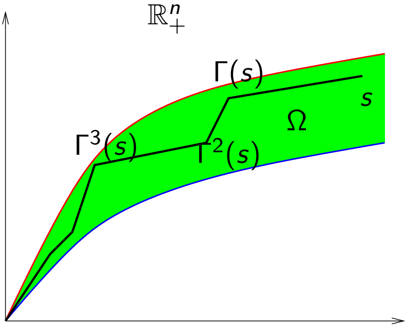
Theorem (Dashkovskiy, Rüffer, W. 2010)
If the small-gain condition

$$D \circ \Gamma(s) \not\geq s \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}$$

is satisfied, then an unbounded sufficiently regular path σ exists in Ω and an ISS Lyapunov function for Σ is given by

$$V(x) = \max_{i=1,\ldots,n} \sigma_i^{-1}(V_i(x_i)) \,. \tag{1}$$

Properties of Ω



$$\Omega = \{s \in \mathbb{R}^n_+ : \big(\Gamma(s)\big)_i < s_i \, \forall i\}$$

- Given a point $s \in \Omega$, the Ω-path can be constructed locally by iteration and linear interpolation.
- The only problem is to find $s \in \Omega$, i.e. a point such that

$$\Gamma_\mu(s) \ll s$$

Where are the algorithms?

For a numerical procedure we are left with the problem of finding a decrease point s for a monotone operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, i.e. a point satisfying

$$\Gamma(s) \ll s$$

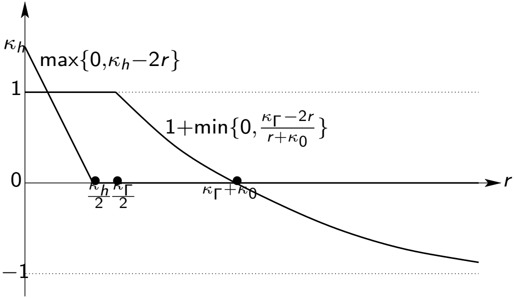
given that a small-gain condition holds.

Step 1: Change the problem to a fixed-point problem

Consider

$$\phi(v) = \Gamma_\mu(v)\left(1 + \min\left\{0, \frac{\kappa_\Gamma - 2\|v\|}{\|v\| + \kappa_0}\right\}\right) + \max\left\{0, \kappa_h - 2\|v\|\right\} e.$$

Here $\kappa_0 > 0$, $\kappa_\Gamma > \kappa_h > 0$ and $e := \sum_{I=1}^{N} e_i$.



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Lemma

If ϕ has a fixed point s^*, then this is a decay point of Γ_μ.
We can regulate the location of the fixed point by choice of the constants.

Step 2: Use homotopy to find a fixed point

Definition

Let $f, g : C \to D$ be continuous. We call f, g *homotopic*, if there exists a continuous mapping

$$\vartheta : C \times [0,1] \to D$$

with

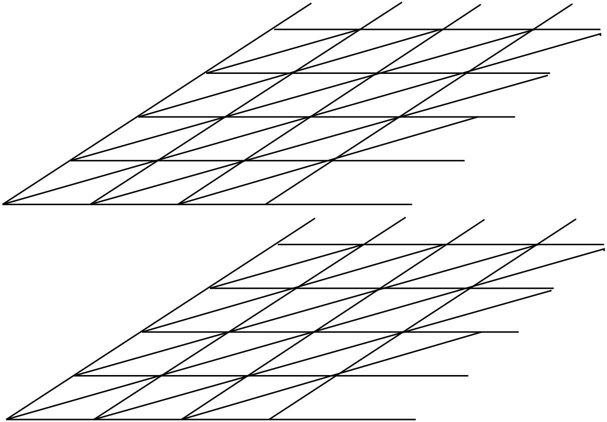
$$\vartheta(s,0) = f(s) \quad \text{and } \vartheta(s,1) = g(s) \quad \forall\ s \in C$$

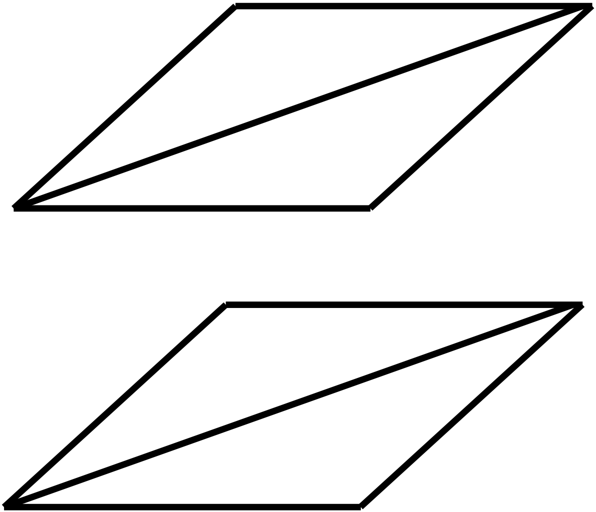
Step 2: Use homotopy to find a fixed point

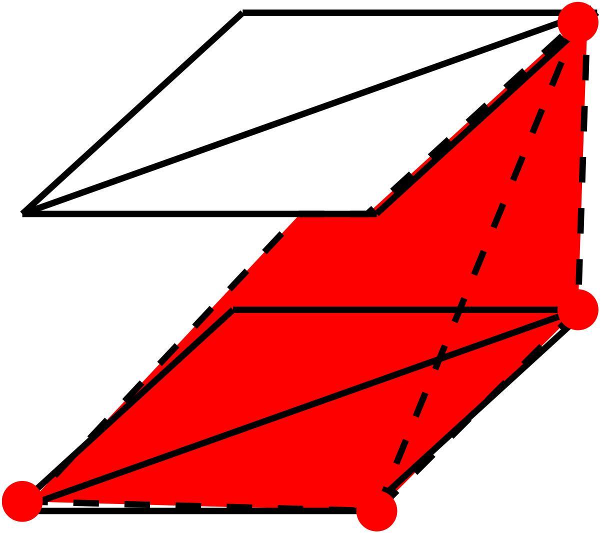
- Fix a triangulation of $\mathbb{R}^n_+ \times [0,1]$ with corner points only $\mathbb{R}^n_+ \times \{0,1\}$. Choose some $c \in \mathbb{R}^n_+$ such that $(c,0)$ lies in the interior of a facet in $\mathbb{R}^n \times \{0\}$.
- Consider the homotopy

$$\vartheta(v,t) = (1-t)c + t\phi(v)$$

- For $t = 0$ the unique fixed point is obviously c.
- Given a sequence $t_k \to 1$ there are fixed points s_{t_k} of $\vartheta(\cdot, t_k)$
- The cluster points of the set of fixed points s_{t_k} of are just the fixed points of $\vartheta(\cdot,1) = \phi$.







The Simplicial Fixed Point Algorithm

Step (0) τ^0: unique N-simplex containing $(c, 0)$.
η^0: the unique $(N+1)$-simplex which has τ^0 as its facet.
y^+ the vertex of η^0 that is not a vertex of τ^0.

Step (1) $k \to k+1$ Compute unique "complete" facet τ^{k+1} of η^k not equal to τ^k.
This is possible, unless $\tau^k \subset \mathbb{R}^N \times \{1\}$.

Step (2) Find unique simplex η^{k+1} sharing the facet τ^{k+1} with η^k and let y^+ be the vertex η^{k+1} not being a vertex of τ^{k+1}. Set $k = k+1$ and return to Step (1).

Convergence

Theorem (Geiselhart, W. 2012)
Let Γ be irreducible and Γ_μ satisfy the small gain condition.
Then there exists an explicit formula for the mesh size below which the simplicial algorithm converges in finitely many steps to a decay point of Γ_μ.

Test example

Rüffer, Dower, Ito, 2010:
System construction: $P \in \mathbb{R}_+^{n \times n}$, $\rho(P) < 1$, $A := -I + P$.

$$S(v)_i = \begin{cases} e^{v_i - 1} & \text{if } v_i > 1 \\ v_i & \text{if } v_i \in [-1, 1] \\ -e^{-v_i - 1} & \text{if } v_i < -1 \end{cases} .$$

Compute decay points for

$$\dot{v} = S'(S^{-1}(v))AS^{-1}(v) =: g(v)$$

Example: Quasi-monotone systems

N	run time	# iterations	simulations
5	0.0277s	20.9	100
10	0.0415s	34.5	100
15	0.0618s	72.3	100
25	0.1710s	187.8	100
50	1.180s	688.4	100
100	13.22s	2711.9	50
150	78.35s	6614.3	10
200	273.6s	11243.8	10

Conclusions

- We have seen a **general procedure** for the **construction** of an (ISS) **Lyapunov** function
- for **large-scale** networks of **interconnected systems**.
- Our result extends the existing result of Jiang, Mareels, Wang (1996) to arbitrarily many systems.
- Local construction of Lyapunov functions can be performed numerically. An algorithm extending an approach by Merrill can be used to determine a point in Ω. The rest of σ is the obtained by iterating Γ