Lyapunov functions for pullback attractors of nonautonomous difference equations

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Aim: *Construct a Lyapunov function characterising pullback attraction and pullback attractors for a discrete-time process generated by a nonautonomous difference equation in* $\mathbb{R}^d$.

Consider a nonautonomous difference equation

$$x_{n+1} = f_n(x_n) \qquad (1)$$

on $\mathbb{R}^d$, where the $f_n : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous mappings.

This generates a process $\phi : \mathbb{Z}^2_{\geq} \times \mathbb{R}^d \to \mathbb{R}^d$ through iteration by

$$\phi(n, n_0, x_0) = f_{n-1} \circ \cdots \circ f_{n_0}(x_0)$$

for all $n \geq n_0$ and each $x_0 \in \mathbb{R}^d$.

This satisfies the initial condition property

$$\phi(n_0, n_0, x_0) = x_0$$

for each $x_0 \in \mathbb{R}^d$ and all $n_0 \in \mathbb{Z}$;

the 2-parameter semigroup property

$$\phi(n_2, n_0, x_0) = \phi(n_2, n_1, \phi(n_1, n_0, x_0))$$

for each $x_0 \in \mathbb{R}^d$ and $n_0 \leq n_1 \leq n_2$ in $\mathbb{Z}$; and

the continuity property

$$x_0 \mapsto \phi(n, n_0, x_0) \quad \text{is Lipschitz continuous for all } n \geq n_0.$$

Pullback attractors

Definition *A ϕ-invariant family of nonempty compact subsets $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ is called a pullback attractor w.r.t. a basin of attraction system $\mathfrak{D}_{att}$ if it is pullback attracting,, i.e.,*

$$\lim_{j\to\infty} dist\,(\phi(n, n-j, D_{n-j}), A_n) = 0 \qquad (2)$$

for all $n \in \mathbb{Z}$ and all $\mathcal{D} = \{D_n : n \in \mathbb{Z}\} \in \mathfrak{D}_{att}$.

ϕ-invariance means that $A_n = \phi(n, n_0, A_{n_0})$ or $A_{n+1} = f_n(A_n)$.

The pullback attraction is taken with respect to a basin of attraction system $\mathfrak{D}_{att}$, which is defined as follows:

Definition *A basin of attraction system* $\mathfrak{D}_{att}$ *consists of families* $\mathcal{D} = \{D_n : n \in \mathbb{Z}\}$ *of nonempty bounded subsets of* $\mathbb{R}^d$ *with the property that* $\mathcal{D}^{(1)} = \{D_n^{(1)} : n \in \mathbb{Z}\} \in \mathfrak{D}_{att}$ *if* $\mathcal{D}^{(2)} = \{D_n^{(2)} : n \in \mathbb{Z}\} \in \mathfrak{D}_{att}$ *and* $D_n^{(1)} \subseteq D_n^{(2)}$ *for all* $n \in \mathbb{Z}$*.*

Although somewhat complicated, the use of a basin of attraction system allows both nonuniform and local attraction regions, which are typical in nonautonomous systems, to be handled.

Obviously $\mathcal{A} \in \mathfrak{D}_{att}$.

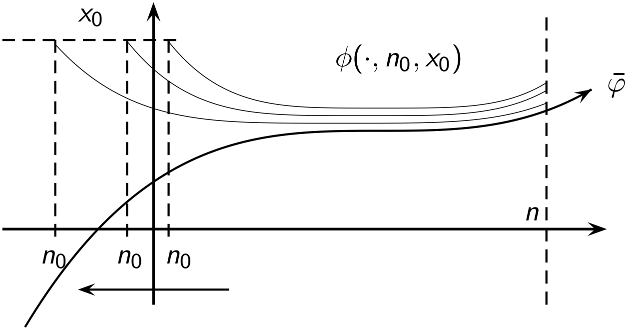

Figure: Pullback attraction.

A pullback absorbing neighbourhood system

The construction of the Lyapunov function requires the existence of a pullback absorbing neighbourhood family.

Lemma *Let $\mathcal{A}$ be a pullback attractor with a basin of attraction system $\mathfrak{D}_{att}$ for a process ϕ.*
Then there exists a pullback absorbing neighbourhood system $\mathcal{B} \subset \mathfrak{D}_{att}$ of $\mathcal{A}$ w.r.t. ϕ. Moreover, $\mathcal{B}$ is ϕ-positive invariant.

Sketch Proof For each $n_0 \in \mathbb{Z}$ pick $\delta_{n_0} > 0$ such that

$$B[A_{n_0};\delta_{n_0}] := \{x \in \mathbb{R}^d : \ \mathrm{dist}(x, A_{n_0}) \leq \delta_{n_0}\}$$

so $\{B[A_{n_0};\delta_{n_0}] : n_0 \in \mathbb{Z}\} \in \mathfrak{D}_{att}$.

Define

$$B_{n_0} := \overline{\bigcup_{j\geq 0} \phi(n_0, n_0 - j, B[A_{n_0-j}; \delta_{n_0-j}])}.$$

Obviously $A_{n_0} \subset \mathrm{int} B[A_{n_0}; \delta_{n_0}] \subset B_{n_0}$.

The positive invariance follows from the definition and the 2-parameter semigroup property to obtain

$$\phi(n_0 + 1, n_0, B_{n_0}) \subseteq B_{n_0+1},$$

and then induction.

The compactness of B_{n_0} follows from the compactness of $B[A_{n_0-j}; \delta_{n_0-j}]$ and hence, by the continuity of $\phi(n_0, n_0 - j, \cdot)$, of $\phi(n_0, n_0 - j, B[A_{n_0-j}; \delta_{n_0-j}])$ for each $j \geq 0$ and $n_0 \in \mathbb{Z}$.

Finally, $\mathcal{B}$ is pullback absorbing w.r.t. $\mathfrak{D}_{att}$ since $\mathcal{A}$ is pullback attracting. □

Necessary and sufficient conditions

The main result is the construction of a Lyapunov function that characterizes this pullback attraction.

Theorem *Let the f_n be uniformly Lipschitz continuous on $\mathbb{R}^d$ for each $n \in \mathbb{Z}$ and let ϕ be the process that they generate. In addition, let $\mathcal{A}$ be a ϕ-invariant family of nonempty compact sets that is pullback attracting with respect to ϕ with a basin of attraction system $\mathfrak{D}_{att}$.*

Then there exists a Lipschitz continuous function $V : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}$ such that

Property 1 (upper bound): *For all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$*

$$V(n_0, x_0) \leq dist(x_0, A_{n_0}); \tag{3}$$

Property 2 (lower bound): For each $n_0 \in \mathbb{Z}$ there exists a function $a(n_0, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with $a(n_0, 0) = 0$ and $a(n_0, r) > 0$ for all $r > 0$ which is monotonically increasing in r such that

$$a(n_0, (x_0, A_{n_0})) \leq V(n_0, x_0), \quad \text{for all } x_0 \in \mathbb{R}^d; \tag{4}$$

Property 3 (Lipschitz condition): For all $n_0 \in \mathbb{Z}$ and x_0, $y_0 \in \mathbb{R}^d$

$$|V(n_0, x_0) - V(n_0, y_0)| \leq \|x_0 - y_0\|; \tag{5}$$

Property 4 (pullback convergence): For all $n_0 \in \mathbb{Z}$ and any $\mathcal{D} \in \mathcal{D}_{att}$

$$\limsup_{n\to\infty} \sup_{z_{n_0-n} \in D_{n_0-n}} V(n_0, \phi(n_0, n_0 - n, z_{n_0-n})) = 0.$$

In addition,

Property 5 (forward convergence): *There exists $\mathcal{N} \in \mathfrak{D}_{att}$, which is positively invariant under ϕ and consists of nonempty compact sets N_{n_0} with $A_{n_0} \subset \operatorname{int} N_{n_0}$ for each $n_0 \in \mathbb{Z}$ such that*

$$V(n_0+1, \phi(n_0+1, n_0, x_0)) \leq e^{-1} V(n_0, x_0) \tag{6}$$

for all $x_0 \in N_{n_0}$, and hence

$$V(n_0+j, \phi(j, n_0, x_0)) \leq e^{-j} V(n_0, x_0), \quad \text{for } x_0 \in N_{n_0},\ j \in \mathbb{N}. \tag{7}$$

Proof

The aim is to construct a Lyapunov function $V(n_0, x_0)$ that characterises a pullback attractor $\mathcal{A}$ and satisfies properties 1–5 of the Theorem.
Define

$$V(n_0, x_0) := \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \text{dist}\left(x_0, \phi(n_0, n_0 - n, B_{n_0-n})\right)$$

for all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$, where

$$T_{n_0,n} = n + \sum_{j=1}^{n} \alpha^+_{n_0-j}$$

with $T_{n_0,0} = 0$.

Here $\alpha_n = \log L_n$, where L_n is the uniform Lipschitz constant of f_n on $\mathbb{R}^d$, and $a^+ = (a + |a|)/2$, i.e., the positive part of a real number a.

<u>Note:</u> $T_{n_0,n} \geq n$ and $T_{n_0,n+m} = T_{n_0,n} + T_{n_0-n,m}$ for n, $m \in \mathbb{N}$, $n_0 \in \mathbb{Z}$.

Proof of property 1

Since $e^{-T_{n_0,n}} \leq 1$ for all $n \in \mathbb{N}$ and $\operatorname{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n}))$ is monotonically increasing from $0 \leq \operatorname{dist}(x_0, \phi(n_0, n_0, B_{n_0}))$ at $n = 0$ to $\operatorname{dist}(x_0, A_{n_0})$ as $n \to \infty$,

$$
\begin{aligned}
V(n_0, x_0) &= \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \operatorname{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n})) \\
&\leq 1 \cdot \operatorname{dist}(x_0, A_{n_0}).
\end{aligned}
$$

Proof of property 3 $|V(n_0, x_0) - V(n_0, y_0)| =$

$$= \left| \sup_{n\in\mathbb{N}} e^{-T_{n_0,n}} \text{dist}\left(x_0, \phi(n_0, n_0 - n, B_{n_0-n})\right) - \sup_{n\in\mathbb{N}} e^{-T_{n_0,n}} \text{dist}\left(y_0, \phi(n_0, n_0 - n, B_{n_0-n})\right) \right|$$

$$\leq \sup_{n\in\mathbb{N}} e^{-T_{n_0,n}} \left|\text{dist}\left(x_0, \phi(n_0, n_0 - n, B_{n_0-n})\right) - \text{dist}\left(y_0, \phi(n_0, n_0 - n, B_{n_0-n})\right)\right|$$

$$\leq \sup_{n\in\mathbb{N}} e^{-T_{n_0,n}} \|x_0 - y_0\| \leq \|x_0 - y_0\|$$

since $|\text{dist}\,(x_0, C) - \text{dist}\,(y_0, C)| \leq \|x_0 - y_0\|$
for any x_0, $y_0 \in \mathbb{R}^d$ and nonempty compact subset C of $\mathbb{R}^d$.

Proof of property 2

If $x_0 \in A_{n_0}$, then $V(n_0, x_0) = 0$ by Property 1, so assume that $x_0 \in \mathbb{R}^d \setminus A_{n_0}$.

Now the supremum in

$$V(n_0, x_0) = \sup_{n \geq 0} e^{-T_{n_0,n}} \text{dist}\left(x_0, \phi(n_0, n_0 - n, B_{n_0-n})\right)$$

involves the product of an exponentially decreasing quantity bounded below by zero and a bounded increasing function, since the sets $\phi(n_0, n_0 - n, B_{n_0-n})$ are a nested family of compact sets decreasing to A_{n_0} with increasing n.

In particular,

$$\text{dist}\left(x_0, A_{n_0}\right) \geq \text{dist}\left(x_0, \phi(n_0, n_0 - n, B_{n_0-n})\right) \quad \text{for } n \in \mathbb{N}.$$

Hence there exists an $N^* = N^*(n_0, x_0) \in \mathbb{N}$ such that

$$\frac{1}{2}\text{dist}(x_0, A_{n_0}) \leq \text{dist}\left(x_0, \phi(n_o, n_0 - n, B_{n_0 - n})\right) \leq \text{dist}(x_0, A_{n_0})$$

for all $n \geq N^*$, but not for $n = N^* - 1$. Then,

$$\begin{aligned} V(n_0, x_0) &\geq e^{-T_{n_0, N^*}} \text{dist}\left(x_0, \phi(n_0, n_0 - N^*, B_{n_0 - N^*})\right) \\ &\geq \frac{1}{2} e^{-T_{n_0, N^*}} \text{dist}\left(x_0, A_{n_0}\right). \end{aligned}$$

Define

$$N^*(n_0, r) := \sup\{N^*(n_0, x_0) : \text{dist}\left(x_0, A_{n_0}\right) = r\}.$$

Now $N^*(n_0, r) < \infty$ for $x_0 \notin A_{n_0}$ with $\text{dist}\,(x_0, A_{n_0}) = r$ and $N^*(n_0, r)$ is nondecreasing with $r \to 0$.

To see this note that by the triangle rule

$$\begin{aligned} \text{dist}(x_0, A_{n_0}) &\leq \text{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0 - n})) \\ &\quad + \text{dist}(\phi(n_0, n_0 - n, B_{n_0 - n}), A_{n_0}). \end{aligned}$$

Also, by pullback convergence, there exists an $N(n_0, r/2)$ such that

$$\text{dist}(\phi(n_0, n_0 - n, B_{n_0 - n}), A_{n_0}) < \frac{1}{2} r$$

for all $n \geq N(n_0, r/2)$.

Hence for $\mathrm{dist}(x_0, A_{n_0}) = r$ and $n \geq N(n_0, r/2)$,

$$r \leq \mathrm{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0 - n})) + \frac{1}{2}r,$$

that is

$$\frac{1}{2}r \leq \mathrm{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0 - n})).$$

Obviously $N^*(n_0, r) \leq N^*(n_0, r/2)$.

Finally, define

$$a(n_0, r) := \frac{1}{2}r \; e^{-T_{n_0, N^*(n_0, r)}}. \tag{8}$$

Note that there is no guarantee here (without further assumptions) that $a(n_0, r)$ does not converge to 0 for fixed $r \neq 0$ as $n_0 \to \infty$.

Proof of property 4

Assume the opposite.

Then there exists an $\varepsilon_0 > 0$, a sequence $n_j \to \infty$ in $\mathbb{N}$ and points $x_j \in \phi(n_0, n_0 - n_j, D_{n_0 - n_j})$ such that $V(n_0, x_j) \geq \varepsilon_0$ for all $j \in \mathbb{N}$.

Since $\mathcal{D} \in \mathfrak{D}_{att}$ and $\mathcal{B}$ is pullback absorbing, there exists an $N = N(\mathcal{D}, n_0) \in \mathbb{N}$ such that

$$\phi(n_0, n_0 - n_j, D_{n_0 - n_j}) \subset B_{n_0} \quad \text{for } n_j \geq N.$$

Hence, $x_j \in B_{n_0}$ for all j such that $n_j \geq N$ and B_{n_0} is a compact set, so there exists a convergent subsequence $x_{j'} \to x^* \in B_{n_0}$.

But also

$$x_{j'} \in \overline{\bigcup_{n \geq n_{j'}} \phi(n_0, n_0 - n, D_{n_0 - n})}$$

and

$$\bigcap_{n_{j'}} \overline{\bigcup_{n \geq n_{j'}} \phi(n_0, n_0 - n, D_{n_0 - n})} \subseteq A_{n_0}$$

by the definition of a pullback attractor. Hence $x^* \in A_{n_0}$ and $V(n_0, x^*) = 0$. But V is Lipschitz continuous in its second variable by property 3, so

$$\varepsilon_0 \leq V(n_0, x_{j'}) = \|V(n_0, x_{j'}) - V(n_0, x^*)\| \leq \|x_{j'} - x^*\|,$$

which contradicts the convergence $x_{j'} \to x^*$.

Proof of property 5

Define

$$N_{n_0} := \{x_0 \in B[B_{n_0};1] \ : \ \phi(n_0+1, n_0, x_0) \in B_{n_0+1}\},$$

where $B[B_{n_0};1] = \{x_0 \ : \ \mathrm{dist}(x_0, B_{n_0}) \leq 1\}$ is bounded because B_{n_0} is compact and $\mathbb{R}^d$is locally compact, so N_{n_0} is bounded. It is also closed, hence compact, since $\phi(n_0+1, n_0, \cdot)$ is continuous and B_{n_0+1} is compact.

Now $A_{n_0} \subset \mathrm{int}B_{n_0}$ and $B_{n_0} \subset N_{n_0}$, so $A_{n_0} \subset \mathrm{int}N_{n_0}$. In addition,

$$\phi(n_0+1, n_0, N_{n_0}) \subset B_{n_0+1} \subset N_{n_0+1},$$

so $\mathcal{N}$ is positive invariant.

It remains to establish the exponential decay inequality (6).

This needs the Lipschitz condition on $\phi(n_0+1, n_0, \cdot) \equiv f_{n_0}(\cdot)$:

$$\|\phi(n_0+1, n_0, x_0) - \phi(n_0+1, n_0, y_0)\| \leq e^{\alpha_{n_0}} \|x_0 - y_0\|$$

for all x_0, $y_0 \in D_{n_0}$ from which it follows that

$$\mathrm{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0+1, n_0, C_{n_0})) \leq e^{\alpha_{n_0}} \mathrm{dist}(x_0, C_{n_0})$$

for any compact subset $C_{n_0} \subset \mathbb{R}^d$.

From the definition of V, we have $V(n_0+1, \phi(n_0+1, n_0, x_0)) =$

$$\begin{aligned} &= \sup_{n \geq 0} e^{-T_{n_0+1,n}} \mathrm{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0, n_0-n, B_{n_0-n})) \\ &= \sup_{n \geq 1} e^{-T_{n_0+1,n}} \mathrm{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0, n_0-n, B_{n_0-n})) \end{aligned}$$

since $\phi(n_0+1, n_0, x_0) \in B_{n_0+1}$ when $x_0 \in N_{n_0}$.

Hence re-indexing and then using the 2-parameter semigroup property and the Lipschitz condition on $\phi(1, n_0, \cdot)$ gives

$$V(n_0+1, \phi(n_0+1, n_0, x_0)) =$$

$$= \sup_{j\geq 0} e^{-T_{n_0+1,j+1}} \mathrm{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0, n_0-j-1, B_{n_0-j-1}))$$

$$= \sup_{j\geq 0} e^{-T_{n_0+1,j+1}} \mathrm{dist}\big(\phi(n_0+1, n_0, x_0), \phi(n_0+1, n_0, \phi(n_0, n_0-j, B_{n_0-j}))\big)$$

$$\leq \sup_{j\geq 0} e^{-T_{n_0+1,j+1}} e^{\alpha_{n_0}} \mathrm{dist}(x_0, \phi(n_0, n_0-j, B_{n_0-j}))$$

Now $T_{n_0+1,j+1} = T_{n_0,j} + 1 - \alpha_{n_0}^+$, so

$$V(n_0+1, \phi(n_0+1, n_0, x_0)) \leq$$

$$\begin{aligned} &\leq \sup_{j\geq 0} e^{-T_{n_0+1,j+1}+\alpha_{n_0}} \mathrm{dist}(x_0, \phi(n_0, n_0-j, B_{n_0-j})) \\ &= \sup_{j\geq 0} e^{-T_{n_0,j}-1-\alpha_{n_0}^+ +\alpha_{n_0}} \mathrm{dist}(x_0, \phi(n_0, n_0-j, B_{n_0-j})) \\ &\leq e^{-1} \sup_{j\geq 0} e^{-T_{n_0,j}} \mathrm{dist}(x_0, \phi(n_0, n_0-j, B_{n_0-j})) \leq e^{-1} V(n_0, x_0). \end{aligned}$$

Moreover, since $\phi(n_0+1, n_0, x_0) \in B_{n_0+1} \subset N_{n_0+1}$, the proof continues inductively to give

$$V(n_0+j, \phi(n_0+j, n_0, x_0)) \leq e^{-j} V(n_0, x_0) \quad \text{for } j \in \mathbb{N}. \qquad \Box$$

Comments on the Theorem

Comment 1: The forward convergence inequality (7) does not imply forward Lyapunov stability or Lyapunov asymptotical stability. Although

$$a(n_0 + j, \mathrm{dist}(\phi(n_0 + j, n_0, x_0), A_{n_0+j})) \leq e^{-j} V(n_0, x_0)$$

there is no guarantee (without additional assumptions) that

$$\inf_{j \geq 0} a(n_0 + j, r) > 0$$

for $r > 0$, so $\mathrm{dist}(\phi(n_0 + j, n_0, x_0), A_{n_0+j})$ need not become small as $j \to \infty$.

Counterexample Consider the process ϕ on $\mathbb{R}$ generated by the nonautonomous difference equation with $f_n = g_1$ for $n \leq 0$ and $f_n = g_2$ for $n \geq 1$, where the mappings $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$g_1(x) := \frac{1}{2}x, \qquad g_2(x) := \max\{0, 4x(1-x)\}$$

for all $x \in \mathbb{R}$.

Then the family $\mathcal{A}$ of subsets $A_{n_0} = \{0\}$ for all $n_0 \in \mathbb{Z}$ is pullback attracting for ϕ, but is not forward Lyapunov asymptotically stable.

Comment 2: The forward convergence inequality (7) can be rewritten as

$$\begin{aligned} V(n_0, \phi(n_0, n_0 - j, x_{n_0-j})) &\leq e^{-j} V(n_0 - j, x_{n_0-j}) \\ &\leq e^{-j} \mathrm{dist}(x_{n_0-j}, A_{n_0-j}) \end{aligned}$$

for all $x_{n_0-j} \in N_{n_0-j}$ and $j \in \mathbb{N}$.

Definition *A family $\mathcal{D} \in \mathfrak{D}_{att}$ is called* <u>*past–tempered*</u> *w.r.t. $\mathcal{A}$ if*

$$\lim_{j\to\infty} \frac{1}{j} \log^+ dist(D_{n_0-j}, A_{n_0-j}) = 0, \quad \text{for } n_0 \in \mathbb{Z},$$

or equivalently if

$$\lim_{j\to\infty} e^{-\gamma j} dist(D_{n_0-j}, A_{n_0-j}) = 0 \quad \text{for } n_0 \in \mathbb{Z},\ \gamma > 0.$$

This says that there is at most sub-exponential growth of the starting sets backwards in time.

For a past-tempered family $\mathcal{D} \subset \mathcal{N}$ it follows that

$$V(n_0, \phi(n_0, n_0 - j, x_{n_0-j})) \leq e^{-j}\text{dist}(D_{n_0-j}, A_{n_0-j}) \longrightarrow 0$$

as $j \to \infty$. Hence

$$a\,(n_0, \text{dist}(\phi(n_0, n_0 - j, x_{n_0-j}), A_{n_0})) \leq e^{-j}\text{dist}(D_{n_0-j}, A_{n_0-j}) \longrightarrow 0$$

as $j \to \infty$.

Since n_0 is fixed in the term on the left, this implies the pullback convergence

$$\lim_{j\to\infty} \text{dist}(\phi(n_0, n_0 - j, D_{n_0-j}), A_{n_0}) = 0.$$

Rate of pullback convergence

Since $\mathcal{B}$ is a pullback absorbing neighbourhood system, for every $n_0 \in \mathbb{Z}$, $n \in \mathbb{N}$ and $\mathcal{D} \in \mathfrak{D}_{att}$ there exists an $N(\mathcal{D}, n_0, n) \in \mathbb{N}$ such that

$$\phi(n_0 - n, n_0 - n - m, D_{n_0-n-m}) \subseteq B_{n_0-n} \quad \text{for } m \geq N.$$

Thus

$$V(n_0, \phi(n_0, n_0 - m, z_{n_0-m})) \leq e^{-T_{n_0,n}} \text{dist}\,(B_{n_0}, A_{n_0})$$

for all $z_{n_0-m} \in D_{n_0-m}$, $m \geq n + N(\mathcal{D}, n_0, n)$ and $n \geq 0$.

It can be assumed that the mapping $n \mapsto n + N(\mathcal{D}, n_0, n)$ is monotonic increasing in n and is hence invertible.

Let the inverse of $m = n + N(\mathcal{D}, n_0, n)$ be $n = M(m) = M(\mathcal{D}, n_0, m)$. Then

$$V(n_0, \phi(n_0, n_0 - m, z_{n_0 - m})) \leq e^{-T_{n_0, M(m)}} \text{dist}\left(B_{n_0}, A_{n_0}\right)$$

for all $m \geq N(\mathcal{D}, n_0, 0) \geq 0$. Usually $N(\mathcal{D}, n_0, 0) > 0$.

This expression can be modified to hold for all $m \geq 0$ by replacing $M(m)$ by $M^*(m)$ defined for all $m \geq 0$ and then introducing a constant $K_{\mathcal{D}, n_0} \geq 1$ to account for the behaviour over the finite time set $0 \leq m < N(\mathcal{D}, n_0, 0)$. For all $m \geq 0$ this gives

$$\boxed{V(n_0, \phi(n_0, n_0 - m, z_{n_0 - m})) \leq K_{\mathcal{D}, n_0} e^{-T_{n_0, M^*(m)}} \text{dist}\left(B_{n_0}, A_{n_0}\right)}$$