Computation of Lyapunov Functions for Differential Inclusions via Linear Programming

collaboration with Lars Grüne (Bayreuth) and Sigurður Hafstein

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UNIVERSITÄT BAYREUTH

ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

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1 Introduction
- Differential Inclusions
- Lyapunov Function

Differential Inclusion

Let $G \subset \mathbb{R}^n$ be compact and consider
a set-valued map $F : G \rightrightarrows \mathbb{R}^n$ with nonempty images, i.e.

$$F(x) \subset \mathbb{R}^n \quad (x \in G).$$

Differential Inclusion

Let $G \subset \mathbb{R}^n$ be compact and consider
a set-valued map $F : G \rightrightarrows \mathbb{R}^n$ with nonempty images, i.e.

$$F(x) \subset \mathbb{R}^n \quad (x \in G) .$$

Let $\mathrm{I} = [0, T]$ or $\mathrm{I} = [0, \infty)$.
$x : \mathrm{I} \to \mathbb{R}^n$ is a solution of the differential inclusion, if

- $x(\cdot)$ is absolutely continuous, i.e. $x(t) = x(0) + \int_0^t v(\tau) d\tau$ for every $t \in \mathrm{I}$ and $v(\cdot) \in L_1(\mathrm{I})$
- $x(t) \in G$ for every $t \in \mathrm{I}$
- $x'(t) \in F(x(t))$ a.e. $t \in \mathrm{I}$

Examples of Differential Inclusions

control problems (parametrized form)

Consider the control problem

$$\begin{aligned} x'(t) &= f(x(t), u(t)) && (t \in \mathrm{I}), \\ u(t) &\in U && (\text{a.e. } t \in \mathrm{I}), \\ x(0) &= x_0 \,. \end{aligned}$$

It is equivalent to solve the differential inclusion (DI)

$$\begin{aligned} x'(t) &\in F(x(t)) && (t \in \mathrm{I}), \\ x(0) &= x_0 \end{aligned}$$

with $F(x) = \bigcup_{u \in U} \{f(x, u)\}$.

Examples of Differential Inclusions (2)

ODE with uncertainty (outer/inner perturbation)

Consider the ODE

$$\begin{aligned} x'(t) &= f(x(t)) \quad (t \in \mathrm{I}), \\ x(0) &= x_0 \end{aligned}$$

and the outer perturbation $F(x) = f(x) + \varepsilon_1 B_1(0)$, $\varepsilon_1 > 0$:

$$\begin{aligned} x'(t) &\in f(x(t)) + \varepsilon_1 B_1(0) \quad (t \in \mathrm{I}), \\ x(0) &= x_0 . \end{aligned}$$

resp. the inner perturbation $F(x) = f(x + \varepsilon_2 B_1(0))$, $\varepsilon_2 > 0$, with

$$\begin{aligned} x'(t) &\in f(x(t) + \varepsilon_2 B_1(0)) = \bigcup_{\eta \in B_1(0)} \{f(x + \varepsilon_2 \eta)\} \quad (t \in \mathrm{I}), \\ x(0) &= x_0 . \end{aligned}$$

Examples of Differential Inclusions (3)

switched systems

Consider the switched system

$$x'(t) = f_\mu(x(t)) \quad (t \in \mathrm{I} \text{ with } x(t) \in G_\mu)\,,$$
$$x(0) = x_0$$

with $f_\mu : G_\mu \to \mathbb{R}^n$ Lipschitz and $G = \bigcup_{\mu=1,\ldots,M} G_\mu$.
Introduce the active set $\mathrm{I}_G(x) = \{\mu \in \{1,\ldots,M\} \,|\, x \in G_\mu\}$ and

$$F(x) = \mathrm{co}\{f_\mu(x) \,|\, \mu \in \mathrm{I}_G(x)\} \quad (x \in G)\,.$$

A piecewise affine system is a switched system with

$$f_\mu(x) = A_\mu x + b_\mu \quad (x \in G_\mu,\ \mu = 1,\ldots,M)\,.$$

Problem Class

Let $G \subset \mathbb{R}^n$ be a compact domain of computation splitted into M compact subregions G_μ

$$G = \bigcup_{\mu=1,\ldots,M} G_\mu \,.$$

With the active set

$$\mathrm{I}_G(x) = \{\mu \in \{1,\ldots,M\} \,|\, x \in G_\mu\}$$

we consider the right-hand side

$$F(x) = \mathrm{co}\{f_\mu(x) \,|\, \mu \in \mathrm{I}_G(x)\} \quad (x \in G)\,,$$
$$x'(t) \in F(x(t)) \quad (\text{a.e. } t \in \mathrm{I}).$$

Problem Class

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$$x'(t) \in F(x(t)) \quad (\text{a.e. } t \in \mathrm{I}).$$

- switched systems, interior of subregions are pairwise disjoint
- polytopic differential inclusions (all subregions equal G)

Possible Subregions

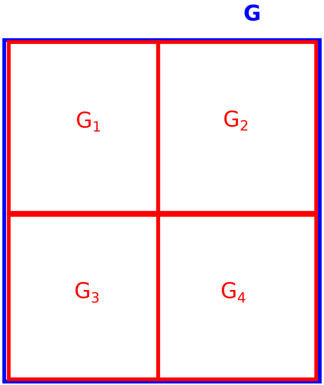

Figure : disjoint interiors of the subregions (partition)

Possible Subregions

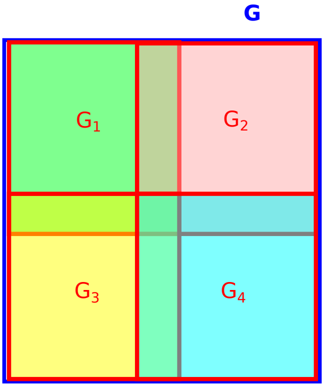

Figure : overlapping subregions

Possible Subregions

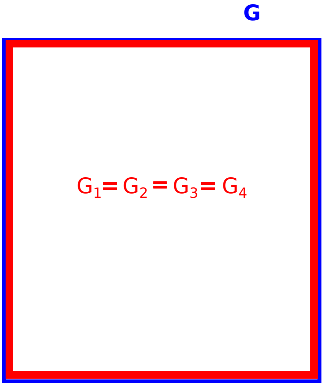

Figure : coinciding subregions

Lyapunov Function

Consider a differential inclusion (DI)

$$\begin{aligned} x'(t) &\in F(x(t)) \quad && (t \in \mathrm{I}), \\ x(t) &\in G && (t \in \mathrm{I}), \\ x(0) &= x_0 \end{aligned}$$

$V : G \to \mathbb{R}$ is a Lyapunov function for (DI), if

- $V(\cdot)$ is Lipschitz continuous
- $V(\cdot)$ is positive definite, i.e. $V(0) = 0$, $V(x) > 0$ for $x \in G \setminus \{0\}$
- there exists a positive function $\alpha : [0, \infty) \to [0, \infty)$ with

$$\nabla V(x)^\top f_1(x) \le -\alpha(\|x\|) \quad (\text{for } x \in G),$$

if $V(\cdot) \in C^1(\mathrm{I})$, $F(x) = \{f_1(x)\}$
(smooth Lyapunov function for ODE).

Lyapunov Function (2)

(i) smooth Lyapunov function

characterization:

(i) $V(\cdot) \in C^1(\mathrm{I})$, $F(x) = \{f_1(x)\}$
$\nabla V(x)^\top f_1(x) \leq -\alpha(\|x\|)$ for $x \in G$

Lyapunov Function (2)

(i) smooth Lyapunov function

(ii) smooth Lyapunov function for (DI)

characterization:

(ii) $V(\cdot) \in C^1(\mathrm{I})$, $F(x)$ not a singleton
$\delta^*(\nabla V(x), F(x)) := \max_{y \in F(x)} \langle \nabla V(x), y \rangle \leq -\alpha(\|x\|)$

Lyapunov Function (2)

(i) smooth Lyapunov function

(ii) smooth Lyapunov function for (DI)

(iii) nonsmooth Lyapunov function for ODE

characterization:

(iii) $V(\cdot) \notin C^1(\mathrm{I})$, $F(x) = \{f_1(x)\}$

$\delta^*(f_1(x), \partial_{\mathrm{Cl}} V(x)) = \max_{d \in \partial_{\mathrm{Cl}} V(x)} \langle d, f_1(x) \rangle \leq -\alpha(\|x\|)$

Lyapunov Function (2)

(i) smooth Lyapunov function

(ii) smooth Lyapunov function for (DI)

(iii) nonsmooth Lyapunov function for ODE

(iv) nonsmooth Lyapunov function for (DI)

characterization:

(iv) $V(\cdot) \notin C^1(\mathrm{I})$, $F(x)$ not a singleton

$\max_{d \in \partial_{\mathrm{Cl}} V(x)} \delta^*(d, F(x)) = \max_{d \in \partial_{\mathrm{Cl}} V(x)} \max_{y \in F(x)} \langle d, y \rangle$
$\leq -\alpha(\|x\|)$

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(i) smooth Lyapunov function

(ii) smooth Lyapunov function for (DI)

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$\nabla V(x)^\top f_1(x) \leq -\alpha(\|x\|)$ for $x \in G$

(ii) $V(\cdot) \in C^1(\mathrm{I})$, $F(x)$ not a singleton
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(iii) $V(\cdot) \notin C^1(\mathrm{I})$, $F(x) = \{f_1(x)\}$
$\delta^*(f_1(x), \partial_{\mathrm{Cl}} V(x)) = \max_{d \in \partial_{\mathrm{Cl}} V(x)} \langle d, f_1(x) \rangle \leq -\alpha(\|x\|)$

(iv) $V(\cdot) \notin C^1(\mathrm{I})$, $F(x)$ not a singleton
$\max_{d \in \partial_{\mathrm{Cl}} V(x)} \delta^*(d, F(x)) = \max_{d \in \partial_{\mathrm{Cl}} V(x)} \max_{y \in F(x)} \langle d, y \rangle$
$\leq -\alpha(\|x\|)$

Contents

2 Continuous Piecewise Linear Approximation

Triangulation

Let $G \subset \mathbb{R}^n$ be compact and consider
a triangulation $\mathcal{T} = \{\, T_\nu \,|, \nu = 1, \ldots, N\}$ with simplices

$$T_\nu = \mathrm{co}\{p_i^{(\nu)} \,|\, i = 1, \ldots, n+1\} \subset \mathbb{R}^n \,,$$

such that

- $G = \bigcup_{\nu=1}^{N} T_\nu$
- $(p_i^{(\nu)})_{i=1,\ldots,n+1}$ are affine independent
- the intersection of two different simplices is either *empty* or a *common face* of both simplices

We denote as active index set

$$\mathrm{I}_{\mathcal{T}}(x) = \{\nu \in \{1, \ldots, N\} \,|\, x \in T_\nu\} \,.$$

Triangulation (2)

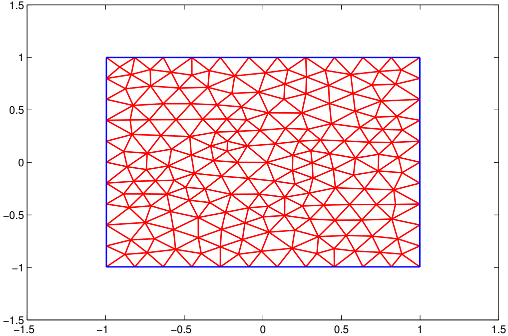

Figure : triangulation of $G = [-1, 1]^2$

Triangulation (2)

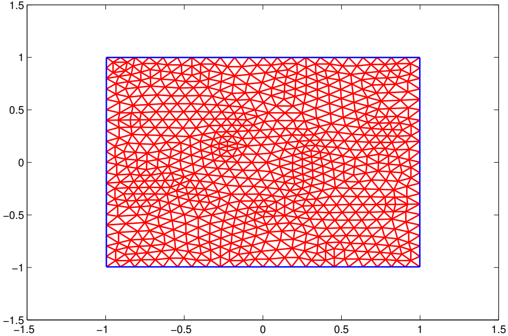

Figure : finer triangulation of $G = [-1, 1]^2$

Continuous Piecewise Linear Interpolation

Let $V : G \to \mathbb{R}$ be given
and a triangulation $\mathcal{T} = \{T_\nu \,|, \nu = 1, \dots, N\}$ of G.
The continuous piecewise linear interpolant $P_1 : G \to \mathbb{R}$ is

$$P_1(x) = \sum_{i=1}^{n+1} \lambda_i V(p_i^{(\nu)}) \quad (x \in T_\nu = \mathrm{co}\{p_i^{(\nu)} \,|\, i = 1, \dots, n+1\}),$$

if $(\lambda_1, \dots, \lambda_{n+1})$ are the barycentric coordinates of x, i.e.

$$x = \sum_{i=1}^{n+1} \lambda_i p_i^{(\nu)}, \quad \sum_{i=1}^{n+1} \underbrace{\lambda_i}_{\geq 0} = 1\,.$$

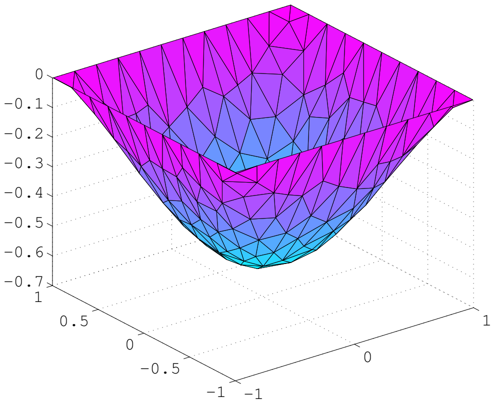

Figure : contin. piecewise linear interpolant on a triangulation

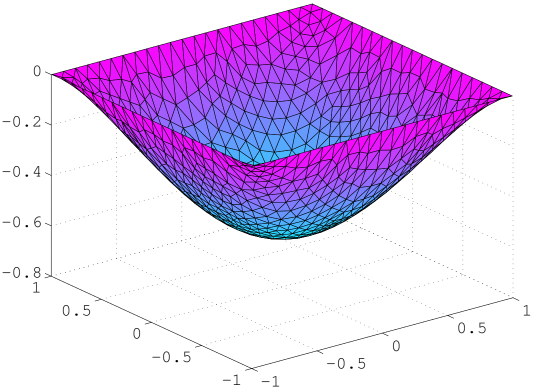

Figure : contin. piecewise linear interpolant on a finer triangulation

$\rightsquigarrow$ interpolation error is small, if simplices are small!

Interpolation Error

error for Lipschitz function in a k-face T

Let $g : G \to \mathbb{R}$ be a function.
For a given triangulation $\mathcal{T} = \{T_\nu \,|, \nu = 1, \dots, N\}$ of G
consider the contin. piecewise linear interpolant $P_1(\cdot)$ of $g(\cdot)$.
Let $T_\nu \in \mathcal{T}$ and $x \in T = \mathrm{co}\{p_i \mid i = 0, \dots, k\} \subset T_\nu$ and $k \leq n$.

(i) If $g(\cdot)$ is Lipschitz with constant L, then

$$|g(x) - P_1(x)| = \left| g\left(\sum_{i=0}^{k} \lambda_i p_i\right) - \sum_{i=0}^{k} \lambda_i g(p_i) \right| \leq Lh,$$

where $h = \mathrm{diam}(T) = \max_{x,y \in T} \|x - y\|_2$ is the diameter of T.

Interpolation Error

error for $\mathcal{C}^2$-function in a k-face T

Let $g : G \to \mathbb{R}$ be a function.
For a given triangulation $\mathcal{T} = \{T_\nu \,|, \nu = 1, \ldots, N\}$ of G consider the contin. piecewise linear interpolant $P_1(\cdot)$ of $g(\cdot)$.
Let $T_\nu \in \mathcal{T}$ and $x \in T = \mathrm{co}\{p_i \,|\, i = 0, \ldots, k\} \subset T_\nu$ and $k \leq n$.

(ii) If $g \in \mathcal{C}^2(U, \mathbb{R})$ with $U \subset \mathbb{R}^n$ is an open set with $U \supset T$, then

$$|g(x) - P_1(x)| = \left| g\left(\sum_{i=0}^{k} \lambda_i p_i\right) - \sum_{i=0}^{k} \lambda_i g(p_i) \right|$$

$$\leq \frac{1}{2} \sum_{i=0}^{k} \lambda_i B_H \|p_i - p_0\|_2 \left(\max_{z \in T} \|z - p_0\|_2 + \|p_i - p_0\|_2 \right) \leq B_H h^2 ,$$

where $B_H := \max_{z \in T} \|H(z)\|_2$, $H(z)$ is the Hessian of g at z.

Contents

3 Asymptotically Stable DI and Lyapunov Theorems

Asymptotically Stable DI

asymptotic stable DI

Consider the differential inclusion

$$
\begin{aligned}
x'(t) &\in F(x(t)) \quad (\text{a.e. } t \in \mathrm{I} = [0, \infty))\,, \\
x(t) &\in G \qquad\quad\;\; (t \in \mathrm{I} = [0, \infty))
\end{aligned}
$$

with $0 \in F(0)$, $0 \in \operatorname{int}(G)$.

(DI) is (strongly) asymptotically stable (at the origin) if

(i) For each $\varepsilon > 0$ there exists $\delta > 0$ such that each solution $x(t)$ with $\|x(0)\| \leq \delta$ satisfies $\|x(t)\| \leq \varepsilon$ for all $t \geq 0$.

(ii) There exists a neighborhood $\mathcal{U}$ of the origin such that for each solution $x(t)$ with $x(0) \in \mathcal{U}$ the convergence $x(t) \to 0$ holds as $t \to \infty$.

Domain of Attraction

Let (DI) be asymptotically stable on G, $\mathrm{I} = [0, \infty)$.
$D \subset \mathbb{R}^n$ is called domain of attraction w.r.t. G,
if

(i) $D := \{x_0 \in \mathbb{R}^n \,|$ every solution with $x(0) = x_0$ is defined on I, i.e., it stays in G, and satisfies $\lim_{t \to \infty} x(t) = 0\}$,

(ii) D is the maximal subset with this property.

Lyapunov Theorem

Clarke/Ledyaev/Stern (1998), Hinrichsen/Pritchard (2005)

If $V(\cdot)$ is a Lyapunov function for (DI), then

(i) (DI) is asymptotically stable

Lyapunov Theorem

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If $V(\cdot)$ is a Lyapunov function for (DI), then

(i) (DI) is asymptotically stable

(ii) if $x(\cdot)$ is a solution of (DI) with

$$x(\tau) \in G \quad (\tau \in [0, t]),$$

then $V(\cdot)$ is monotone decreasing along $x(\cdot)$, i.e.

$$V(x(t)) \leq V(x(0)) - \int_0^t \alpha(\|x(\tau)\|) d\tau$$

Lyapunov Theorem

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$$V(x(t)) \leq V(x(0)) - \int_0^t \alpha(\|x(\tau)\|) d\tau$$

(iii) for $c > 0$ and a subset $C \subset \mathbb{R}^n$ of the sublevel set with

- $C \subset V^{-1}([0, c]) = \{x \in G \mid V(x) \in [0, c]\}$ connected,
- $0 \in \operatorname{int} C$,
- $V^{-1}([0, c]) \subset \operatorname{int} G$,

it follows that C is contained in the domain of attraction w.r.t. G.

The inequality

$$V(x(t)) \leq V(x(0)) - \int_0^t \alpha(\|x(\tau)\|)d\tau$$

follows from

$$\frac{d}{dt}(V \circ x)(t) \leq \max_{d \in \partial_{\mathrm{Cl}} V(x)} \delta^*(d, F(x)) \leq -\alpha(\|x(t)\|_2)\,.$$

Converse Lyapunov Theorem by Teel/Praly (2000)

If

- (DI) is asymptotically stable
- with D as domain of attraction w.r.t. G,

then there exists a Lyapunov function $V : D \to \mathbb{R}$ which lies in $\mathcal{C}^\infty$.

Contents

4 Numerical Construction of Lyapunov Functions

Lyapunov Function

properties for a Lyapunov function

$V : G \to \mathbb{R}$ is a Lyapunov function for (DI), if

(P1) $V(\cdot)$ is Lipschitz continuous

(P2) $V(\cdot)$ is positive definite, i.e. $V(0) = 0$,
$V(x) > 0$ for $x \in G \setminus \{0\}$

(P3) there exists a positive function $\alpha : [0, \infty) \to [0, \infty)$ with

$$\max_{d \in \partial_{\mathrm{Cl}} V(x)} \max_{y \in F(x)} \langle d, y \rangle \le -\alpha(\|x\|) \quad (\text{for } x \in G),$$

Lyapunov Function

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(P3) there exists a positive function $\alpha : [0, \infty) \to [0, \infty)$ with

$$\max_{d \in \partial_{\mathrm{Cl}} V(x)} \max_{y \in F(x)} \langle d, y \rangle \le -\alpha(\|x\|) \quad (\text{for } x \in G),$$

If we fix $\varepsilon > 0$ and consider $x \in G_\varepsilon := G \setminus (\operatorname{int} B_\varepsilon(0))$,
then we can choose $\alpha(r) = r$ for $r > 0$
and demand for a *rescaled* function $\widetilde{V}(\cdot) = s \cdot V(\cdot)$:

$$\widetilde{V}(x) \ge \|x\|_2 \qquad (x \in G_\varepsilon),$$

$$\max_{d \in \partial_{\mathrm{Cl}} \widetilde{V}(x)} \max_{y \in F(x)} \langle d, y \rangle \le -\|x\|_2 \quad (x \in G_\varepsilon)$$

Auxiliary Results in Nonsmooth Optimization and for (DI)

Kummer (1988), Scholtes (1994)

If $V : G \to \mathbb{R}$ is contin. piecew. lin. on a triangulation $\mathcal{T} = (T_\nu)_\nu$, it is Lipschitz continuous and with $\nabla V_\nu := \nabla V|_{\text{int } T_\nu}$

$$\partial_{\text{Cl}} V(x) = \text{co}\{\nabla V_\nu \mid \nu \in \mathrm{I}_{\mathcal{T}}(x)\}\,.$$

Auxiliary Results in Nonsmooth Optimization and for (DI)

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$$\partial_{\text{Cl}} V(x) = \text{co}\{\nabla V_\nu \mid \nu \in \mathrm{I}_{\mathcal{T}}(x)\}.$$

Filippov (1988), Stewart (1990)

Assume that the subregions $(\text{int } G_\mu)_{\mu=1,\ldots,M}$ are pairwise disjoint. The Filippov regularization

$$x'(t) \in F(x(t)) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\text{co}}(f((B_\delta(x(t)) \cap G) \setminus N))$$

of the right-hand side

$$f(x) = f_\mu(x) \quad (x \in G_\mu)$$

of a switched system equals $F(x) = \text{co}\{f_\mu(x) \mid \mu \in \mathrm{I}_G(x)\}$.

Linear Optimization Problem

We consider

- $\varepsilon > 0$ and set $G_\varepsilon := G \setminus (\text{int}\, B_\varepsilon(0))$,
- a triangulation $\mathcal{T}_\varepsilon = \{ T_\nu \,|\, \nu = 1, \ldots, N\}$ with $T_\nu \subset \text{int}\, G_\varepsilon$,
- a family of subregions $(G_\mu)_{\mu=1,\ldots,M}$ of G
- the subregions G_μ and the triangulation $\mathcal{T}_\varepsilon$ have to satisfy the compatibility condition:

either $G_\mu \cap T_\nu$ is empty or a k-face of T_ν

Linear Optimization Problem (2)

We calculate the function $V : G \to \mathbb{R}$
via the linear optimization problem
based on the ▸ properties of a Lyapunov function:

(D1) $V(\cdot)$ is continous piecewise linear

(D2) $V_i^{(\nu)} \geq \|p_i^{(\nu)}\|_2$ for each vertex $p_i^{(\nu)}$, $i = 1, \ldots, n+1$, of $T_\nu \in \mathcal{T}_\varepsilon$

(D3) $\langle \nabla V_\nu, f_\mu(p_i^{(\nu)}) \rangle \leq -\|p_i^{(\nu)}\|_2$ for each $p_i^{(\nu)}$ as in (D2), where

$$\nabla V_\nu := \nabla V|_{\text{int } T_\nu} \equiv \text{const} \quad \text{for all } T_\nu \in \mathcal{T}_\varepsilon.$$

We additionally introduce for $\nabla V^{(\nu)} = (\nabla V_j^{(\nu)})_j$

(D4) $|\nabla V_j^{(\nu)}| \leq C_j^{(\nu)}$ for $j = 1, \ldots, n$

continuous piecewise linear function:

- given by $V(p_i^{(\nu)}) = V_i^{(\nu)}$

Linear Optimization Problem (3)

less than $(2n+1)N$ variables:

- $V_i^{(\nu)} > 0$ for $i = 1, \ldots, n+1$, $T_\nu \in \mathcal{T}$, $\nu = 1, \ldots, M$
- $C_j^{(\nu)} \geq 0$ for ν as above, $j = 1, \ldots, n$

maximal $((n+1)M + 2n+1)N$ linear constraints (sparse matrix):

- $V_i^{(\nu)} \geq \|p_i^{(\nu)}\|_2$ for i, ν as before
- $\langle \nabla V^{(\nu)}, f_\mu(p_i^{(\nu)}) \rangle \leq -\|p_i^{(\nu)}\|_2$ for i, ν as before, $\mu \in \mathrm{I}_G(p_i^{(\nu)})$
- $|\nabla V_j^{(\nu)}| \leq C_j^{(\nu)}$ for $j = 1, \ldots, n$

objective function:

- is arbitrary $\rightsquigarrow$ feasibility problem
- can be tuned to choose a specific Lyapunov function

Taking into Account: Interpolation Errors

Remark

The presented linear optimization problem
only calculates an approximate Lyapunov function.

▸ error estimates for linear interpolation

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The presented linear optimization problem
only calculates an approximate Lyapunov function.

▸ error estimates for linear interpolation

new constraints with relaxation $A_{\nu\mu} > 0$:

$$\text{(D3')} \quad \langle \nabla V^{(\nu)}, f_\mu(p_i^{(\nu)}) \rangle + A_{\nu\mu} \|\nabla V_\nu\|_1 \leq -\|p_i^{(\nu)}\|_2$$

Let $h_\nu = \mathrm{diam}(T_\varepsilon)$.
If $V(\cdot)$ is Lipschitz and if we choose $Lh_\nu \leq A_{\nu\mu}$
or if $V(\cdot) \in \mathcal{C}^2(U, \mathbb{R})$ with $U \supset T$ and $B_{\mu h} h_\nu^2 \leq A_{\nu\mu}$,
where $B_{\mu H}$ bounds the second partial derivatives of f_μ, then

$$\langle \nabla V_\nu, f_\mu(x) \rangle \leq -\|x\|_2 \quad \text{for every } x \in T,\ T\ k\text{-face of } T_\nu.$$

example for constraints

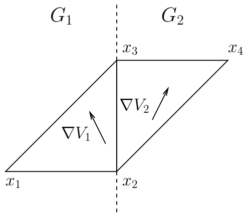

Figure : gradient conditions for two adjacent simplices

switched system with $f_i : G_i \to \mathbb{R}$, $i = 1, 2$
two simplices: $T_1 = \text{co}\{x_1, x_2, x_3\}$, $T_2 = \text{co}\{x_2, x_3, x_4\}$
$T_1 \cap T_2 = \text{co}\{x_2, x_3\}$ which leads to the following inequalities:

$$\langle \nabla V_1, f_1(x) \rangle \le -\|x\| \quad \text{for every } x \in \{x_1, x_2, x_3\} \subset T_1,$$
$$\langle \nabla V_2, f_2(x) \rangle \le -\|x\| \quad \text{for every } x \in \{x_2, x_3, x_4\} \subset T_2,$$
$$\langle \nabla V_1, f_2(x) \rangle \le -\|x\| \quad \text{for every } x \in \{x_2, x_3\} \subset T_1 \cap T_2,$$
$$\langle \nabla V_2, f_1(x) \rangle \le -\|x\| \quad \text{for every } x \in \{x_2, x_3\} \subset T_2 \cap T_1.$$

Construction of the Lyapunov Function

Proposition on linear optimization problem

If the following problem

(D1) $V(\cdot)$ is continuous piecewise linear

(D2) $V_i^{(\nu)} \geq \|p_i^{(\nu)}\|_2$ for each vertex $p_i^{(\nu)} \in T_\nu \in \mathcal{T}_\varepsilon$

(D3')

$$\langle \nabla V^{(\nu)}, f_\mu(p_i^{(\nu)}) \rangle \leq -\|p_i^{(\nu)}\|_2$$

has a feasible solution and each $f_\mu(\cdot)$ is Lipschitz,
then the contin. piecewise linear function is uniquely defined
and for every $x \in T_\nu$

$$\langle \nabla V_\nu, f_\mu(x) \rangle \leq -\|x\|_2 \quad \text{for all } \mu \in \mathrm{I}_G(x),\ \nu \in \mathrm{I}_{\mathcal{T}}(x)\,.$$

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(D3')

$$\langle \nabla V^{(\nu)}, f_\mu(p_i^{(\nu)}) \rangle + A_{\nu\mu} \|\nabla V_\nu\|_1 \leq -\|p_i^{(\nu)}\|_2$$

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Construction of the Lyapunov Function (2)

Proposition

If

- each $f_\mu(\cdot)$ is Lipschitz on G_μ,
- (DI) possesses a $\mathcal{C}^2$-Lyapunov function on G,

then for each $\varepsilon > 0$ there exists a triangulation $\mathcal{T}_\varepsilon$
such that the linear optimization problem has a solution
and yields a contin. piecewise linear Lyapunov function.

in the proof:

important estimates only hold,
if the simplices of the triangulation have small diameters h_μ
and they are not too "flat", cf. FEM.

Contents

5 Examples

Pendulum with uncertain friction

Grüne/Junge (2009)

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$f(x_1, x_2) = (x_2, -k x_2 - g \sin(x_1))^\top,$$

where $g = 9.81$ is the earth gravitation
and $k \geq 0$ models the friction of the pendulum.
(DI) is asymptotic stable for $k > 0$,
e.g. with $k \in [k_1, k_2] = [0.2, 1]$.
Setting

$$G_1 = G_2\,,$$
$$f_\mu(x) = (x_2, -k_\mu x_2 - g \sin(x_1))^\top,\ \ \mu = 1, 2\,,$$

we can allow time dependent friction with the (DI)

$$x'(t) \in F(x(t)) = \mathrm{co}\{f_\mu(x(t)) \,|\, \mu = 1, 2\}\,.$$

Pendulum with uncertain friction (2)

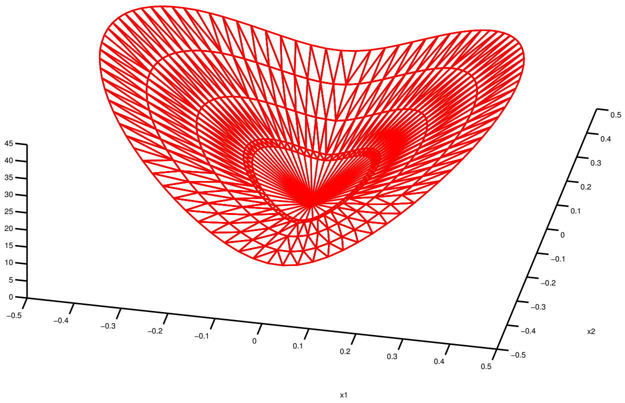

Figure : Lyapunov function for pendulum with uncertain friction

Pendulum with uncertain friction (3)

Remarks to example

- multivalued in whole domain
- compatibility condition trivially holds
- linear optimization problem has a feasible point
  $\rightsquigarrow$ continuous piecewise linear Lyapunov function exists
- Lyapunov function exists even for $\varepsilon = 0$
  due to triangle fans in the triangulation

Nonsmooth harmonic oscillator with nonsmooth friction

Bacciotti/Ceragioli (1999)

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$f(x_1, x_2) = \left(-\mathrm{sgn}(x_2) - \frac{1}{2}\mathrm{sgn}(x_1), \mathrm{sgn}(x_1)\right)^\top$$

with

$$\mathrm{sgn}(x_i) = \begin{cases} 1 & (x_i \geq 0)\,, \\ -1 & (x_i < 0)\,. \end{cases}$$

The vector field is constant on the four subregions

$$G_1 = [0, \infty) \times [0, \infty), \qquad G_2 = (-\infty, 0] \times [0, \infty),$$
$$G_3 = (-\infty, 0] \times (-\infty, 0], \qquad G_4 = [0, \infty) \times (-\infty, 0],$$

$\rightsquigarrow$ switched system $\rightsquigarrow$ Filippov regularization applies

Nonsmooth harmonic oscillator with nonsmooth friction (2)

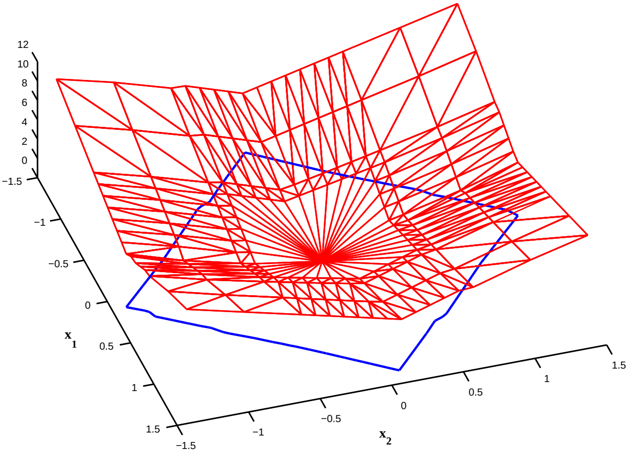

Figure : Lyapunov function for nonsmooth harmonic oscillator

Nonsmooth harmonic oscillator with nonsmooth friction (3)

Remarks to example

- a.e. single-valued
- compatibility condition trivially holds
- linear optimization problem succeeds to compute a continuous piecewise linear Lyapunov function
- Lyapunov function exists for arbitrary small $\varepsilon > 0$, but $\varepsilon = 0$ is not possible
- $x = (0, 1) \in G_1 \cap G_2$, hence

$$\mathrm{I}_G(x) = \{1, 2\} \text{ and } \partial_{\mathrm{Cl}} f(x) = \mathrm{co}\left\{ \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1 \end{pmatrix} \right\}$$

Nonsmooth harmonic oscillator with nonsmooth friction (4)

Remarks to example (2)

- in Bacciotti/Ceragioli (1999) another invariance property $\rightsquigarrow V(x) = |x_1| + |x_2|$ does not fulfill our invariance property:

$$\partial_{\mathrm{Cl}} V(x) = \mathrm{co}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\},$$

$$\max_{d \in \partial_{\mathrm{Cl}} V(x)} \delta^*(d, F(x)) \geq \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \right\rangle = 5/2 > 0$$

Hence, no monotone decrease along solutions is guaranteed.

Nonsmooth harmonic oscillator with nonsmooth friction (4)

Remarks to example (3)

- the inequality

$$\max_{d \in \partial_{\mathrm{Cl}} V(0)} \delta^*(d, F(0)) \leq 0 = -\|0\|_2$$

cannot hold, since

$$0 \in \operatorname{int} F(0) = \operatorname{co}\left\{ \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3/2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\},$$

$$\partial_{\mathrm{Cl}} V(0) = [-1, 1]^2$$

- nevertheless, the algorithm produces a similar Lyapunov function to $V(x) = |x_1| + |x_2|$

Conclusions

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- several extension were or will be presented in this workshop

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