Local Lyapunov Functions for periodic and finite-time ODEs

Peter Giesl and Sigurdur Hafstein

Abstract Lyapunov functions for general systems are difficult to construct. However, for autonomous linear systems with exponentially stable equilibrium, there is a classical way to construct a global Lyapunov function by solving a matrix equation. Consequently, the same function is a local Lyapunov function for a nonlinear system.
In this paper, we generalise these results to time-periodic and, in particular, finite-time systems with an exponentially attractive zero solution. We show the existence of local Lyapunov functions for nonlinear systems. For finite-time systems, we consider a generalised notion of a Lyapunov function, which is not necessarily continuously differentiable, but just locally Lipschitz continuous; the derivative is then replaced by the Dini derivative.

Dedicated to Jürgen Scheurle on the occasion of his 60th birthday.

1 Introduction

Lyapunov functions were introduced by Lyapunov in 1892 [22] to study stability of equilibria or other invariant sets. They can also be used to study the basin of attraction of attractors by their sublevel sets. For simplicity, we will in the following focus on an equilibrium or the zero solution as an attractor. The main features of a

Peter Giesl
Department of Mathematics, University of Sussex, Falmer, BN1 9QH, UK, e-mail: p.a.giesl@sussex.ac.uk

Sigurdur Hafstein
School of Science and Engineering, Reykjavik University, Menntavegi 1, IS-101 Reykjavik, Iceland, e-mail: sigurdurh@ru.is

Lyapunov function are that it (i) decreases (strictly) along solutions and (ii) attains its minimum on the attractor.

Lyapunov functions characterise certain attractivity properties and the basin of attraction; the necessity, i.e. the existence of Lyapunov functions, has been shown in so-called converse theorems. However, the construction of a Lyapunov function still remains a challenging problem. Recently, several algorithmic methods have been proposed to construct a Lyapunov function for a given system. Many of these methods face difficulties near the equilibrium or zero solution, since here the Lyapunov function does not decrease, but is constant.

Let us consider three modern construction methods, the so-called SOS method (sum of squares) [23, 24, 25], where a Lyapunov function that is presentable as a sum of squares of polynomials is constructed by convex optimization, the CPA (continuous and piecewise affine) method [16, 10, 11, 12], where a Lyapunov function that is continuous and locally affine on each simplex of a suitable triangulation is constructed by linear programming, and the RBF (radial basis functions) method using radial basis functions to numerically solve the Zubov equation [7]. All three methods can compute Lyapunov functions on compact neighborhoods of exponentially stable equilibria of autonomous systems and include the equilibrium in the domain of the Lyapunov function computed, given that the equilibrium is exponentially stable. These methods are, however, very different in nature. The SOS method is basically a local method, where the domain of the Lyapunov function can be enhanced by increasing the order of the polynomial Lyapunov function at the cost of greater computational complexity. The CPA and RBF methods are not local in nature and have no problems computing Lyapunov functions with large domains, if an arbitrary small neighborhood of the equilibrium is excluded [16], [7], respectively.

The problem of including the equilibrium at the origin in the domain of Lyapunov functions for CPA and RBF for the nonlinear system $\dot{x} = f(x)$ can be overcome by studying the linearised problem $\dot{x} = Ax$, where $A = Df(0)$. For such a linear equation, there is a classical method to construct a Lyapunov function $v(x) = x^T Qx$. This function is a **local** Lyapunov function for the nonlinear system $\dot{x} = f(x)$, i.e. v decreases along solutions only in a (small) neighborhood of the origin. Hence, the local Lyapunov function can be used to determine a local basin of attraction and close the gap between the implications of the nonlocal Lyapunov function and the local behaviour. Moreover, it can be combined with a global construction method to construct a Lyapunov function which is a true Lyapunov function even near the equilibrium. For the RBF method this was done in [8].

For the CPA method it was shown that a modified CPA method can always compute a CPA Lyapunov function including the equilibrium in its domain, first for planar systems [10, 11] and then for general n-dimensional systems [12, 13]. The key to the existence of a CPA Lyapunov function close to the equilibrium was to use the Lyapunov function $W(x) = \sqrt{x^T Qx}$, which satisfies $A\|x\|_2 \leq W(x)$ and $W'(x) \leq -B\|x\|_2$, and to interpolate this function on the edges of a suitably fine triangulation around the origin.

This paper generalises these ideas to **time-periodic** systems of the form $\dot{x} = f(t,x)$, where $f(t,x)$ is a T-periodic function, i.e. $f(t+T,x) = f(t,x)$, as well as to **finite-time** systems of the form $\dot{x} = f(t,x)$, considered over the finite-time interval $[0,T]$. The reason why we are enhancing the Lyapunov stability theory in this direction is because the CPA method has some nice properties like only assuming $f \in C^2$ and is extendable to switched systems [17] and differential inclusions [1] in a straight forward manner. Hence, the results of this paper will, besides the theoretical insight into Lyapunov functions, provide the starting point to develop a CPA construction method for Lyapunov functions for time-periodic and finite-time systems on domains, which include the attractive solution.

Lyapunov functions for periodic systems are functions $v(t,x)$ where the orbital derivative $v'(t,x) = \nabla_x v(t,x) \cdot f(t,x) + v_t(t,x)$ is negative. Such a Lyapunov function can be considered to be T-periodic without loss of generality.

Finite-time systems consider a nonautonomous equation $\dot{x} = f(t,x)$ over a finite-time interval $\mathbb{I} = [0,T]$. Finite-time dynamics were first studied in applications, in particular in fluid dynamics. The first mathematical theory was introduced by George Haller, who defined a *Lagrangian coherent structure* [20], i.e. time-evolving surfaces which can serve as boundaries of attraction areas. The relation of Lagrangian coherent structures to *finite-time Lyapunov exponents* as well as computational aspects are studied in [18, 19]. Furthermore, hyperbolicity and stable/unstable cones, which adapt the classical, infinite-time concepts of hyperbolicity and stable/unstable manifolds to the finite-time case, have been studied in [2, 6, 5].

While in the definition of hyperbolicity, attractivity is supposed to occur at every instance within the time domain under consideration, in [26, 14], a concept of attraction has been introduced, which allows that trajectories near an attracting solution move away from it, provided they return before the end of the time period. In this paper, we will use this notion of attractivity, where the distance of a solution $x(t)$ to the zero solution at time T is smaller than the distance of the solution at time 0, i.e. $\|x(T)\| < \|x(0)\|$. Note that for finite-time stability, the chosen norm is crucial, since different norms lead to different notions of attractivity; this is not the case for autonomous or periodic systems with infinite time because all norms on $\mathbb{R}^n$ are equivalent. Lyapunov functions for general nonautonomous systems have been studied in [21, 15], whereas Lyapunov functions for finite-time systems have been considered in [14, 9].

To characterise stability of zero solutions in periodic systems, one can use Floquet theory. We will show that Floquet theory is also helpful in the finite-time case, however, similar results to the periodic case can only be obtained under conditions that are stronger than assuming the attractivity of the zero solution and only for a specific type of vector norm. It turns out, that in the general case a finite-time Lyapunov function can be constructed by a different approach.

An autonomous system can be regarded as a periodic system, and a periodic system can also be considered over a finite-time interval; hence, we can compare the different notions of attractivity in these cases. It turns out that finite-time attractivity implies periodic-time attractivity, whereas the notions for autonomous and periodic-time are equivalent.

The paper is structured in the following way: In Sections 2-4 we study autonomous, periodic and finite-time systems, respectively. In each section, we start with linear systems, characterise exponential stability of the zero solution (equilibrium in the autonomous case), and show the existence of global Lyapunov functions. Furthermore, we consider nonlinear systems and prove similar results for local Lyapunov functions. While the results in the autonomous case are classical, parts of the periodic case are new. The main advance of the paper is the study of the finite-time case. In Section 5, we compare the notion of attractivity in periodic systems with the same system regarded as a finite-time system, and then we compare all three notions for an autonomous system. We end the paper with conclusions and an outlook for further work and applications of the results.

Notations

Definition 1. Consider a matrix $A \in \mathbb{C}^{n\times n}$.

1. A is called Hurwitz if all its eigenvalues have a strictly negative real part.
2. A is called Hermitian if A is equal to its conjugate transpose $A^* := \overline{A^T}$.
3. A is called positive definite if all its eigenvalues are real-valued and strictly positive.

Note that a Hermitian matrix A has real eigenvalues and $v^T Av$ is a real number for all $v \in \mathbb{R}^n$.

We denote by $\|\cdot\|_2 \colon \mathbb{C}^n \to \mathbb{R}$ the Euclidean norm $\|v\|_2 = \sqrt{\langle v, v\rangle}$, where $\langle v, w\rangle = \overline{v}^T w$, and by $\|\cdot\| \colon \mathbb{C}^n \to \mathbb{R}$ an arbitrary vector norm on $\mathbb{C}^n$. As usual, for $A \in \mathbb{C}^{n\times n}$, we denote by $\|A\|$ the induced matrix norm

$$\|A\| := \max_{x\in\mathbb{C}^n, \|x\|=1} \|Ax\|,$$

so that $\|Ax\| \le \|A\|\|x\|$ holds for all $x \in \mathbb{C}^n$. For $x_0 \in \mathbb{R}^n$ and $\eta > 0$, we define the open ball with respect to a norm on $\mathbb{R}^n$ by $B_\eta(x_0) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < \eta\}$.

2 Autonomous system

The section about autonomous systems does not contain any new results, but collects classical results that are needed for the periodic and finite-time case. It is included for the convenience of the reader, and for comparison with the other two cases.

Lemma 1. *Let $C \in \mathbb{C}^{n\times n}$ be a Hermitian, positive definite matrix, and $L \in \mathbb{C}^{n\times n}$ be Hurwitz. Then there is a unique solution $Q \in \mathbb{C}^{n\times n}$ of the matrix equation*

$$QL + L^* Q = -C$$

and Q is Hermitian and positive definite. If C and L are real-valued, then so is Q.

The proof is similar to the real case, cf. [21, Theorem 4.6].

Definition 2. A (strict) local Lyapunov function for the equilibrium at the origin of system $\dot{x} = f(x)$, where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $f(0) = 0$ is a function $V \in C(U, \mathbb{R}) \cap C^1(U \setminus \{0\}, \mathbb{R})$, where $U \subset \mathbb{R}^n$ is an open neighborhood of 0, which satisfies

1. $V(x) > 0$ for all $x \in U \setminus \{0\}$ and $V(0) = 0$ and
2. $V'(x) < 0$ for all $x \in U \setminus \{0\}$

where the orbital derivative is defined by $V'(x) = \nabla V(x) \cdot f(x)$. If $U = \mathbb{R}^n$, then the Lyapunov function is called global.

Note that the condition on differentiability of V can be dropped if the orbital derivative is replaced by the Dini derivative; this will be considered in Section 4.1. A Lyapunov function gives important information about the stability and the basin of attraction of the equilibrium 0.

Theorem 1. *Let V be a local Lyapunov function. Then the equilibrium 0 is asymptotically stable and any compact set $V^{-1}([0,c])$ with $c > 0$, contained in U, is a subset of the basin of attraction of 0.*

Theorem 2. *Consider the autonomous, linear system*

$$\dot{x} = Ax, \quad \text{where } A \in \mathbb{R}^{n \times n}. \tag{1}$$

Every fundamental matrix solution $\Phi(t)$ of (1) can be expressed in the form

$$\Phi(t) = e^{tA} P_0$$

where $P_0 \in \mathbb{R}^{n \times n}$ and the zero solution of (1) is globally exponentially stable if and only if A is Hurwitz.

Let $C \in \mathbb{R}^{n \times n}$ be a symmetric, positive definite matrix and $Q \in \mathbb{R}^{n \times n}$ be the solution of the matrix equation $QA + A^T Q = -C$ given by Lemma 1; note that this implies that Q is also symmetric and positive definite.

Then $V \colon \mathbb{R}^n \to \mathbb{R}$, $V(x) := x^T Q x$, and $W \colon \mathbb{R}^n \to \mathbb{R}$, $W(x) := \sqrt{V(x)} = \sqrt{x^T Q x}$, are both global Lyapunov functions for (1), satisfying

$$\begin{aligned} a_1 \|x\|_2^2 \leq V(x) &\leq b_1 \|x\|_2^2, \qquad & V'(x) &\leq -c_1 \|x\|_2^2, \\ a_2 \|x\|_2 \leq W(x) &\leq b_2 \|x\|_2, \qquad & W'(x) &\leq -c_2 \|x\|_2. \end{aligned}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ with constants $a_1, b_1, c_1, a_2, b_2, c_2 > 0$.

In the nonlinear case, we have the following theorem, cf. [21, Corollary 4.3, Proof of Theorem 4.7].

Theorem 3. *Consider the autonomous, nonlinear system*

$$\dot{x} = f(x) \tag{2}$$

with $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$ and $A := Df(0)$.

The equilibrium 0 of (2) is locally exponentially stable if and only if the equilibrium 0 of (1) is globally exponentially stable, i.e. by Theorem 2 if A is Hurwitz. The functions V and W from Theorem 2 are local Lyapunov functions for (2) in some open neighborhood U of 0 and satisfy the same inequalities as in Theorem 2.

3 Periodic time

Most results of this section are classical, however, the explicit form of the Lyapunov functions in Theorems 6, 7 using Floquet theory is, to the best of our knowledge, new. We start with a fundamental lemma, concerning the matrix logarithm, cf. [3, Theorem 2.47].

Lemma 2. *Let $M \in \mathbb{R}^{n\times n}$ be invertible. Then the matrix equation*

$$e^X = M \tag{3}$$

has a solution $X \in \mathbb{C}^{n\times n}$.

It is important to notice that, in general, even if the matrix M is real-valued, the matrix X can be complex-valued. Moreover, the solution X is not unique. A characterisation of all real-valued matrices M for which the matrix equation (3) has a real solution is given in [4].

3.1 Linear systems

The classical Floquet Theorem gives a representation of the fundamental solution in terms of complex matrices, even if $A(t)$ is real, cf. [3, Theorem 2.48].

Theorem 4. *Consider the T-periodic system*

$$\dot{x} = A(t)x \tag{4}$$

where $A(t) \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is T-periodic.

Then every fundamental matrix solution $\Phi(t)$ of (4) can be expressed in the form

$$\Phi(t) = P(t)e^{tL} \tag{5}$$

where $P(t)$ is continuously differentiable and T-periodic, $P(t) \in \mathbb{C}^{n\times n}$ is invertible for all $t \in \mathbb{R}$ and $L \in \mathbb{C}^{n\times n}$.

Definition 3. A T-periodic (strict) local Lyapunov function for the zero solution of system $\dot{x} = f(t,x)$, where $f \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ with $f(t,0) = 0$ for all $t \in \mathbb{R}$ and

$f(t+T,x) = f(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, is a function $V \in C(\mathbb{R}\times U,\mathbb{R}) \cap C^1(\mathbb{R}\times U\setminus\{0\},\mathbb{R})$, where $U \subset \mathbb{R}^n$ is an open neighborhood of 0, which satisfies

1. $V(t+T,x) = V(t,x)$ for all $x \in U$ and $t \in \mathbb{R}$, i.e. V is T-periodic,
2. $V(t,x) > 0$ for all $x \in U\setminus\{0\}$ and $V(t,0) = 0$ for all $t \in \mathbb{R}$ and
3. $V'(t,x) < 0$ for all $x \in U\setminus\{0\}$ for all $t \in \mathbb{R}$

where the orbital derivative is defined by

$$V'(t,x) = \nabla_x V(t,x)\cdot f(t,x) + V_t(t,x).$$

If $U = \mathbb{R}^n$, then the Lyapunov function is called global.

Theorem 5. *Let V be a T-periodic local Lyapunov function. Then the zero solution is asymptotically stable and any compact set $V^{-1}([0,c])\big|_{[0,T]\times\mathbb{R}^n}$ with $c > 0$, contained in $[0,T]\times U$, is a subset of the basin of attraction of the zero solution.*

In the following theorem we construct a T-periodic Lyapunov function for the linear system (4). Note that this Lyapunov function is the same as the one constructed in [21, Theorem 4.12] for the general nonautonomous case. In the periodic case, to which we restrict ourselves here, however, one can drop some assumptions on the uniformity with respect to t and, moreover, we can give a more explicit expression for V, using Floquet theory.

Theorem 6. *Consider the T-periodic linear equation*

$$\dot{x} = A(t)x \tag{6}$$

where $A(t) \in C(\mathbb{R},\mathbb{R}^{n\times n})$ is T-periodic. Then the zero solution of (6) is globally exponentially stable if and only if L is Hurwitz, where L is defined in Theorem 4.

Let $C \in \mathbb{C}^{n\times n}$ be a Hermitian, positive definite matrix and $Q \in \mathbb{C}^{n\times n}$ be the solution of the matrix equation $QL + L^*Q = -C$, see Lemma 1; note that also Q is Hermitian and positive definite.

Then $V,W\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$,

$$\begin{aligned} V(t,x) &:= x^T(P^{-1}(t))^*QP^{-1}(t)x \\ W(t,x) &:= \sqrt{V(t,x)} = \sqrt{x^T(P^{-1}(t))^*QP^{-1}(t)x} \end{aligned}$$

are both T-periodic global Lyapunov functions for (6), satisfying

$$\begin{aligned} a_1\|x\|_2^2 \le V(t,x) \le b_1\|x\|_2^2, &\qquad V'(t,x) \le -c_1\|x\|_2^2, \\ a_2\|x\|_2 \le W(t,x) \le b_2\|x\|_2, &\qquad W'(t,x) \le -c_2\|x\|_2, \end{aligned}$$

for all $x \in \mathbb{R}^n\setminus\{0\}$ and $t \in \mathbb{R}$ with constants $a_1,b_1,c_1,a_2,b_2,c_2 > 0$.

Proof. Using the transformation $y = P^{-1}(t)x$, the system is transformed into the autonomous system $\dot{y} = Ly$; the characterisation of exponential stability now follows from Theorem 2.

Using Theorem 4 we express the fundamental matrix solution with initial condition $\Phi(0) = I$ by
$$\Phi(t) = P(t)e^{tL}$$
where $P(0) = P(T) = I$, $P(t+T) = P(t)$ and $P(t), L \in \mathbb{C}^{n\times n}$. Note that since $\Phi(t)$ is a solution, we have

$$\dot{\Phi}(t) = A(t)\Phi(t) = A(t)P(t)e^{tL}$$

On the other hand,

$$\dot{\Phi}(t) = \dot{P}(t)e^{tL} + P(t)Le^{tL},$$
$$\text{which yields } \dot{P}(t) = -P(t)L + A(t)P(t).$$

Moreover, since $0 = \frac{d}{dt}\left(P(t)P^{-1}(t)\right) = \dot{P}(t)P^{-1}(t) + P(t)\dot{P}^{-1}(t)$, we have

$$\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t) = LP^{-1}(t) - P^{-1}(t)A(t). \tag{7}$$

Note that
$$V(t,x) = x^T(P^{-1}(t))^*QP^{-1}(t)x$$
is T-periodic and real-valued since $(P^{-1}(t))^*QP^{-1}(t)$ is Hermitian. Moreover, since Q is positive definite and $P^{-1}(t)$ is non-singular and T-periodic, there are constants $a_1, b_1 > 0$ such that $a_1\|x\|_2^2 \le V(t,x) \le b_1\|x\|_2^2$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

We show the statement for $V'(t,x)$. Using (7), we have

$$\begin{aligned}
V'(t,x) &= x^T(A(t))^T(P^{-1}(t))^*QP^{-1}(t)x + x^T(\dot{P}^{-1}(t))^*QP^{-1}(t)x \\
&\quad + x^T(P^{-1}(t))^*Q\dot{P}^{-1}(t)x + x^T(P^{-1}(t))^*QP^{-1}(t)A(t)x \\
&= x^T[(A(t))^T(P^{-1}(t))^* + (P^{-1}(t))^*L^* - (A(t))^*(P^{-1}(t))^*]QP^{-1}(t)x \\
&\quad + x^T(P^{-1}(t))^*Q[LP^{-1}(t) - P^{-1}(t)A(t) + P^{-1}(t)A(t)]x \\
&= x^T(P^{-1}(t))^*[L^*Q + QL]P^{-1}(t)x \\
&= -(P^{-1}(t)x)^*CP^{-1}(t)x \\
&\le -c_1\|x\|_2^2
\end{aligned}$$

for a suitable $c_1 > 0$, since C is positive definite. For the function W, we use

$$W'(t,x) = \frac{1}{2W(t,x)}V'(t,x) \le \frac{1}{2\sqrt{b_1}\|x\|_2}(-c_1\|x\|_2^2) = -\frac{c_1}{2\sqrt{b_1}}\|x\|_2.$$

3.2 Nonlinear systems

Theorem 7. *Consider the T-periodic nonlinear equation*

$$\dot{x} = f(t,x) \tag{8}$$

where $f \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, $f(t+T,x) = f(t,x)$, $f(t,0) = 0$ *for all* $t \in \mathbb{R}$ *and* $A(t) := D_x f(t,0)$.

Consider (6) with the same $A(t)$. *The zero solution of (8) is locally exponentially stable if and only if the zero solution of (6) is globally exponentially stable, i.e.* L *is Hurwitz, where* L *is defined in Theorem 6.*

The functions V *and* W *defined in Theorem 6 are local Lyapunov functions for (8) and satisfy the same inequalities as in Theorem 6.*

Proof. The zero solution is exponentially stable if and only if L is Hurwitz, cf. e.g. [21, Theorem 4.15] – note that the assumptions of that theorem, which holds in the more general nonautonomous case, can be relaxed, since we are focussing on the periodic case. In particular, $D_x f(t,x)$ is bounded uniformly in t, since it is periodic in t, and the Lipschitz continuity, which was used in Taylor's Theorem, can be dropped by the following argument: Using Taylor's Theorem, we can write $f(t,x) = A(t)x + \psi(t,x)$, where $\psi(t,x) = (D_x f(t,\theta x) - D_x f(t,0))x$ by the mean value theorem, where $\theta \in [0,1]$, i.e. $\psi(t,x) = o(\|x\|)$ as $x \to 0$, uniformly in t, since $D_x f(t,x)$ is continuous and T-periodic. Hence, for all $\varepsilon > 0$ there is a $r > 0$ such that $\|\psi(t,x)\|_2 \le \varepsilon\|x\|_2$ holds for all $\|x\|_2 < r$ and all $t \in \mathbb{R}$.

We show that V is a local Lyapunov function, fulfilling the inequalities. Note that the inequalities on $V(t,x)$ are clear, by Theorem 6, so that we only have to prove $V'(t,x) \le -c_1\|x\|_2^2$; note that in the nonlinear case, $V'(t,x)$ is different to the linear case.

Since C is Hermitian and positive definite, there is a smallest eigenvalue $\lambda > 0$ of C such that $y^T C y \ge \lambda\|y\|_2^2$ for all $y \in \mathbb{R}^n$. Set $\varepsilon := \frac{\lambda}{4\|Q\|_2 \max_{t\in[0,T]}(\|P^{-1}(t)\|_2\|P(t)\|_2)}$ and choose $r > 0$ as above such that $\|\psi(t,x)\|_2 \le \varepsilon\|x\|_2$ holds for all $\|x\|_2 < r$ and all $t \in \mathbb{R}$. Then, similar to the theorem in the linear case, we have, using (7)

$$\begin{aligned}
V'(t,x) &= x^T(A(t))^T(P^{-1}(t))^* Q P^{-1}(t)x + \psi(t,x)^T(P^{-1}(t))^* Q P^{-1}(t)x \\
&\quad + x^T(\dot{P}^{-1}(t))^* Q P^{-1}(t)x + x^T(P^{-1}(t))^* Q \dot{P}^{-1}(t)x \\
&\quad + x^T(P^{-1}(t))^* Q P^{-1}(t)A(t)x + x^T(P^{-1}(t))^* Q P^{-1}(t)\psi(t,x) \\
&\le x^T[(A(t))^T(P^{-1}(t))^* + (P^{-1}(t))^* L^* - (A(t))^*(P^{-1}(t))^*]QP^{-1}(t)x \\
&\quad + x^T(P^{-1}(t))^* Q[LP^{-1}(t) - P^{-1}(t)A(t) + P^{-1}(t)A(t)]x \\
&\quad + 2\|Q\|_2\|P^{-1}(t)\|_2\|P^{-1}(t)x\|_2\|\psi(t,x)\|_2 \\
&= x^T(P^{-1}(t))^*[L^*Q + QL]P^{-1}(t)x \\
&\quad + 2\varepsilon\|Q\|_2\|P^{-1}(t)\|_2\|P^{-1}(t)x\|_2\|x\|_2 \\
&= -(P^{-1}(t)x)^* C P^{-1}(t)x \\
&\quad + 2\varepsilon\|Q\|_2 \max_{t\in[0,T]}(\|P^{-1}(t)\|_2\|P(t)\|_2)\|P^{-1}(t)x\|_2^2 \\
&\le \left(-\lambda + \frac{\lambda}{2}\right)\|P^{-1}(t)x\|_2^2 \\
&\le -c_1\|x\|_2^2
\end{aligned}$$

for all $\|x\|_2 < r$ with a suitable $c_1 > 0$. The argumentation for W is as in Theorem 6.

4 Finite time

For this section, we fix an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$. We consider the nonautonomous ODE

$$\dot{x} = f(t,x) \tag{9}$$

where $f \in C^1([0,T] \times \mathbb{R}^n, \mathbb{R}^n)$ over the finite-time interval $\mathbb{I} = [0,T]$. We denote the solution of (9) with initial value $x(t_0) = x_0$ by $\varphi(t,t_0,x_0) := x(t)$ and assume that it exists in the whole interval $[0,T]$. This is e.g. the case if $D_x f(t,x)$ is bounded. We will later assume that $\mu(t) = 0$ is a solution, i.e. $f(t,0) = 0$ for all $t \in \mathbb{I}$. We use the following definition of finite-time attractivity from [26, 14].

Definition 4 (Finite-time attractivity, domain of attraction). Let $\mu : \mathbb{I} \to \mathbb{R}^n$ be a solution of (9).

1. μ is called *attractive on* $\mathbb{I}$ with respect to the norm $\|\cdot\|$ if there exists an $\eta > 0$ such that
$$\|\varphi(T,0,\xi) - \mu(T)\| < \|\xi - \mu(0)\| \; \forall \xi \in B_\eta(\mu(0)) \setminus \{\mu(0)\}\,.$$
2. μ is called *exponentially attractive on* $\mathbb{I}$ with respect to the norm $\|\cdot\|$ if
$$\limsup_{\eta \searrow 0} \frac{1}{\eta} \sup_{\xi \in B_\eta(0)} (\|\varphi(T,0,\xi) - \mu(T)\|) < 1\,,$$
and the negative number
$$\frac{1}{T} \ln \left(\limsup_{\eta \searrow 0} \frac{1}{\eta} \sup_{\xi \in B_\eta(0)} (\|\varphi(T,0,\xi) - \mu(T)\|) \right)$$
is called *rate of exponential attraction*.
3. Let $\mu : \mathbb{I} \to \mathbb{R}^n$ be an attractive solution on $\mathbb{I}$. Then a connected and invariant nonautonomous (i.e. $G_\mu(t) := \{x \in \mathbb{R}^n \mid (t,x) \in G_\mu\}$ is nonempty for all $t \in \mathbb{I}$) set $G_\mu \subset \mathbb{I} \times \mathbb{R}^n$ is called *domain of attraction of* μ if
$$\|\varphi(T,0,x) - \mu(T)\| < \|x - \mu(0)\| \text{ holds for all } x \in G_\mu(0) \setminus \{\mu(0)\}\,,$$
and G_μ is the maximal set containing $\text{graph}(\mu)$ with this property.

In order to study the local properties of linear and nonlinear systems, we can use a Floquet-like theorem to define a local Lyapunov function. The following theorem is similar to the classical Floquet Theorem, but does not require $A(t)$ to be periodic. Thus, $P(t)$ is not periodic either, but we can still show that $P(0) = P(T)$ holds.

Theorem 8. *Consider the nonautonomous linear system*

$$\dot{x} = A(t)x \tag{10}$$

where $A \in C(\mathbb{I}, \mathbb{R}^{n\times n})$. The principal solution, i.e. satisfying $\Phi(0) = I$, of (10) can be expressed in the form

$$\Phi(t) = P(t)e^{tL}$$

where $P(t)$ is continuously differentiable, $P(t) \in \mathbb{C}^{n\times n}$ is invertible for all $t \in \mathbb{I}$, $P(0) = P(T) = I$ and $L \in \mathbb{C}^{n\times n}$.

Proof. Define $M := \Phi^{-1}(0)\Phi(T)$. By Lemma 2, there is a matrix $L \in \mathbb{C}^{n\times n}$ such that $e^{TL} = M$. With $P(t) := \Phi(t)e^{-tL}$ we have

$$P(T) = \Phi(T)e^{-TL} = \Phi(T)M^{-1} = \Phi(0) = P(0) = I$$

using $\Phi(0) = I$, and $P(t)$ fulfills all the stated properties.

We will first reprove the characterisation of finite-time exponential stability which was given in [14] for the Euclidean norm, now for a general norm $\|\cdot\|$.

Theorem 9. *Denote by $F_T : \mathbb{R}^n \to \mathbb{R}^n$ the time–T map of (9), which is defined by $F_T(x) := \varphi(T, 0, x)$. Moreover, let $\mu : \mathbb{I} \to \mathbb{R}^n$, $\mu(t) = 0$ be a solution of (9). Then μ is exponentially attractive on $\mathbb{I}$ if and only if $\|DF_T(0)\| < 1$, where DF_T is the Jacobian of F_T with respect to x. The rate of exponential attraction is given by*

$$\frac{1}{T}\ln\|DF_T(0)\|.$$

If the principal solution $\Phi(t)$ of the linearised equation

$$\dot{x} = D_x f(t, 0)x$$

with $\Phi(0) = I$ is expressed $\Phi(t) = P(t)e^{tL}$ as in Theorem 8, we have $DF_T(0) = e^{TL}$.

In particular the zero solution μ is exponentially attractive on $\mathbb{I}$ if and only if $\|e^{TL}\| < 1$.

Proof. We consider $\mu(t) = 0$ and the solution $\varphi(t, 0, w)$ starting in $w \in \mathbb{R}^n$. Using Taylor's Theorem, we obtain

$$\varphi(T, 0, w) - 0 = F_T(w) - F_T(0) = DF_T(0)w + \psi(w),$$

where $\lim_{\|w\|\to 0} \frac{\psi(w)}{\|w\|} = 0$. Thus,

$$\limsup_{\|w\|\to 0} \frac{\|\varphi(T, 0, w)\|}{\|w\|} = \limsup_{\|w\|\to 0} \frac{\|DF_T(0)w\|}{\|w\|} = \|DF_T(0)\|. \tag{11}$$

$$\text{Now } \frac{1}{\eta} \sup_{\xi\in B_\eta(0)} \|\varphi(T, 0, \xi)\| = \sup_{\|w\|<\eta} \frac{\|\varphi(T, 0, w)\|}{\|w\|} \frac{\|w\|}{\eta}. \tag{12}$$

From (12) we can conclude

$$\frac{1}{\eta} \sup_{\xi\in B_\eta(0)} \|\varphi(T, 0, \xi)\| \leq \sup_{\|w\|<\eta} \frac{\|\varphi(T, 0, w)\|}{\|w\|}$$

which implies with (11) that

$$\limsup_{\eta\searrow 0}\frac{1}{\eta}\sup_{\xi\in B_\eta(0)}\|\varphi(T,0,\xi)\| \le \limsup_{\|w\|\to 0}\frac{\|\varphi(T,0,w)\|}{\|w\|} = \|DF_T(0)\|.$$

Furthermore, (12) and (11) yield for all fixed $\theta\in(0,1)$

$$\frac{1}{\eta}\sup_{\xi\in B_\eta(0)}\|\varphi(T,0,\xi)\| \ge \sup_{\|w\|=\theta\eta}\frac{\|\varphi(T,0,w)\|}{\|w\|}\theta$$

and

$$\limsup_{\eta\searrow 0}\frac{1}{\eta}\sup_{\xi\in B_\eta(0)}\|\varphi(T,0,\xi)\| \ge \limsup_{\|w\|\to 0}\frac{\|\varphi(T,0,w)\|}{\|w\|}\theta = \theta\|DF_T(0)\|.$$

Since this inequality holds for all $\theta\in(0,1)$, we have

$$\limsup_{\eta\searrow 0}\frac{1}{\eta}\sup_{\xi\in B_\eta(0)}\|\varphi(T,0,\xi)\| \ge \|DF_T(0)\|.$$

This shows $\limsup_{\eta\searrow 0}\frac{1}{\eta}\sup_{\xi\in B_\eta(0)}\|\varphi(T,0,\xi)\| = \|DF_T(0)\|$.

Furthermore, we can relate DF_T to the solution of the linearised equation. Denote by $F_t\colon\mathbb{R}^n\to\mathbb{R}^n$ the time–t map of (9), which is defined by $F_t(x) := \varphi(t,0,x)$. Then $\Phi(t,x) = DF_t(x)$ solves the first variation equation

$$\dot{\Phi}(t,x) = D_x f(t,\varphi(t,0,x))\Phi(t,x).$$

In particular, as $\mu(t)=0$ is a solution of (9), we obtain

$$\dot{\Phi}(t,0) = D_x f(t,0)\Phi(t,0) = A(t)\Phi(t,0)$$

with solution $\Phi(t,0) = P(t)e^{tL}$. Thus, $DF_T(0) = \Phi(T,0) = P(T)e^{TL} = e^{TL}$. Hence, the zero solution of the nonlinear equation is exponentially stable if and only if $\|e^{TL}\| < 1$.

4.1 Dini derivative

Due to the general norm $\|\cdot\|$, the assumption that a Lyapunov function $V(t,x)$ is C^1 is too restrictive. For example, for the system $\dot{x} = -x$, $x\in\mathbb{R}$ and $\|x\| := |x|$, the function $V(t,x) = |x|$ is not C^1 at 0, but it is a Lyapunov function in the sense that it is decreasing along trajectories. We will give a precise definition in Definition 5. We only assume that a Lyapunov function is continuous and locally Lipschitz in x, and we have to replace the orbital derivative by a weaker notion, the Dini derivative. Note that this can also be done in the autonomous and periodic case.

We define a finite-time Lyapunov function. The definition is similar to the periodic case, but V is fixed at times 0 and T by the norm.

Definition 5. A finite-time (strict) local Lyapunov function for the zero solution of the system (9) is a continuous function $V\colon \mathbb{I}\times U \to \mathbb{R}^n$, where $U\subset\mathbb{R}^n$ is an open neighborhood of 0, which satisfies the following properties:

1. $V(t,x)$ is locally Lipschitz in x.
2. $V(0,x)=\|x\|^p$ and $V(T,x)=\|x\|^p$ for all $x\in U$, where $p\geq 1$.
3. $V(t,x)>0$ for all $x\in U\setminus\{0\}$ and $V(t,0)=0$ for all $t\in\mathbb{I}$.
4. $V^+(t,x)<0$ for all $x\in U\setminus\{0\}$ and all $t\in\mathbb{I}\setminus\{T\}=[0,T)$.

Here the orbital derivative is defined by the Dini derivative

$$V^+(t,x)=\limsup_{h\searrow 0}\frac{V(t+h,x+h\cdot f(t,x))-V(t,x)}{h}. \tag{13}$$

If $U=\mathbb{R}^n$, then the function is a global finite-time Lyapunov function.

Remark 1. Locally Lipschitz in x in 1. is defined as follows: For every compact $C\subset\mathbb{I}\times U$ there exists a constant $L>0$, such that $|V(t,x)-V(t,y)|\leq L\|x-y\|$ for all $(t,x),(t,y)\in C$. It is needed to define the orbital derivative using $f(t,x)$ in 4. by (13) as shown on page 48 in [17]. Without this property

$$V^+(t,x)=\limsup_{h\searrow 0}\frac{V(t+h,\varphi(t+h,t,x))-V(t,x)}{h}$$

is **not** necessarily true and we would actually have to know the solution trajectories $t\mapsto\phi(t,0,x)$ to take the Dini derivative along orbits.

Remark 2. If V is differentiable along orbits, i.e. the limit

$$\lim_{h\to 0}\frac{V(t+h,\varphi(t+h,t,x))-V(t,x)}{h}$$

exists for all relevant t and x, then clearly $V^+(t,x)$ is equal to this limit. Note however, that this does not imply that $\nabla_x V(t,x)$ or $V_t(t,x)$ exist, e.g. consider the example at the beginning of Section 4.1.

Theorem 10. *Let V be a finite-time local Lyapunov function for the system (9). Then the zero solution of (9) is attractive and any compact set $V^{-1}([0,C])$ with $C>0$, contained in $\mathbb{I}\times U$ is a subset of the domain of attraction of the zero solution.*

Let $V(t,x)$ additionally fulfill: There exist constants $b\geq 1$ and $c>0$ such that

1. *$V(t,x)\leq b\|x\|^p$ for all $t\in\mathbb{I}$ and all $x\in U$, where p is the same as in Definition 5.*
2. *$V^+(t,x)\leq -c\|x\|^p$ for all $t\in[0,T)=\mathbb{I}\setminus\{T\}$ and all $x\in U$.*

Then the zero solution of (9) is exponentially attractive with rate of exponential attraction $\leq -c/(bp)$.

Proof. Since $x(t) = 0$ is a solution, there is an open neighborhood $U' \subset U$ of 0 such that $x \in U'$ implies $\varphi(t,0,x) \in U$ for all $t \in \mathbb{I}$.

Because V is locally Lipschitz it follows by [17] (page 48 and Corollary 3.10) that $t \mapsto V(t,\varphi(t,0,x))$ is a strictly decreasing function on $\mathbb{I}$, if $x \neq 0$. Hence,

$$\|x\|^p = V(0,x) > V(T,\varphi(T,0,x)) = \|\varphi(T,0,x)\|^p$$

for all $x \in U' \setminus \{0\}$ and the zero solution is attractive.

The same argument shows that $V^{-1}([0,C])$, if it is contained in $\mathbb{I} \times U$, is positively invariant. Now let $(t_0,x_0) \in V^{-1}([0,C])$ with $x_0 \neq 0$. Then either the trajectory $\varphi(t,t_0,x_0)$ stays in $V^{-1}([0,C])$ for all $t \in \mathbb{I}$, or there is a $\tau \in \mathbb{I}$ such that $\varphi(\tau,t_0,x_0) \notin V^{-1}([0,C])$. In the first case, (t_0,x_0) is in the domain of attraction by the fact that $t \mapsto V(t,\varphi(t,t_0,x_0))$ is a strictly decreasing function. In the second case, note that $\varphi(\tau,t_0,x_0) \notin V^{-1}([0,C])$ implies that $\varphi(0,t_0,x_0) \notin V^{-1}([0,C])$, since $V^{-1}([0,C])$ is positively invariant. Again, by the positive invariance, we have $\varphi(T,t_0,x_0) \in V^{-1}([0,C])$. Hence,

$$\|\varphi(T,t_0,x_0)\|^p = V(T,\varphi(T,t_0,x_0)) \leq C < V(0,\varphi(0,t_0,x_0)) = \|\varphi(0,t_0,x_0)\|^p$$

which shows that also in this case (t_0,x_0) lies in the domain of attraction. This proves the first claim of the theorem.

Now, assume that 1. and 2. are also fulfilled. Then V fulfills the Dini differential inequality

$$V^+(t,\varphi(t,0,x)) \leq -\frac{c}{b}V(t,\varphi(t,0,x))$$

and by taking Lemma 6.10 in [17] into consideration, we get $V(T,\varphi(T,0,x)) \leq V(0,x)e^{-cT/b}$ and thus $\|\varphi(T,0,x)\|^p \leq \|x\|^p e^{-cT/b}$ so for $x \neq 0$

$$\frac{\|\varphi(T,0,x)\|}{\|x\|} \leq e^{-cT/(bp)} < 1$$

and by Definition 4, 2., the zero solution is exponentially attractive with rate of exponential attraction $\leq -c/(bp)$.

Remark 3. Note the obvious error in the statement of Lemma 6.10 in [17]. Of course $L_{\mathscr{C}}$ is a local Lipschitz constant for s and not y as should be clear from the text and from the proof. y does not have to be locally Lipschitz. This is an important point for otherwise $V(t,x)$ would have to be locally Lipschitz in (t,x) and not only x.

Note that the Dini derivative does in general not obey the chain-rule. To see this consider e.g. $f(x) = |x|$, $g(x) = -x$ and $h(x) = (f \circ g)(x) = |-x|$. Then

$$h^+(0) = \limsup_{\eta \searrow 0} \frac{|-\eta| - |0|}{\eta} = 1 \text{ but}$$

$$f^+(g(0)) \cdot g^+(0) = \limsup_{\eta \searrow 0} \frac{|-0+\eta| - |-0|}{\eta} \cdot \limsup_{\eta \searrow 0} \frac{-\eta - (-0)}{\eta} = 1 \cdot (-1) = -1.$$

However, for our needs the following simple lemma suffices.

Lemma 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\limsup_{h\searrow 0} f(h) = S < 0$. Then there is a $\tau > 0$ such that $f(x) < 0$ for all $x \in (0,\tau)$. If $g : \mathbb{R} \to \mathbb{R}$ is a further function such that $\lim_{h\searrow 0} g(h) = L \neq 0$, then $\limsup_{h\searrow 0} f(h)g(h) = SL$.*

Proof. Assume there is no such $\tau > 0$. Then

$$\limsup_{h\searrow 0} f(h) = \lim_{h\searrow 0}[\sup\{f(x) \mid x \in (0,h)\}] \geq \lim_{h\searrow 0} 0 = 0,$$

which is a contradiction to $\limsup_{h\searrow 0} f(h) = S < 0$.

Now assume $L > 0$ and let $0 < \varepsilon < L/2$ be arbitrary. Then, for all $\tau > h > 0$ small enough, we have $0 < L-\varepsilon < g(h) < L+\varepsilon$, i.e. $(L+\varepsilon)f(h) \leq g(h)f(h) \leq (L-\varepsilon)f(h)$, and therefore

$$(L+\varepsilon)S \leq \limsup_{h\searrow 0} f(h)g(h) \leq (L-\varepsilon)S,$$

i.e. $\limsup_{h\searrow 0} f(h)g(h) = SL$ by lack of alternatives. The case $L < 0$ follows similarly.

4.2 Linear systems

Let us first focus on the linear case, i.e.

$$\dot{x} = A(t)x. \tag{14}$$

We can use the construction method in [14], which uses linear interpolation along a trajectory between the values at times 0 and T, to construct finite-time Lyapunov functions in the following two theorems.

Theorem 11. *Let the zero solution of*

$$\dot{x} = A(t)x, \tag{15}$$

where $A \in C(\mathbb{I}, \mathbb{R}^{n\times n})$, be exponentially stable.

Then there exists a finite-time Lyapunov function which satisfies

$$a_1\|x\|^2 \leq V(t,x) \leq b_1\|x\|^2, \qquad V^+(t,x) \leq -c_1\|x\|^2 \tag{16}$$

for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{I}\setminus\{T\}$, where $a_1, b_1, c_1 > 0$.

Proof. We define $V(t,\varphi(t,0,x))$ by linear interpolation of the values at time T and 0. Note that the principal fundamental solution $\Phi(t)$ (with $\Phi(0) = I$) of (15) can be, by Theorem 8, expressed in the form $\Phi(t) = P(t)e^{tL}$, where $P(t)$ is continuously differentiable, $P(t) \in \mathbb{C}^{n\times n}$ is invertible for all $t \in \mathbb{I}$, $P(0) = P(T) = I$ and $L \in \mathbb{C}^{n\times n}$. Hence, $\varphi(t_1,t_2,x) = \Phi(t_1)\Phi^{-1}(t_2)x = P(t_1)e^{L(t_1-t_2)}P^{-1}(t_2)x$. We define

$$V(t,x) := [\|\varphi(T,t,x)\|^2 - \|\varphi(0,t,x)\|^2]\frac{t}{T} + \|\varphi(0,t,x)\|^2 \tag{17}$$

$$= \frac{t}{T}\|\varphi(T,t,x)\|^2 + \left(1-\frac{t}{T}\right)\|\varphi(0,t,x)\|^2$$

$$= \frac{t}{T}\|e^{TL}\Phi^{-1}(t)x\|^2 + \left(1-\frac{t}{T}\right)\|\Phi^{-1}(t)x\|^2 \tag{18}$$

It is easy to see that $V(0,x) = V(T,x) = \|x\|^2$. Since $e^{TL} \neq 0$, $\Phi^{-1}(t)$ is non-singular and continuous for $t \in \mathbb{I}$ and the norm $\|\cdot\|$ is continuous, the mappings $(t,x) \mapsto \|e^{TL}\Phi^{-1}(t)x\|$ and $(t,x) \mapsto \|\Phi^{-1}(t)x\|$ are both continuous functions from the compact set $\mathbb{I}\times\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ into the real numbers. Hence, there are $a_1, b_1 > 0$ such that

$$a_1\|x\|^2 \le \|e^{TL}\Phi^{-1}(t)x\|^2 \le b_1\|x\|^2 \tag{19}$$

$$a_1\|x\|^2 \le \|\Phi^{-1}(t)x\|^2 \le b_1\|x\|^2 \tag{20}$$

for all $t \in \mathbb{I}$. Together with (18) this shows the first part of (16).

To show that $V(t,x)$ is locally Lipschitz in x let $\eta > 0$ be an arbitrary constant and $x,y \in \overline{B_\eta(0)}$ and $t \in \mathbb{I}$. By (18), (19) and (20) we get

$$\begin{aligned}
&|V(t,x) - V(t,y)| \\
&= \Big|\frac{t}{T}(\|e^{TL}\Phi^{-1}(t)x\| + \|e^{TL}\Phi^{-1}(t)y\|)(\|e^{TL}\Phi^{-1}(t)x\| - \|e^{TL}\Phi^{-1}(t)y\|) \\
&\quad + \left(1-\frac{t}{T}\right)(\|\Phi^{-1}(t)x\| + \|\Phi^{-1}(t)y\|)(\|\Phi^{-1}(t)x\| - \|\Phi^{-1}(t)y\|)\Big| \\
&\le \sqrt{b_1}(\|x\|+\|y\|)\left[\frac{t}{T}\|e^{TL}\Phi^{-1}(t)(x-y)\| + \left(1-\frac{t}{T}\right)\|\Phi^{-1}(t)(x-y)\|\right] \\
&\le \sqrt{b_1}(\|x\|+\|y\|)\left[\frac{t}{T}\sqrt{b_1}\|x-y\| + \left(1-\frac{t}{T}\right)\sqrt{b_1}\|x-y\|\right] \\
&= b_1(\|x\|+\|y\|)\|x-y\| \qquad (21) \\
&\le 2\eta b_1\|x-y\|,
\end{aligned}$$

which proves that $V(t,x)$ is locally Lipschitz in x.

Finally, we show the second part of (16). For every $(t_0,x_0) \in \mathbb{I}\times\mathbb{R}^n$ define the function $\psi_{(t_0,x_0)}(t) := V(t,\varphi(t,t_0,x_0))$ on $\mathbb{I}\setminus\{T\}$. We claim that for every $(t_0,x_0) \in \mathbb{I}\times\mathbb{R}^n$ the function $\psi_{(t_0,x_0)}(t)$ is differentiable with respect to t. To see this note that by (17) and the semigroup property $\varphi(t_1,t_2,\varphi(t_2,t_3,x)) = \varphi(t_1,t_3,x)$

$$\begin{aligned}
\psi_{(t_0,x_0)}(t) &= [\|\varphi(T,t,\varphi(t,t_0,x_0))\|^2 - \|\varphi(0,t,\varphi(t,t_0,x_0))\|^2]\frac{t}{T} + \|\varphi(0,t,\varphi(t,t_0,x_0))\|^2 \\
&= [\|\varphi(T,t_0,x_0)\|^2 - \|\varphi(0,t_0,x_0)\|^2]\frac{t}{T} + \|\varphi(0,t_0,x_0)\|^2
\end{aligned}$$

so that

$$\psi'_{(t_0,x_0)}(t) = [\|\varphi(T,t_0,x_0)\|^2 - \|\varphi(0,t_0,x_0)\|^2]\frac{1}{T}.$$

By Theorem 9 we have $\|e^{TL}\| =: \nu \in (0,1)$, since the zero solution is exponentially stable. Since $V(t,x)$ is locally Lipschitz in x we have by Remark 2, the product rule for differentiation, (17) and (20) that

$$\begin{aligned}
V^+(t_0,x_0) &= \limsup_{h\searrow 0} \frac{V(t_0+h,\varphi(t_0+h,t_0,x_0)) - V(t_0,x_0)}{h} \\
&= \limsup_{h\searrow 0} \frac{\psi_{(t_0,x_0)}(t_0+h) - \psi_{(t_0,x_0)}(t_0)}{h} \\
&= \psi'_{(t_0,x_0)}(t_0) \\
&= \frac{1}{T}\left(\|\varphi(T,t_0,x_0)\|^2 - \|\varphi(0,t_0,x_0)\|^2\right) \\
&= \frac{1}{T}\left(\|e^{TL}\Phi^{-1}(t_0)x_0\|^2 - \|\Phi^{-1}(t_0)x_0\|^2\right) \\
&\le \frac{\nu^2-1}{T}\|\Phi^{-1}(t_0)x_0\|^2 \\
&\le -\frac{1-\nu^2}{T}a_1\|x_0\|^2
\end{aligned}$$

Hence, with $c_1 := (1-\nu^2)a_1/T > 0$, the rest of (16) is shown.

We will show the existence of another Lyapunov function; note that this is particularly useful if one wants to approximate it by a continuous piecewise affine function. Indeed, the authors showed in [12] that such a function can be approximated by a continuous piecewise affine function so closely that the approximation is a true CPA Lyapunov function for autonomous systems, even in a neighborhood of the equilibrium.

Theorem 12. *Let the zero solution of the system (15) be exponentially stable. Then there exists a finite-time Lyapunov function $W(t,x)$ which satisfies*

$$a_2\|x\| \le W(t,x) \le b_2\|x\|, \qquad W^+(t,x) \le -c_2\|x\| \tag{22}$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{I}\setminus\{T\}$, where $a_2, b_2, c_2 > 0$. W is globally Lipschitz in x.

Proof. We define $W(t,x) := \sqrt{V(t,x)}$, where $V(t,x)$ is the function from Theorem 11, and notice that immediately from (16) $\sqrt{a_1}\|x\| \le W(t,x) \le \sqrt{b_1}\|x\|$ follows. It is easy to see that $W(0,x) = W(T,x) = \|x\|$. To show the second part of (22) fix $t \in [0,T)$. The case $x = 0$ follows from

$$\frac{\sqrt{V(t+h,0+h\cdot A(t)0)} - \sqrt{V(t,0)}}{h} = \frac{0-0}{h} = 0,$$

i.e. $W^+(t,0) = 0$. If $x \neq 0$ we have by Lemma 3

$$\begin{aligned} W^+(t,x) &= \limsup_{h\searrow 0} \frac{\sqrt{V(t+h,x+h\cdot A(t)x)}-\sqrt{V(t,x)}}{h} \\ &= \limsup_{h\searrow 0} \frac{V(t+h,x+h\cdot A(t)x)-V(t,x)}{h\cdot(\sqrt{V(t,x+h\cdot A(t)x)}+\sqrt{V(t,x)})} \\ &\leq \frac{-c_1\|x\|^2}{2\sqrt{V(t,x)}} \leq -\frac{c_1}{2\sqrt{b_1}}\|x\|. \end{aligned}$$

It remains to show that $W(t,x)$ is globally Lipschitz in x. The case $x=y=0$ is trivial and otherwise, by (21) and (16), we have

$$\begin{aligned} |W(t,x)-W(t,y)| &= \left|\sqrt{V(t,x)}-\sqrt{V(t,y)}\right| = \frac{|V(t,x)-V(t,y)|}{\sqrt{V(t,x)}+\sqrt{V(t,y)}} \\ &\leq \frac{b_1(\|x\|+\|y\|)}{\sqrt{a_1}\cdot(\|x\|+\|y\|)}\cdot\|x-y\| \leq \frac{b_1}{\sqrt{a_1}}\cdot\|x-y\|. \end{aligned}$$

Remark 4. A different function W with the properties as in Theorem 12 is

$$W_2(t,x) := [\|\varphi(T,t,x)\|-\|\varphi(0,t,x)\|]\frac{t}{T}+\|\varphi(0,t,x)\|.$$

One can prove the properties, following the proof of Theorem 12, dropping the squares.

We give an example that these two definitions lead to two different functions W_1 and W_2. Consider $\dot{x}=-x$ on the interval $[0,1]$ with the Euclidean norm. Then

$$\begin{aligned} V(t,x) &= \left(te^{2(t-1)}+(1-t)e^{2t}\right)x^2, \\ W_1(t,x) &= \sqrt{te^{2(t-1)}+(1-t)e^{2t}}|x|, \\ W_2(t,x) &= \left(te^{t-1}+(1-t)e^t\right)|x|. \end{aligned}$$

4.3 Nonlinear systems

Now we consider the nonlinear system

$$\dot{x} = f(t,x) \tag{23}$$

over the finite-time interval $\mathbb{I}=[0,T]$ and show the existence of local finite-time Lyapunov functions.

Theorem 13. *Consider the nonlinear system*

$$\dot{x} = f(t,x) \tag{24}$$

where $f \in C^1([0,T]\times\mathbb{R}^n,\mathbb{R}^n)$*,* $f(t,0)=0$ *for all* $t\in[0,T]$ *over the finite-time interval* $\mathbb{I}=[0,T]$*. Define* $A(t):=D_x f(t,0)$*.*

Consider (15) with the same $A(t)$*. Then the Lyapunov functions* V *and* W *in Theorems 11 and 12 respectively are also finite-time Lyapunov functions for (24) satisfying* $V^+(t,x)\le -c_v\|x\|^2$ *and* $W^+(t,x)\le -c_w\|x\|$*,* $c_v,c_w>0$*, for all* $t\in\mathbb{I}\setminus\{T\}$ *and all* x *in some open neighborhood* $U\subset\mathbb{R}^n$ *of* 0*.*

Proof. It suffices to show that $V^+(t,x)\le -c_v\|x\|^2$ and $W^+(t,x)\le -c_w\|x\|$ for all $t\in\mathbb{I}\setminus\{T\}$ and all x in some open neighborhood $U\subset\mathbb{R}^n$ of 0.

We first show $W^+(t,x)\le -c_w\|x\|$. By Taylor's Theorem we can write $f(t,x)=A(t)x+\psi(t,x)$, where for all $\varepsilon>0$ there is a $\eta>0$ such that $\|\psi(t,x)\|\le\varepsilon\|x\|$ holds for all $\|x\|<\eta$ and all $t\in\mathbb{I}$, cf. the proof of Theorem 7. Because $W(t,x)$ is globally Lipschitz in x by Theorem 12, there is a constant $L>0$ such that $|W(t,x)-W(t,y)|\le L\|x-y\|$ for all $t\in\mathbb{I}$ and $x,y\in\mathbb{R}^n$. Hence by (22)

$$
\begin{aligned}
W^+(t,x) &= \limsup_{h\searrow 0}\frac{W(t+h,x+h\cdot f(t,x))-W(t,x)}{h}\\
&\le \limsup_{h\searrow 0}\frac{W(t+h,x+h\cdot[A(t)x+\psi(t,x)])-W(t+h,x+h\cdot A(t)x)}{h}\\
&\qquad+\limsup_{h\searrow 0}\frac{W(t+h,x+h\cdot A(t)x)-W(t,x)}{h}\\
&\le \limsup_{h\searrow 0}\frac{L\|h\psi(t,x)\|}{h}-c_2\|x\|\\
&\le L\|\psi(t,x)\|-c_2\|x\|.
\end{aligned}
$$

With $\varepsilon:=c_2/(2L)$ and $c_w:=c_2/2$ it follows that there exists an $\eta>0$ such that $W^+(t,x)\le -c_w\|x\|$ for all $t\in\mathbb{I}\setminus\{T\}$ and all $x\in B_\eta(0)=:U$.

Now consider $V(t,x)=[W(t,x)]^2$. The case $x=0$ is trivial. For $x\neq 0$ we have by Lemma 3 and the above estimate

$$
\begin{aligned}
V^+(t,x) &= \limsup_{h\searrow 0}\frac{W^2(t+h,x+h\cdot f(t,x))-W^2(t,x)}{h}\\
&\le \limsup_{h\searrow 0}[W(t+h,x+h\cdot f(t,x))+W(t,x)]\frac{W(t+h,x+h\cdot f(t,x))-W(t,x)}{h}\\
&= 2W(t,x)\cdot[-c_w\|x\|]\le -a_2c_w\|x\|^2
\end{aligned}
$$

for all $t\in\mathbb{I}\setminus\{T\}$ and all $x\in B_\eta(0)=:U$. That is $V^+(t,x)\le -c_v\|x\|^2$ with $c_v=a_2c_w>0$.

4.4 Norm $\|x\|^2 = x^T N x$

In this section we restrict ourselves to the class of norms $\|x\|^2 = x^T N x$, where $N \in \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix.

In the following Theorem 14 we consider a nonlinear system and give a sufficient condition for the exponential stability of the zero solution. The construction of the Lyapunov function is similar to the periodic-time case. Note that the assumptions of Theorem 14 are sufficient, but not necessary for the exponential attraction of the zero solution, see Theorem 15, 2.

Theorem 14. *Consider*

$$\dot{x} = f(t,x), \tag{25}$$

where $f \in C^1([0,T]\times\mathbb{R}^n,\mathbb{R}^n)$*,* $f(t,0) = 0$ *for all* $t \in [0,T]$ *over the finite-time interval* $\mathbb{I} = [0,T]$*. Define* $A(t) := D_x f(t,0)$*. Let* L *be defined as in Theorem 8.*

Let the norm $\|\cdot\|$ *be defined by*

$$\|x\|^2 = x^T N x,$$

where $N \in \mathbb{R}^{n\times n}$ *is a symmetric, positive definite matrix.*

If the Hermitian matrix $L^*N + NL$ *is Hurwitz, then the zero solution of (25) is exponentially stable. In this case,*

$$V(t,x) := x^T (P^{-1}(t))^* N P^{-1}(t) x \text{ and } W(t,x) = \sqrt{V(t,x)}$$

are finite-time local Lyapunov function, satisfying

$$\begin{aligned} a_1\|x\|^2 \le V(t,x) \le b_1\|x\|^2, &\qquad V'(t,x) \le -c_1\|x\|^2, \\ a_2\|x\| \le W(t,x) \le b_2\|x\|, &\qquad W'(t,x) \le -c_2\|x\|, \end{aligned}$$

for all $x \in U\setminus\{0\}$ *and* $t \in \mathbb{I}\setminus\{T\}$*, where* U *is an open neighborhood of* 0*, with constants* $a_1,b_1,c_1,a_2,b_2,c_2 > 0$*.*

Proof. Using Theorem 8, we express the fundamental matrix solution with initial condition $\Phi(0) = I$ by

$$\Phi(t) = P(t)e^{tL}$$

where $P(0) = P(T) = I$ and $P(t), L \in \mathbb{C}^{n\times n}$. The Hermetian matrix $(L^*N + NL)$ is negative definite. Denote the maximal eigenvalue by $-\nu < 0$, which gives us

$$z^T(L^*N + NL)z \le -\nu\|z\|_2^2 \tag{26}$$

for all $z \in \mathbb{C}^n$.

We define the functions V and W as in the theorem. The inequalities for $V(t,x)$ and $W(t,x)$ follow from the fact that $P^{-1}(t)$ is non-singular and N is positive definite. As $P(0) = P(T) = I$, we have $V(0,x) = V(T,x) = \|x\|^2$ and $W(0,x) = W(T,x) = \|x\|$.

Now we show the inequality for $V'(t,x)$. We us $\dot{P}^{-1}(t) = LP^{-1}(t) - P^{-1}(t)A(t)$, which is shown as in the periodic case (see (7) in the proof of Theorem 6) and $(A(t))^* = (A(t))^T$ since $A(t) \in \mathbb{R}^{n\times n}$. Furthermore, by Taylor $f(t,x) = A(t)x + \psi(x)$, where for $\varepsilon := \frac{\nu}{2\|N\|_2}$ there is $r > 0$ such that $\frac{\|P^{-1}(t)\psi(t,x)\|_2}{\|P^{-1}(t)x\|_2} < \varepsilon$ for all $t \in \mathbb{I}$ and $\|x\|_2 < r$. Hence, we obtain

$$\begin{aligned}
V'(t,x) &= x^T(A(t))^T(P^{-1}(t))^*NP^{-1}(t)x + (\psi(t,x))^*(P^{-1}(t))^*NP^{-1}(t)x \\
&\quad + x^T(\dot{P}^{-1}(t))^*NP^{-1}(t)x + x^T(P^{-1}(t))^*N\dot{P}^{-1}(t)x \\
&\quad + x^T(P^{-1}(t))^*NP^{-1}(t)A(t)x + x^T(P^{-1}(t))^*NP^{-1}(t)\psi(t,x) \\
&= x^T[(A(t))^T(P^{-1}(t))^*N + (P^{-1}(t))^*L^*N - (A(t))^*(P^{-1}(t))^*N]P^{-1}(t)x \\
&\quad + x^T(P^{-1}(t))^*[NLP^{-1}(t) - NP^{-1}(t)A(t) + NP^{-1}(t)A(t)]x \\
&\quad + 2\|P^{-1}(t)x\|_2\|N\|_2\|P^{-1}(t)\psi(t,x)\|_2 \\
&= ((P^{-1}(t))x)^*[L^*N + NL]P^{-1}(t)x + \frac{\nu}{2}\|P^{-1}(t)x\|_2^2 \\
&\leq -\frac{\nu}{2}\|P^{-1}(t)x\|_2^2 \\
&\leq -c_1\|x\|^2
\end{aligned}$$

for a suitable $c_1 > 0$ and for all $t \in \mathbb{I}$ and $0 < \|x\|_2 < r$ by (26). The proof for W follows as in Theorem 6.

5 Relations between autonomous, periodic and finite-time systems

5.1 Periodic systems as finite-time systems

If we consider a time-periodic system

$$\dot{x} = f(t,x),$$

then we can also regard this system as a finite-time system. We discuss the stability of the zero solution with respect to the different notions.

Theorem 15. *Consider a T-periodic system $\dot{x} = f(t,x)$ with $f \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) = 0$ for all $t \in \mathbb{R}$. We can also consider the system as a finite-time system over the interval $\mathbb{I} = [0,T]$.*

If the zero solution is exponentially stable with respect to the finite-time case, then it is exponentially stable with respect to the periodic-time case.

By Theorems 4 and 8 there is a matrix $L \in \mathbb{C}^{n\times n}$ such that the principal solution of the linearised equation with $\Phi(0) = I$ can be expressed as $\Phi(t) = P(t)e^{tL}$, where $P(0) = P(T) = I$. Now the following statements hold true for L:

1. *Let $\|\cdot\|$ be a arbitrary norm on $\mathbb{R}^n$. If $\|e^{TL}\| < 1$, then L is Hurwitz.*
2. *Let $N \in \mathbb{R}^{n\times n}$ be a symmetric, positive definite matrix and let $\|\cdot\|$ be the induced matrix norm corresponding to the vector norm $\|x\|^2 = x^T Nx$.*
   If $L^*N + NL$ is Hurwitz, then $\|e^{TL}\| < 1$ for all $T > 0$.

Proof. Assume that the zero solution is exponentially stable with respect to the finite-time case. Then, by Theorem 9 we have $\|DF_T(0)\| = \mu \in (0,1)$. Since $F_T(x) = DF_T(0)x + \psi(x)$ with a function $\psi(x) = o(\|x\|)$ as $x \to 0$, there exists $\eta > 0$ such that $\|\psi(x)\| \le \frac{1-\mu}{2}\|x\|$ for all $x \in B_\eta(0)$. Thus,

$$\|F_T(x)\| \le \|DF_T(0)\|\|x\| + \|\psi(x)\| \le \mu\|x\| + \frac{1-\mu}{2}\|x\| = \frac{1+\mu}{2}\|x\|.$$

Denoting $\nu := \frac{1+\mu}{2} \in (0,1)$, since $\mu \in (0,1)$, we have now

$$\|F_T(x)\| \le \nu\|x\|$$

for all $x \in B_\eta(x)$. This is also the Poincaré map $P\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \to \varphi(T,0,x)$ of the periodic system and thus

$$\|P^k x\| \le \nu^k \|x\|$$

for all $k \in \mathbb{N}$, which implies the exponential stability with respect to the periodic system as $\nu \in (0,1)$. Part 2. is a direct consequence of Theorems 14, 7 and 9.

5.2 *Autonomous systems as periodic and finite-time systems*

An autonomous system

$$\dot{x} = f(x)$$

can be considered as a periodic system with any period T, and also on a finite-time interval $[0,T]$ with any $T > 0$. We discuss the relations between the different notions of attractivity for such a system. We start with an example, and we later prove a general theorem.

Example 1. Consider the linear system with $f(x) = Ax$, where $A := \begin{pmatrix} -1 & c \\ 0 & -1 \end{pmatrix}$ with $c \in \mathbb{R}$ and $x \in \mathbb{R}^2$. Then the principal solution can be written as $\Phi(t) = e^{tL}$, where in the previous notation we have $P(t) = I$ and $L = \begin{pmatrix} -1 & c \\ 0 & -1 \end{pmatrix}$. The eigenvalues of L are -1 and have negative real part. Thus, as an autonomous example, the origin is exponentially asymptotically stable for all $c \in \mathbb{R}$ and so is the zero solution if we regard it as a periodic system.

Now we consider the system as a finite-time system on the interval $\mathbb{I} = [0,T]$ with the Euclidean norm $\|x\|^2 = x^T x$, i.e. $N = I$. In this case, $\|e^{TL}\|^2$ is given by the maximal eigenvalue of $e^{TL^*}e^{TL}$. We have

$$e^{TL} = e^{-T}\begin{pmatrix} 1 & Tc \\ 0 & 1 \end{pmatrix}$$

and thus $e^{TL^*}e^{TL} = e^{-2T}\begin{pmatrix} 1 & Tc \\ Tc & T^2c^2+1 \end{pmatrix}.$

The eigenvalues are

$$\lambda_{1,2} = e^{-2T}\left(1+\frac{T^2c^2}{2} \pm \sqrt{T^2c^2+\frac{T^4c^4}{4}}\right).$$

Both eigenvalues are < 1 if and only if

$$|c| < 2\frac{\sinh T}{T} =: c^*(T).$$

Note that $\lim_{T\to 0} c^*(T) = 2$ and $\lim_{T\to\infty} c^*(T) = \infty$. Hence, depending on the finite-time interval $[0,T]$ under consideration, the zero solution is exponentially asymptotically stable, if and only if $|c| < c^*(T)$. As $T \to \infty$, the zero solution is exponentially attractive for all c.

Now we consider the condition that L^*N+NL is Hurwitz of Theorem 14, which is sufficient for the exponential attractivity. In this example, $L^*N+NL = L^*+L = \begin{pmatrix} -2 & c \\ c & -2 \end{pmatrix}$. The eigenvalues are $\mu_{1,2} = -2 \pm c$, and they are both negative if $|c| < 2$. Hence, the condition that the zero solution is finite-time attractive **for all** $T > 0$ ($|c| \leq 2$) is nearly equivalent to L^*N+NL being Hurwitz ($|c| < 2$).

We can prove the following general lemma.

Lemma 4. *Consider the autonomous system*

$$\dot{x} = f(x) \tag{27}$$

with $f \in C^1(\mathbb{R}^n,\mathbb{R}^n)$, $f(0) = 0$ and $Df(0) =: A \in \mathbb{R}^{n\times n}$.

The zero solution of this T-periodic system for any $T > 0$ is exponentially stable if and only if the equilibrium 0 is exponentially stable for the autonomous system.

If the zero solution is finite-time exponentially attractive for a $T > 0$, then it is exponentially stable both for the autonomous and periodic system.

Now consider the norm $\|x\|^2 = x^T Nx$ with symmetric positive definite matrix $N \in \mathbb{R}^{n\times n}$. Then we have the following implications

$$(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv).$$

(i) All eigenvalues λ of A^TN+NA satisfy $\lambda < 0$.
(ii) The zero solution of (27) is exponentially stable over the finite-time interval $[0,T]$ for all $T > 0$.
(iii) For all $T > 0$, $\|e^{TA}\| < 1$.
(iv) All eigenvalues λ of A^TN+NA satisfy $\lambda \leq 0$.

Proof. The first parts follows from the fact that $L = A$ and Theorems 3 and 7 as well as Theorem 15.

For the second part, note that (i)⇒(ii) follows from Theorem 14 and (ii)⇔(iii) follows from Theorem 9. It is left to show (iii)⇒(iv).

Let us assume that for all $T > 0$, we have $\|e^{TA}\| < 1$ and, in contradiction to the statement, that there is an eigenvalue $\lambda > 0$ and $v \in \mathbb{R}^n \setminus \{0\}$ such that

$$(A^T N + NA)v = \lambda v.$$

Since $\|e^{TA}\| < 1$, we have

$$\begin{aligned}
\|v\|^2 &> v^T e^{TA^T} N e^{TA} v \\
&= v^T \left(I + TA^T + \frac{1}{2}T^2(A^T)^2 + \dots \right) N \left(I + TA + \frac{1}{2}T^2A^2 + \dots \right) v \\
&= v^T \left(N + T(A^T N + NA) + \varphi(T) \right) v \\
&= \|v\|^2 + \|v\|_2^2 (T\lambda + \varphi(T))
\end{aligned}$$

where $\varphi(T) = o(T)$ as $T \to 0$. Hence, there is a $T > 0$ such that $|\varphi(T)| \le T\lambda/2$ and $0 > \|v\|_2^2 T\lambda/2 > 0$ due to $\lambda > 0$, which is a contradiction.

6 Conclusions and Outlook

In this article, we have generalised the construction of local Lyapunov functions for general nonlinear systems to periodic-time and finite-time systems. As in the classical autonomous case, we have constructed two types of Lyapunov functions V and W, satisfying

$$\begin{aligned}
a_1\|x\|^2 \le V(t,x) \le b_1\|x\|^2, &\qquad V'(t,x) \le -c_1\|x\|^2, \\
a_2\|x\| \le W(t,x) \le b_2\|x\|, &\qquad W'(t,x) \le -c_2\|x\|.
\end{aligned}$$

They are global Lyapunov functions for linear systems, and local Lyapunov functions for nonlinear ones.

Although we give explicit formulas for V and W, we are using the Floquet representation of solutions, so that in explicit examples their calculation requires the solution of the first variation equation. The practical use of the results, besides the theoretical existence, is to derive an algorithm for the construction of CPA Lyapunov functions for periodic and finite-time systems, where the results of this paper will be important to close the gap between the local and the global part of the Lyapunov function. We envisage that, as in the autonomous case [11, 13], where we have used similar results to show the existence and to algorithmically construct a CPA Lyapunov function, we can use the results of this paper for similar algorithms in the time-periodic and finite-time cases.

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