# ON THE NUMBER OF SQUARE-FREE PERMUTATIONS

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Abstract. A permutation is square-free if it does not contain two consecutive factors of length more than one that coincide in the reduced form (as patterns). We prove that the number of square-free permutations of length $n$ is $n^{n(1-\varepsilon_n)}$ where $\varepsilon_n \to 0$ when $n \to \infty$.

**Keywords:** square freeness, consecutive pattern, enumeration, permutation

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## 1. Introduction

A *square* in a word is a classical concept in combinatorics on words meaning two equal consecutive factors in the word. For example, the word 213413413 contains the square 134134, whereas the word 2141231 is square-free. It was first established by Thue [6] that there are arbitrary long square-free words over 3 (or more) letter alphabets, whereas it is easy to see that square-free words over 2 and 1 letter alphabets are of length at most 3 and 1, respectively. A question on the number of different square-free words of length $n$ is rather complicated: for example, for 3 letter alphabets, it is shown [4, 5] that the number of such words is $3^{cn(1-\epsilon_n)}$ where $1.30173\ldots < c < 1.30178\ldots$ and $\epsilon_n \to 0$ when $n \to \infty$.

In this paper, we extend, in a natural way, the notion of a square in a word to that in a permutation (see the definition below). We provide a construction of a class of arbitrary long square-free permutations and use it to find asymptotically the number of square-free permutations.

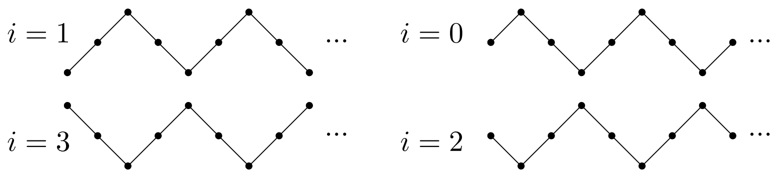
## 2. Definitions, notations and the main construction

The *reduced form* of a permutation is obtained by substituting the $i$-th largest entrance of it by $i$. For example, the reduced form of 2648 is 1324. For the last example, we also say that the permutation 2648 forms the *pattern* 1324. A permutation is *square-free* if it does not contain two consecutive factors of length more than one that are equal in the reduced form (as patterns). For example, the permutation 246153 contains the square 4615 (in the reduced form the first and the last two letters form the pattern 12), whereas the permutation 246513 is square-free.

It is easy to see that in order to avoid squares of length 2, a permutation $\pi$ must have such an index $i \in \{0, 1, 2, 3\}$ so that for every non-negative integer $t$ the inequalities

$$\pi_{i+4t} \leq \pi_{i+4t\pm 1} \text{ and } \pi_{i+4t+2} \geq \pi_{i+4t\pm 1} \qquad (1)$$

hold. Schematically, the four possible kinds of permutations (according to the choice of $i$) can be shown as follows:



Here a dot represents an element and the order of elements represented by two non-consecutive dots is irrelevant, whereas each pair of consecutive dots is comparable (a lower dot represents a smaller number).

If $\pi$ is a permutation satisfying (1) then we say that the elements with the indices $4t+i$ form the *lower* level, the elements with the indices $4t+i\pm 1$ form the *medium* level, and the elements with the indices $4t+i+2$ form the *upper* level. For example, the square-free permutation 1574236 has 1 and 2 on the lower level, 7 and 6 on the upper level, and 3, 4, and 5 on the medium level. We denote the permutation induced by the medium level of the permutation $\pi$ by $\pi'$.

The key thing in what follows is the following construction of a square-free permutation of length $n$. Choose $i \in \{0,1,2,3\}$ and let $k \approx \lfloor n/4 \rfloor, \ell \approx \lfloor 3n/4 \rfloor$ (the exact values of $k$ and $\ell$ depend on the parities of $i$ and $n$). Take any square-free permutation $\pi'$ over the elements $k, k+1, \dots, \ell$, place it on the medium level of the permutation $\pi$ to be created. Afterwards, fill in the upper (respectively lower) level with an arbitrary permutation over the elements $\ell+1, \ell+2, \dots, n$ (respectively $1, 2, \dots, k-1$) according to the choice of $i$ and satisfying (1).

For example, using the permutation 4356 for the medium level, we can create, for instance, the following square-free permutations using the construction above: 42375168, 841375269, 14832576, etc.

**Lemma 1.** *The construction above is valid, that is, $\pi$ does not contain any squares.*

*Proof.* Assume that $\pi$ has a square $W_1W_2$ where $W_1$ and $W_2$ coincide as patterns. Since $\pi$ satisfies (1), the length of $W_j, j = 1, 2$ is at least 4. In particular, these patterns must satisfy (1) and have the same $i$, i. e. the same lower, medium, and upper levels. Moreover, the patterns $W_1'$ and $W_2'$, induced by the corresponding medium levels must also coincide, contradicting the assumption that $\pi'$ is square-free. □

## 3. The main result

Our main result is the following

**Theorem 2.** *The number $f(n)$ of square-free permutations of length $n$ is $n^{n(1-\varepsilon_n)}$ where $\varepsilon_n \to 0$ when $n \to \infty$.*

*Proof.* As we know from Stirling's formula, $n! = (n/e)^n o(n)$ when $n \to \infty$. Therefore, $f(n) \leq n! \leq n^{n(1-\varepsilon_n)}$ for some $\varepsilon_n \to 0$ when $n \to \infty$.

To obtain a lower bound on $f(n)$ we use the construction presented in the previous section. Since we deal with asymptotic enumeration, we can assume that $n$ is a power of 2, the medium level contains $n/2$ elements while each of the remaining two levels contains $n/4$ elements. Now, the medium level can be occupied by an arbitrary square-free permutation of length $n/2$, whereas on the lower and upper levels there can be any permutations of length $n/4$, which leads to

$$f(n) \geq ((n/2^2)!)^2 f(n/2) \geq \cdots \geq \prod_{i=1}^{\log n} ((n/2^{i+1})!)^2$$

where we used the fact that $f(1) = 1$.

By Stirling's formula,

$$((n/2^{i+1})!)^2 \geq (n/(e2^{i+1}))^{2n/2^{i+1}} \geq (n/2^{i+3})^{n/2^i} = 2^{(\log n - i - 3)(n/2^i)}.$$

So,

$$f(n) \geq \prod_{i=1}^{\log n} 2^{((n \log n)/2^i - n(i+3)/2^i)} = 2^{\sum_{i=1}^{\log n}((n \log n)/2^i - n(i+3)/2^i)}.$$

Clearly, $\sum_{i=1}^{\log n}(1/2^i) = (n-1)/n$ and

$$\sum_{i=1}^{\log n}((i+3)/2^i) \leq \sum_{i=0}^{\infty}((i+3)/2^i) = 5.$$

Hence,

$$f(n) \geq 2^{n \log n - \log n - 5n} = 2^{(n \log n - o(n \log n))} = n^{n(1-\varepsilon_n)}$$

for some $\varepsilon_n \to 0$ when $n \to \infty$. □

## 4. Concluding remarks

A direction of possible research is to find an upper and lower bounds for the function $\epsilon_n$ appearing in Theorem 2. For a lower bound, it would be of help to discover new (rich) classes of square-free permutations (we do not expect a full characterization to be obtained). For example, one can take any permutation obtained by the construction above with the property that the minimum element $m$ on the medium level has a neighbor $a$ less than it; then one can substitute $a$ with 1, $m$ with 2, each element $i$, $i < a$ in the original permutation with $i + 2$ and each element $i$, $a < i < m$ in the original permutation with $i + 1$ keeping elements larger than $m$ unchanged. The resulting permutation will be square-free and in most of the cases not from the class of square-free permutations considered in this paper.

For obtaining an upper bound, it is helpful to realize an intimate connection to permutation pattern avoidance theory. Indeed, if a permutation satisfies (1) it must avoid 12 *consecutive* (also known as *segmented*) *patterns* of length 4: 1234, 4321, 2143, 3412, 3142, 2413, 4231, 1324, 4132, 2314, 1423, and 3241. In fact, avoiding these 12 patterns is the same as avoiding two *partially ordered consecutive patterns* $121'2'$ and $2'1'21$ where the only relations between the patterns letters are $1 < 2$ and $1' < 2'$ (see [3] to learn more on partially ordered patterns). In any case, several results and approaches are known on avoidance of consecutive patterns of length 4 (e.g., see [1, 2]), which can be used for obtaining a rough upper bound in question (instead of prohibiting 12 patterns, one can start with prohibiting 1, 2, etc, patterns out of them), and then an upper bound could be improved by including more patterns to avoid (out of the 12 patterns). It is conceivable that a direct enumeration of permutations satisfying (1) can be done without working with length 4 patterns to be avoided.

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