

Crucial words for abelian powers

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Outline

- 1 Background
 - Repetitions & patterns in words
 - Crucial words & abelian powers
- 2 Minimal crucial words avoiding abelian cubes
 - Upper bound for length
 - Lower bound for length
- 3 Minimal crucial words avoiding abelian k -th powers
 - Upper bound for length
 - Lower bound for length
- 4 Further research

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- That is:

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gives (in the limit) the infinite word

$$cbacabcbabcacbacabcbabcacbabcbacabca\dots$$

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- Now called the *Thue-Morse word* as it was rediscovered by **Morse in 1921** (in the context of symbolic dynamics).

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*Does there exist an infinite word over a fixed finite alphabet containing no **abelian squares**, i.e., avoiding factors of the form XX' where X' is a permutation of X ?*
- **Answer: YES**. Existence was established for alphabets of size:
 - **25** and improved to **7** (A. Evdokimov, 1968 & 1971);
 - **5** (P.A.B. Pleasants, 1970);
 - **4** (Keränen, 1992), the **optimal result** (such a word does not exist over a 3-letter alphabet).

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- **Examples:**

- V contains the abelian square **43232 32324**.
- **123 312 213** is an abelian cube.

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In fact: W is a *minimal crucial word* over $\{1, 2\}$ with respect to abelian cubes.

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- The *k-generalised Zimin word* $Z_n^k = X_n$ is defined as

$$X_1 = 1^{k-1} = 11 \dots 1, \quad X_n = (X_{n-1}n)^{k-1}X_{n-1} = X_{n-1}nX_{n-1}n \dots nX_{n-1}$$

where the number of 1's, as well as the number of n 's, is $k - 1$.

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- Z_n^k gives the length of a minimal crucial word avoiding k -th powers.

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- Z_n^k is crucial with respect to (abelian) k -th powers.
- Z_n^k has length $k^n - 1$.
- Z_n^k gives the length of a minimal crucial word avoiding k -th powers.

Much less is known in the case of abelian k -th powers ...

Minimal crucial words avoiding abelian powers

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- Now we extend the study of crucial abelian k -power-free words to the case of $k > 2$.
 - We provide a complete solution to the problem of determining the length of a minimal crucial abelian cube-free word (the case $k = 3$).
 - And we conjecture a solution in the general case.
- Let $\ell_k(n)$ denote the length of a minimal crucial word over \mathcal{A}_n avoiding abelian k -th powers.

Outline

- 1 Background
 - Repetitions & patterns in words
 - Crucial words & abelian powers
- 2 Minimal crucial words avoiding abelian cubes
 - Upper bound for length
 - Lower bound for length
- 3 Minimal crucial words avoiding abelian k -th powers
 - Upper bound for length
 - Lower bound for length
- 4 Further research

Upper bound for $\ell_3(n)$

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Sketch Proof: “Greedy” construction of a crucial abelian cube-free word $X = X_n$ over \mathcal{A}_n , defined recursively as follows:

$$X_1 = 11 \quad \text{and} \quad X_n = \phi_1(\sigma(X_{n-1}))1 \quad \text{for } n \geq 2,$$

where $\sigma : x \mapsto x + 1$ and $\phi_1 : x \mapsto x1$ for all letters x .

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That is: Set $X_1 = 11$ and assume X_{n-1} has been constructed. Then:

- 1 Increase all letters of X_{n-1} by 1 to obtain X'_{n-1} .
- 2 Insert the letter 1 to the right of each letter of X'_{n-1} and adjoin one extra 1 to the right of the resulting word to obtain X_n .

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- $|X_n| = 3 \cdot 2^{n-1} - 1$.
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- X_n is crucial with respect to abelian cubes: X_n avoids abelian cubes, whereas $X_n i$ ends with an abelian cube for each $i \in \mathcal{A}_n$ (by induction).
- Hence $\ell_3(n) \leq 3 \cdot 2^{n-1} - 1$.

An optimal construction

- Let X be a crucial word over \mathcal{A}_n with respect to abelian k -th powers.

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- If X is **minimal**, we may assume w.l.o.g. that Xn is an **abelian k -th power**, and we write:

$$X = \Omega_{n,1}\Omega_{n,2}\cdots\Omega'_{n,k}$$

where $\Omega_{n,k} = \Omega'_{n,k}n$ and the k **blocks** $\Omega_{n,j}$ are equal up to permutation.

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A construction of crucial abelian cube-free words over \mathcal{A}_n for $n \geq 4$:

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A construction of crucial abelian cube-free words over \mathcal{A}_n for $n \geq 4$:

- Basis:** Minimal crucial abelian square-free words $W_n = W_{n,2}$ given by **Evdokimov & Kitaev (2004)**. For $n = 4, 5, 6, 7$:

$$W_{4,2} = 34231\ 3231,$$

$$W_{5,2} = 4534231\ 432341,$$

$$W_{6,2} = 564534231\ 54323451,$$

$$W_{7,2} = 67564534231\ 6543234561, \text{ where spaces separate the blocks.}$$

An optimal construction . . .

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- **1st block $\Omega_{n,1}$** : adjoin the factors $i(i+1)$ for $i = n-1, n-2, \dots, 2$, followed by the letter 1.

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For $n = 4, 5, 6, 7$, we have:

$$W_{4,2} = 3423$$

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$$W_{4,2} = 34231\ 32$$

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$$W_{4,2} = 34231\ 323$$

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Example: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$$W_{4,2} = 34231 \ 3231,$$

An optimal construction . . .

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$$W_{5,2} = 4534231 \ 432341,$$

$$W_{6,2} = 564534231 \ 54323451,$$

$$W_{7,2} = 67564534231 \ 6543234561.$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow \underline{34231}\ \underline{34231}\ 3231$

Duplicate 1st block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow 34231\ 34231\underline{134}\ 3231$

Append 134 to 2nd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34\underline{4}231\ 34231134\ 3231$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423\underline{3}1\ 34231134\ 3231$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 3442331\underline{1}\ 34231134\ 3231$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 323\underline{3}1$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 32331\underline{1}$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233\underline{4}11$$

Insert 4 before leftmost 1 in 3rd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

\longrightarrow

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow \underline{4534231}\ \underline{4534231}\ 432341$$

Duplicate 1st block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 4534231\ 4534231\mathbf{1345}\ 432341$$

Append 1345 to 2nd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45\mathbf{5}34231\ 45342311345\ 432341$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534\mathbf{4}231\ 45342311345\ 432341$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423\mathbf{3}1\ 45342311345\ 432341$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 4553442331\underline{1}\ 45342311345\ 432341$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323\underline{3}41$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 432334\underline{4}1$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 43233441\underline{1}$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344\underline{5}11$$

Insert 5 before leftmost 1 in 3rd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

\longrightarrow

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow \underline{564534231}\ \underline{564534231}\ 54323451$$

Duplicate 1st block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 564534231\ 564534231\mathbf{13456}\ 54323451$$

Append 13456 to 2nd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 56\bar{6}45\bar{5}344\bar{4}23\bar{3}1\bar{1}\ 56453423113456\ 54323\bar{3}4\bar{4}5\bar{5}1\bar{1}$$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 56645534423311\ 56453423113456\ 5432334455\mathbf{6}11$$

Insert 6 before leftmost 1 in 3rd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$$

$$W_{7,2} = 67564534231\ 6543234561$$

\longrightarrow

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$$

$$W_{7,2} = 67564534231\ 6543234561$$

$$\longrightarrow \underline{67564534231}\ \underline{67564534231}\ 6543234561$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$$

$$W_{7,2} = 67564534231\ 6543234561$$

$$\longrightarrow 67564534231\ 67564534231\ 134567\ 6543234561$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$$

$$W_{7,2} = 67564534231\ 6543234561$$

$$\longrightarrow 677566455344423311\ 67564534231134567\ 654323344556611$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$$

$$W_{7,2} = 67564534231\ 6543234561$$

$$\longrightarrow 67756645534423311\ 67564534231134567\ 6543233445566\underline{7}11$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$$

$$W_{7,2} = 67564534231\ 6543234561$$

$$\longrightarrow 67756645534423311\ 67564534231134567\ 6543233445566711$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$$

$$W_{7,2} = 67564534231\ 6543234561$$

$$\longrightarrow 67756645534423311\ 67564534231134567\ 6543233445566711$$

Note:

$$W_{n,3} = \underbrace{(n-1)nn\Omega_{n-1,1}}_{\Omega_{n,1}} \underbrace{(n-1)n\Omega_{n-1,2}n}_{\Omega_{n,2}} \underbrace{(n-1)\Omega'_{n-1,3}[11]^{-1}(n-1)n11}_{\Omega'_{n,3}}.$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$$W_{4,2} = 34231\ 3231 \longrightarrow \underline{344}23311\ \underline{34}231134\ \underline{3}233\underline{4}11 = W_{4,3}$$

$$W_{5,2} = 4534231\ 432341$$

$$\longrightarrow \underline{455}34423311\ \underline{45}34231134\underline{5}\ \underline{4}32334\underline{45}11 = W_{5,3}$$

$$W_{6,2} = 564534231\ 54323451$$

$$\longrightarrow \underline{566}45534423311\ \underline{56}45342311345\underline{6}\ \underline{5}43233445\underline{56}11 = W_{6,3}$$

$$W_{7,2} = 67564534231\ 6543234561$$

$$\longrightarrow \underline{677}56645534423311\ \underline{67}56453423113456\underline{7}\ \underline{6}54323344556\underline{67}11$$

Note:

$$W_{n,3} = \underbrace{(n-1)nn\Omega_{n-1,1}}_{\Omega_{n,1}} \underbrace{(n-1)n\Omega_{n-1,2}n}_{\Omega_{n,2}} \underbrace{(n-1)\Omega'_{n-1,3}[11]^{-1}(n-1)n11}_{\Omega'_{n,3}}.$$

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n 's, and three x 's for $x = 3, \dots, n - 1$.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n 's, and three x 's for $x = 3, \dots, n-1$.

- Hence, $|W_{n,3}| = 3(3(n-3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n 's, and three x 's for $x = 3, \dots, n - 1$.

- Hence, $|W_{n,3}| = 3(3(n - 3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.
- Moreover, $W_{n,3}$ is crucial with respect to abelian cubes.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n 's, and three x 's for $x = 3, \dots, n-1$.

- Hence, $|W_{n,3}| = 3(3(n-3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.
- Moreover, $W_{n,3}$ is crucial with respect to abelian cubes.

For instance: it is easy to check that

$$W_{4,3} = 34423311 \ 34231134 \ 3233411$$

is an abelian cube-free crucial word on 4 letters.

Use same arguments for $n > 4$.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n 's, and three x 's for $x = 3, \dots, n - 1$.

- Hence, $|W_{n,3}| = 3(3(n - 3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.
- Moreover, $W_{n,3}$ is crucial with respect to abelian cubes.
- Thus, a minimal crucial word avoiding abelian cubes has length at most $9n - 13$ for $n \geq 4$.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n 's, and three x 's for $x = 3, \dots, n - 1$.

- Hence, $|W_{n,3}| = 3(3(n - 3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.
- Moreover, $W_{n,3}$ is crucial with respect to abelian cubes.
- Thus, a minimal crucial word avoiding abelian cubes has length at most $9n - 13$ for $n \geq 4$.

Theorem (Glen-Halldórsson-Kitaev, 2008)

For $n \geq 4$, we have $\ell_3(n) \leq 9n - 13$.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n 's, and three x 's for $x = 3, \dots, n - 1$.

- Hence, $|W_{n,3}| = 3(3(n - 3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.
- Moreover, $W_{n,3}$ is crucial with respect to abelian cubes.
- Thus, a minimal crucial word avoiding abelian cubes has length at most $9n - 13$ for $n \geq 4$.

Theorem (Glen-Halldórsson-Kitaev, 2008)

For $n \geq 4$, we have $\ell_3(n) \leq 9n - 13$.

This upper bound is optimal . . .

Lower bound for $\ell_3(n)$

Let X be a **crucial abelian cube-free word** over \mathcal{A}_n such that Xn is an **abelian cube**.

- Sort in non-decreasing order the # of occurrences of the letters $1, 2, \dots, n-1$ in X to obtain the sequence $(a_1 \leq a_2 \leq \dots \leq a_{n-1})$.

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We prove that for $n \geq 5$ this sequence **cannot** be “improved” by decreasing one or more of its terms, no matter how the crucial word is constructed.

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That is, for a crucial abelian cube-free word X over \mathcal{A}_n :

- $(a_1, a_2) \neq (3, 3)$;
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For example: 11, 21211, 11231321211, 42131214231211321211.

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Note:

- $|W_{n,2}| = 4n - 7$ and $|W_{n,3}| = 9n - 13 \longrightarrow W_{n,2}$ and $W_{n,3}$ are minimal crucial words over \mathcal{A}_n avoiding abelian squares and abelian cubes, respectively.

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- In the case of $k \geq 4$, we make the following conjecture.

Conjecture (Glen-Halldórsson-Kitaev, 2008)

For $k \geq 4$ and sufficiently large n , the length of a minimal crucial word over \mathcal{A}_n avoiding abelian k -th powers is given by $k^2(n - 1) - k - 1$.

Lower bound for $\ell_k(n)$

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- A slight improvement using results in the case of abelian cubes ...

Theorem (Glen-Halldórsson-Kitaev, 2008)

For $n \geq 5$ and $k \geq 4$, we have $\ell_k(n) \geq k(3n - 4) - 1$.

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- **Example:** 323121 is a **maximal abelian square-free word** over $\{1, 2, 3\}$ of minimal length.
- The length of a minimal crucial word gives a **lower bound** for the length of a shortest maximal word.

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Question: Can our approach improve Bullock's result or can it provide an alternative solution?

Thank you for your attention!