Existence of piecewise linear Lyapunov functions in arbitrary dimensions

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Abstract

Lyapunov functions are an important tool to determine the basin of attraction of exponentially stable equilibria in dynamical systems. In Marinosson (2002), a method to construct Lyapunov functions was presented, using finite differences on finite elements and thus transforming the construction problem into a linear programming problem. In Hafstein (2004), it was shown that this method always succeeds in constructing a Lyapunov function, except for a small, given neighbourhood of the equilibrium.

For two-dimensional systems, this local problem was overcome by choosing a fan-like triangulation around the equilibrium. In Giesl/Hafstein (2010) the existence of a piecewise linear Lyapunov function was shown, and in Giesl/Hafstein (subm.) it was shown that the above method with a fan-like triangulation always succeeds in constructing a Lyapunov function, without any local exception. However, the previous papers only considered two-dimensional systems. This paper generalises the existence of piecewise linear Lyapunov functions to arbitrary dimensions.

1 Introduction

In this paper we study the autonomous system of differential equations $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, and assume that the origin is an exponentially stable equilibrium with basin of attraction denoted by $\mathcal{A}$. A Lyapunov function $w\colon \mathbb{R}^n \to \mathbb{R}$ is a function which decreases along solutions of the differential equation. This can be expressed by a negative orbital derivative $w'(\mathbf{x}) < 0$, where the orbital derivative is the derivative along solutions and is given by $w'(\mathbf{x}) = \nabla w(\mathbf{x}) \cdot f(\mathbf{x})$. Lyapunov

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functions can be used to determine subsets of the basin of attraction $\mathcal{A}$ through their sublevel sets.

The standard method to obtain a local Lyapunov function and thus a subset of the basin of attraction is to solve the Lyapunov equation, i.e. to find a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ which is the solution to $J^TQ + QJ = -P$, where $J := Df(\mathbf{0})$ is the Jacobian of f at the origin and $P \in \mathbb{R}^{n\times n}$ is an arbitrary positive definite matrix. Then the quadratic function $\mathbf{x} \mapsto \mathbf{x}^TQ\mathbf{x}$ is a local Lyapunov function for the system $\dot{\mathbf{x}} = f(\mathbf{x})$, i.e. it is a Lyapunov function for the system in some neighborhood of the origin. The size of this neighborhood is a priori not known and is, except for linear f, in general a poor estimate of $\mathcal{A}$ (see, for example, [8] for more details). This method to compute local Lyapunov functions is constructive because there is an algorithm to solve the Lyapunov equation that succeeds whenever it possesses a solution, cf. Bartels and Stewart [2].

In the last decades there have been several proposals to construct Lyapunov functions numerically. To name a few, Johansson and Rantzer proposed a construction method in [12] for piecewise quadratic Lyapunov functions for piecewise affine autonomous systems. Julian, Guivant, and Desages in [14] and Julian in [13] presented a linear programming problem to construct piecewise affine Lyapunov functions for autonomous piecewise affine systems. This method can be used for autonomous, nonlinear systems if some a posteriori analysis of the generated Lyapunov function is done. Garcia and Agamennoni [4] recently published a paper based on similar ideas. In [11], Johansen uses linear programming to parameterise Lyapunov functions for autonomous nonlinear systems, but does not give error estimates. Parrilo in [19] and Papachristodoulou and Prajna in [18] consider the numerical construction of Lyapunov functions that are presentable as sums of squares for autonomous polynomial systems under polynomial constraints. These ideas have been taken further by a recent publications of Peet [20], where he proves the existence of a polynomial Lyapunov function on bounded regions for exponentially stable systems.

Giesl proposed in [5] a method to construct Lyapunov functions for autonomous systems with an exponentially stable equilibrium by solving numerically a generalised Zubov equation, cf. [21],

$$\nabla V(\mathbf{x}) \cdot f(\mathbf{x}) = -p(\mathbf{x}), \tag{1.1}$$

where usually $p(\mathbf{x}) = \|\mathbf{x}\|_2^2$ for the equilibrium at the origin. A solution to the partial differential equation (1.1) is a Lyapunov function for the system. He uses radial basis functions to find a numerical solution to (1.1) and there are error estimates given.

In [17], Hafstein (alias Marinosson) presented a method to compute piecewise affine Lyapunov function. In this method one first triangulates a compact neighborhood $\mathcal{C} \subset \mathcal{A}$ of the origin and then constructs a linear programming problem with the property, that a continuous Lyapunov function V, affine on each triangle of the triangulation, can be constructed from any feasible solution to it. In [8] it was proved that for exponentially stable equilibria this method is always capable of generating a Lyapunov function $V : \mathcal{C} \setminus \mathcal{N} \longrightarrow \mathbb{R}$, where $\mathcal{N} \subset \mathcal{C}$ is an arbitrary small, a priori determined neighborhood of the origin. In [9] these results were generalised to asymptotically stable systems, in [10] to asymptotically stable, arbitrary switched, non-autonomous systems, and in [1] to asymptotically stable differential inclusions.

In [6], we have shown that the triangulation scheme used in [17, 8, 9, 10] in general does not allow for piecewise affine Lyapunov functions near the equilibrium. However, in the same paper we have proposed a new, fan-like triangulation around the equilibrium, and we have proved that a piecewise affine Lyapunov function with respect to this new triangulation always exists. In the above mentioned paper, however, we have only dealt with the two-dimensional case.

In this paper, we obtain a similar result for arbitrary dimensions, but using a different approach. In particular, we show that for any system with an exponentially stable equilibrium, there exists a local, piecewise linear Lyapunov function. We give a constructive proof of this fact by first describing the triangulation. The piecewise linear Lyapunov function w is then constructed by the values of the function $v(\mathbf{x}) := \sqrt{\mathbf{x}^T Q\mathbf{x}}$ on all vertices, where Q satisfies the Lyapunov equation $J^TQ + QJ = -I$, $J := Df(\mathbf{0})$. For all other points the function w is uniquely defined by the fact that it is linear on all simplices.

The main part of this paper is the proof of the existence of this piecewise linear Lyapunov function $w(\mathbf{x})$. The main step is the characterisation of $\nabla w(\mathbf{x})$ as a multiple of the vector $\mathbf{c}$, which satisfies a system of $(n-1)$ linear equations. We then estimate the difference of $\nabla w(\mathbf{x})$ to $\nabla v(\mathbf{x})$ and show that it tends to zero as the triangulation becomes finer. Hence, we show that a piecewise linear Lyapunov function exists if the triangulation is fine enough.

In the two-dimensional case, the existence of a piecewise affine Lyapunov function led to an improvement of the algorithm in [17, 8, 9, 10]: using the advanced triangulation scheme with a fan-like triangulation around the equilibrium, one can construct and compute a piecewise affine Lyapunov function $V : \mathcal{C} \longrightarrow \mathbb{R}$ for any system with an exponentially stable equilibrium, cf. [7]. We are confident that, based on the results of this paper, a similar construction method for arbitrary dimensions will be possible.

Notations

For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we denote the Euclidean scalar product by $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$ and the Euclidean norm by $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. We further use the maximum norm $\|\mathbf{x}\|_\infty = \max |x_i|$. The induced matrix norm for a matrix $A \in \mathbb{R}^{n\times n}$ is given by $\|A\|_2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2$. The convex hull of the vectors $\mathbf{x}_0, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ is denoted by

$$\mathrm{co}\{\mathbf{x}_0, \ldots, \mathbf{x}_k\} := \left\{ \sum_{i=0}^k \lambda_i \mathbf{x}_i \,:\, \sum_{i=0}^k \lambda_i = 1,\ 0 \leq \lambda_i \leq 1 \text{ for } i = 0, \ldots, k \right\}.$$

Note that if the vectors $\mathbf{x}_0, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ are affinely independent, i.e. $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_k - \mathbf{x}_0$ are linearly independent, then $\mathrm{co}\{\mathbf{x}_0, \ldots, \mathbf{x}_k\}$ is polyhedron with a positive k-

dimensional volume, i.e. a k-simplex. A vector $\mathbf{x} \in \mathbb{R}^n$ is assumed to be a column vector and $\mathbf{x}^T$ is the corresponding row vector.

2 Preliminaries

2.1 A Lyapunov function v

We consider the differential equation

$$\dot{\mathbf{x}} \quad = \quad f(\mathbf{x}), \tag{2.1}$$

where $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$. We denote the solution $\mathbf{x}(t)$ of $\dot{\mathbf{x}} = f(\mathbf{x})$ at time t with initial condition $\mathbf{x}(0) = \boldsymbol{\xi}$ by $\boldsymbol{\phi}(t, \boldsymbol{\xi})$.

Furthermore, we assume that $\mathbf{x}_0 = \mathbf{0}$ is an exponentially stable equilibrium, i.e. that the rate of convergence of solutions to it is exponential. An equilibrium is exponentially stable if and only if it is linearly asymptotically stable, i.e. the linearised system $\dot{\mathbf{x}} = Df(\mathbf{0})\mathbf{x}$ is asymptotically stable, which is also equivalent to the linearised system being exponentially asymptotically stable [15, Theorem 4.15]. This is again equivalent to the condition that all eigenvalues of $Df(\mathbf{0})$ have strictly negative real part [15, Corollary 4.3].

We consider $n \geq 3$, since the two-dimensional case has already been solved in [6]. The Lyapunov equation

$$J^T Q + QJ = -I, \text{ where } J := Df(\mathbf{0}),$$

has a unique solution $Q \in \mathbb{R}^{n \times n}$ which is symmetric and positive definite. Hence, the square root $Q^{\frac{1}{2}}$ exists and is also symmetric and positive definite. Define the norm

$$\|\mathbf{x}\|_Q := \|Q^{\frac{1}{2}}\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T Q \mathbf{x}}.$$

Note that we also have $\|Q^{\frac{1}{2}}\|_2^2 = \|Q\|_2$ since Q is symmetric and positive definite.

A (strict) Lyapunov function V for the equilibrium $\mathbf{0}$ of (2.1) is a positive definite function of the state space which is decreasing along the solution trajectories of the system. More precisely, V is a continuous function $V : \mathcal{C} \to \mathbb{R}$, where $\mathcal{C} \neq \emptyset$ is an open neighborhood of the origin, the closure of which is compact, fulfilling $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{C} \setminus \{\mathbf{0}\}$, as well as

$$D^+V(\mathbf{x}) := \limsup_{h \to 0+} \frac{V(\boldsymbol{\phi}(h, \mathbf{x})) - V(\mathbf{x})}{h} < 0 \text{ for all } \mathbf{x} \in \mathcal{C} \setminus \{\mathbf{0}\}, \tag{2.2}$$

where D^+ denotes the Dini derivative, cf. e.g. [16, Part I]. Note that if V is continuously differentiable, then the Dini derivative is equal to the orbital derivative, i.e.

$$D^+V(\mathbf{x}) = \frac{d}{dt} V(\boldsymbol{\phi}(t, \mathbf{x}))\bigg|_{t=0} = \nabla V(\mathbf{x}) \cdot f(\mathbf{x}) = V'(\mathbf{x}).$$

The following proposition is taken from [6, Proposition 4.1] with $r = 1$ and a slightly different set $B_\delta(\mathbf{0})$. The proposition shows that the function $v(\mathbf{x}) := \|\mathbf{x}\|_Q$ is a Lyapunov function in a neighborhood of the equilibrium. This function will be interpolated at the vertices of each simplex of a certain triangulation, and thus we will construct a piecewise linear Lyapunov function w in the next section.

Proposition 2.1 *Consider $\dot{\mathbf{x}} = f(\mathbf{x})$, where $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and assume that $\mathbf{x}_0 = \mathbf{0}$ is an exponentially stable equilibrium. Let the positive definite matrix $Q \in \mathbb{R}^{n\times n}$ be the unique solution of the Lyapunov equation $J^T Q + QJ = -I$, $J := Df(\mathbf{0})$.*

Then there is a number $\delta > 0$, such that the function $v \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}) \cap \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$, given by

$$v(\mathbf{x}) \ := \ \|\mathbf{x}\|_Q \ = \ \sqrt{\mathbf{x}^T Q \mathbf{x}}, \tag{2.3}$$

satisfies

$$\begin{aligned} &v(\mathbf{x}) \geq C\|\mathbf{x}\|_2 && \text{for all } \mathbf{x} \in B_\delta(\mathbf{0}) \text{ and} \\ &v'(\mathbf{x}) = \nabla v(\mathbf{x}) \cdot f(\mathbf{x}) \ \leq \ -2c\|\mathbf{x}\|_2 && \text{for all } \mathbf{x} \in B_\delta(\mathbf{0}) \setminus \{\mathbf{0}\}, \end{aligned}$$

where

$$c := \frac{1}{8\left\|Q^{\frac{1}{2}}\right\|_2}, \ C := \frac{1}{\left\|Q^{-\frac{1}{2}}\right\|_2} \ \text{ and } \ B_\delta(\mathbf{0}) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 < \delta\}.$$

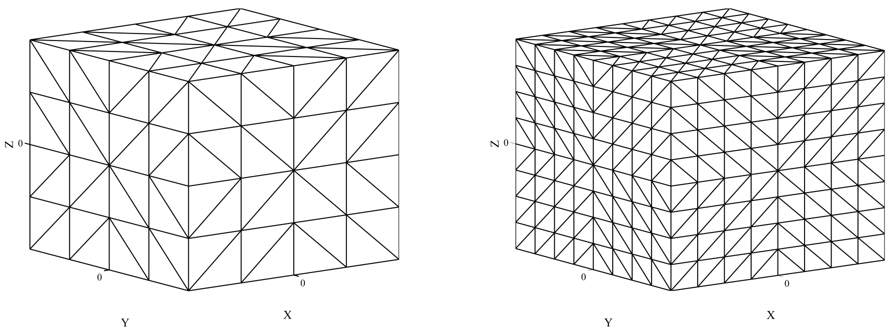
2.2 Triangulation

To construct a piecewise linear Lyapunov function from the Lyapunov function in Proposition 2.1 we need to fix our triangulation, i.e. a subdivision of $\mathbb{R}^n$ into n-simplices, such that the intersection of two different simplices in the subdivision is either empty or a k-simplex, $0 \leq k < n$, and its vertices are the common vertices of the two different n-simplices. Such a structure is often referred to as a simplicial n-complex.

We do this by modifying the simplicial n-complex used in [10] locally at the origin in a similar way as we did in [6], adapted to n and not only two dimensions. The main idea is to take the intersection of the boundary of a box $[-b, b]^n$, $b > 0$, with the simplices in a simplicial n-complex as in [10], such that the boundary is subdivided into a simplicial $(n-1)$-complex. To all the simplices in this new simplicial $(n-1)$-complex we then add the origin as a vertex to get a new simplicial n-complex locally at the origin, cf. Figure 1, where this is depicted for $n = 3$, and Figure 2, where four exemplary simplices of such a triangulation are shown.

For the construction we use the set S_n of all permutations of the numbers $1, 2, \ldots, n$, the characteristic functions $\chi_{\mathcal{J}}(i)$ equal to one if $i \in \mathcal{J}$ and equal to zero if $i \notin \mathcal{J}$, and the standard orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathbb{R}^n$. Further, we use the functions $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^n \to \mathbb{R}^n$, defined for every $\mathcal{J} \subset \{1, 2, \ldots, n\}$ by

$$\mathbf{R}^{\mathcal{J}}(\mathbf{x}) := \sum_{i=1}^{n} (-1)^{\chi_{\mathcal{J}}(i)} x_i \mathbf{e}_i.$$



(a) The triangulation $\mathcal{T}_{1,b}$ for $n = 3$. (b) The triangulation $\mathcal{T}_{2,b}$ for $n = 3$.

Figure 1: Schematic pictures of the local triangulations for $n = 3$. Note that for $n = 3$ the elements of $\mathcal{T}_{K,b}$, $K \in \mathbb{N}$, are tetrahedra with the origin as a vertex. The intersection of the boundary of a box $[-b, b]^3$ with a suitable simplicial 3-complex from [10] delivers a simplicial 2-complex (triangles). By adding the origin as a vertex to all the simplices in the simplicial 2-complex, we get a simplicial 3-complex (tetrahedra) locally at the origin.

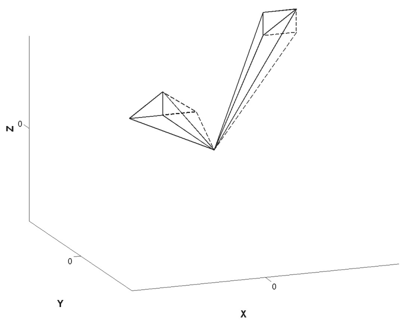


Figure 2: A few exemplary simplices from the triangulation $\mathcal{T}_{1,b}$ for $n = 3$. The simplices depicted are, from left to right, $\text{co}\{(0,0,0), (-b/2, b, 0), (0, b, 0), (0, b, b/2)\}$, $\text{co}\{(0,0,0), (0, b, 0), (0, b, b/2), (b/2, b, 0)\}$, $\text{co}\{(0,0,0), (b/2, b/2, b), (b/2, b, b), (b, b, b)\}$, and $\text{co}\{(0,0,0), (b/2, b/2, b), (b, b/2, b), (b, b, b)\}$.

Thus $\mathbf{R}^{\mathcal{J}}(\mathbf{x})$ puts a minus in front of the coordinate x_i of $\mathbf{x}$ if $i \in \mathcal{J}$.

Note that the two parameters b and K of the triangulation $\mathcal{T}_{K,b}$ refer to the size of the box $[-b, b]^n$ covered by it and to the fineness, respectively.

Definition 2.2 *To construct the triangulation $\mathcal{T}_{K,b}$, we first define the triangulations $\mathcal{T}$ and $\mathcal{T}_K$ as intermediate steps.*

1. *The triangulation $\mathcal{T}$ consists of the simplices*

$$T_{\mathbf{z},\mathcal{J},\sigma} := \operatorname{co}\left\{\mathbf{R}^{\mathcal{J}}\left(\mathbf{z}+\sum_{i=1}^{j}\mathbf{e}_{\sigma(i)}\right) \;:\; j=0,1,2,\dots,n\right\}$$

   for all $\mathbf{z} \in \mathbb{Z}^n_{\geq 0}$, all $\mathcal{J} \subset \{1,2,\dots,n\}$, and all $\sigma \in S_n$.

2. *Choose a positive integer K and consider the $(n-1)$-simplices obtained by taking the intersection of the n-simplices $T_{\mathbf{z},\mathcal{J},\sigma}$ in $\mathcal{T}$ and the boundary of the cube $[-2^K, 2^K]^n$. A new triangulation $\mathcal{T}_K$, local at the origin, is now obtained by taking every such $(n-1)$-simplex and adding the origin as a vertex to it.*

3. *The final step is now to choose a constant $b > 0$ and scale the triangulation $\mathcal{T}_K$, such that the vertices of the n-simplices in $\mathcal{T}_K$, with the exception of the origin, are in the boundary of the cube $[-b,b]^n$. Thus, transform every simplex $T \in \mathcal{T}_K$ with the mapping $\mathbf{x} \mapsto \rho\mathbf{x}$, where $\rho := 2^{-K}b$. The set of the resulting simplices is denoted by $\mathcal{T}_{K,b}$.*

We will refer to $\mathcal{T}_{K,b}$ as the *standard triangulation* of $[-b,b]^n$ with fineness $\rho = 2^{-K}b$. We have four remarks on this triangulation:

Remark 2.3 *The triangulation $\mathcal{T}$ is studied in more detail in [16, Sections 4.1 and 4.2]. A sometimes more intuitive description of $T_{\mathbf{z},\mathcal{J},\sigma}$ is the simplex $\{\mathbf{x} : 0 \leq x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)} \leq 1\}$ translated by $\mathbf{x} \mapsto \mathbf{x}+\mathbf{z}$ and then a minus-sign is put in front of the i-th entry of the resulting vector whenever $i \in \mathcal{J}$.*

Remark 2.4 *For dimension $n = 2$, this construction is the same as the one in [6]. While the construction above only defines the local part of the triangulation around $\mathbf{0}$, it can be expanded to a global triangulation of $\mathbb{R}^n$ by using the simplices from $\mathcal{T}$ scaled with ρ outside of $[-b,b]^n$. This will be needed in a subsequent paper to derive an algorithm for the construction of a local and global piecewise affine Lyapunov function.*

Remark 2.5 *For every $T \in \mathcal{T}_{K,b}$ and every vertex $\mathbf{x} \neq \mathbf{0}$ of T we have $\|\mathbf{x}\|_\infty = b$, i.e. there is at least one $k \in \{1,\dots,n\}$ such that $|x_k| = b$. Further, if $T \in \mathcal{T}_{K,b}$ and $\mathbf{x} \neq \mathbf{0}$ is an arbitrary vertex of T, then the other nonzero vertices of T are given by*

$$\mathbf{x} + \rho\sum_{i=1}^{j-1}\mathbf{u}_i, \quad j = 2,\dots,n, \tag{2.4}$$

where $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}$ is a suitable paraxial orthonormal basis of the hyperplane $\{\mathbf{z} \in \mathbb{R}^n : z_k = 0\}$, where k is such that $|x_k| = b$. By defining $\mathbf{u}_n := \mathbf{e}_k$, there *is a permutation $\sigma \in S_n$ such that $\mathbf{u}_i = \pm \mathbf{e}_{\sigma(i)}$ for every $i = 1, 2, \ldots, n$, where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is the standard orthonormal basis for $\mathbb{R}^n$. In particular $\sigma(n) = k$ and $|\mathbf{x} \cdot \mathbf{u}_n| = |\mathbf{x} \cdot \mathbf{e}_k| = b$.*

Remark 2.6 *$\mathcal{T}_{K,b}$ is indeed a simplicial n-complex as shown in Lemma 2.7.*

Lemma 2.7 *Consider the set of simplices $\mathcal{T}_{K,b}$ from Definition 2.2 and let $T_1 = \text{co}\{\mathbf{0}, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and $T_2 = \text{co}\{\mathbf{0}, \mathbf{y}_1, \ldots, \mathbf{y}_n\}$ be two of its simplices. Then*

$$T_1 \cap T_2 = T_3 := \text{co}\{\mathbf{0}, \mathbf{z}_1, \ldots, \mathbf{z}_m\},$$

where $\mathbf{z}_1, \ldots, \mathbf{z}_m$ are the vertices $\neq \mathbf{0}$ that are common to T_1 and T_2, i.e. $\mathbf{z}_i = \mathbf{x}_{\alpha(i)} = \mathbf{y}_{\beta(i)}$ for $\alpha, \beta \in S_n$ and $i = 1, \ldots, m$.

Proof: The inclusion "$\supseteq$" is obvious so we only show the inclusion "$\subseteq$". Let $\mathbf{x} \in T_1 \cap T_2$. If $\mathbf{x} = \mathbf{0}$ then clearly $\mathbf{x} \in T_3$ so we assume that $\mathbf{x} \neq \mathbf{0}$. By the definition of a convex hull we can write $\mathbf{x}$ as a convex combination of the vertices of T_1 and as a convex combination of the vertices of T_2, i.e., with $\mathbf{x}_0 = \mathbf{y}_0 = \mathbf{0}$,

$$\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i = \sum_{i=1}^{n} \mu_i \mathbf{y}_i,$$

where the numbers λ_i and μ_i are all non-negative and $\sum_{i=0}^{n} \lambda_i = \sum_{i=0}^{n} \mu_i = 1$. Consider the vector $\widetilde{\mathbf{x}} := (b/\|\mathbf{x}\|_\infty)\,\mathbf{x}$. We claim that

$$\widetilde{\mathbf{x}} \in \text{co}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \cap \text{co}\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}.$$

By construction

$$\|\mathbf{x}_i + \mathbf{x}_j\|_\infty = \|\mathbf{x}_i\|_\infty + \|\mathbf{x}_j\|_\infty = 2b \quad i, j = 1, \ldots, n,$$

so

$$\|\mathbf{x}\|_\infty = \left\| \sum_{i=1}^{n} \lambda_i \mathbf{x}_i \right\|_\infty = \sum_{i=1}^{n} \lambda_i \|\mathbf{x}_i\|_\infty = b \sum_{i=1}^{n} \lambda_i.$$

Hence,

$$\widetilde{\mathbf{x}} = \sum_{i=1}^{n} \left(\frac{\lambda_i b}{\|\mathbf{x}\|_\infty} \right) \mathbf{x}_i,$$

where

$$\sum_{i=1}^{n} \left(\frac{\lambda_i b}{\|\mathbf{x}\|_\infty} \right) = 1 \quad \text{and} \quad 0 \leq \frac{\lambda_i b}{\|\mathbf{x}\|_\infty} \leq 1 \text{ for } i = 1, \ldots, n,$$

i.e. $\widetilde{\mathbf{x}} \in \text{co}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. By symmetry $\widetilde{\mathbf{x}} \in \text{co}\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ follows. This proves the claim.

Hence

$$\widetilde{\mathbf{x}}/\rho \in \mathrm{co}\{\mathbf{x}_1/\rho,\ldots,\mathbf{x}_n/\rho\} \cap \mathrm{co}\{\mathbf{y}_1/\rho,\ldots,\mathbf{y}_n/\rho\},$$

where both simplices on the right are in $\mathcal{T}$. By Theorem 4.11 in [16] $\widetilde{\mathbf{x}}/\rho$ can be written as a convex combination of the vertices common to these simplices

$$\widetilde{\mathbf{x}}/\rho \in \mathrm{co}\{\mathbf{z}_1/\rho,\ldots,\mathbf{z}_m/\rho\}, \quad \mathbf{z}_i = \mathbf{x}_{\alpha(i)} = \mathbf{y}_{\beta(i)}$$

and it follows that $\widetilde{\mathbf{x}}$ has a unique representation as a convex sum of the vertices $\mathbf{z}_1,\ldots,\mathbf{z}_m$,

$$\widetilde{\mathbf{x}} = \sum_{i=1}^{m} \nu_i \mathbf{z}_i.$$

But then

$$\mathbf{x} = \frac{\|\mathbf{x}\|_\infty}{b}\widetilde{\mathbf{x}} = \left(1 - \frac{\|\mathbf{x}\|_\infty}{b}\right)\underbrace{\mathbf{z}_0}_{=\mathbf{0}} + \sum_{i=1}^{m} \frac{\nu_i\|\mathbf{x}\|_\infty}{b}\mathbf{z}_i$$

so $\mathbf{x}$ can be represented as a convex combination of vertices common to T_1 and T_2, i.e. $\mathbf{x} \in T_3$ which proves the lemma. □

3 Main result

We prove the existence of a piecewise linear Lyapunov function $w : [-b,b]^n \to \mathbb{R}$ for any $\mathcal{C}^1$ system with an exponentially stable equilibrium at the origin. This is achieved by defining w through the properties: for every $T \in \mathcal{T}_{K,b}$

$$w \text{ is linear on } T \text{ and for every vertex } \mathbf{x} \text{ of } T \text{ we have } w(\mathbf{x}) = v(\mathbf{x}), \tag{3.1}$$

where v is the Lyapunov function from Proposition 2.1. The function w is continuous but not differentiable, however, it is $\mathcal{C}^\infty$ except for the intersections of simplices in $T \in \mathcal{T}_{K,b}$.

Remark 3.1 *Denote by $\mathcal{V}_{T,b}$ be the set of all nonzero vertices of all the simplices in $\mathcal{T}_{K,b}$. A main part of the proof is to show, with ∇w appropriately interpreted, that*

$$\max_{\mathbf{x}\in\mathcal{V}_{K,b}} \|\nabla w(\mathbf{x}) - \nabla v(\mathbf{x})\|_2 \to 0 \tag{3.2}$$

as $K \to \infty$. With $b > 0$ fixed this is equivalent to $\rho = 2^{-K}b \to 0$. We quantify this convergence in (3.30) as

$$\max_{\mathbf{x}\in\mathcal{V}_{K,b}} \|\nabla w(\mathbf{x}) - \nabla v(\mathbf{x})\|_2 \le \frac{\rho}{b}C_0 = 2^{-K}C_0. \tag{3.3}$$

To convince the reader that this is a nontrivial problem, let us consider two examples.

1. *The limit $b \to 0$ does in general not imply the limit (3.2). For example, in $\mathbb{R}^2$ we consider the cube $[-b,b]^2$ and the simplex (triangle) with the vertices $\mathbf{x}_0 = (0,0)$, $\mathbf{x}_1 = (0,b)$ and $\mathbf{x}_2 = (b,b)$. Then for $v(\mathbf{x}) = \|\mathbf{x}\|$ we have $w(\mathbf{x}) = \mathbf{w}^T\mathbf{x}$ with $\mathbf{w} = (\sqrt{2}-1,1)^T$ as then $v(\mathbf{x}_i) = w(\mathbf{x}_i)$ for all $i = 0,1,2$. Since $\nabla v(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$, we have $\nabla v(\mathbf{x}_1) = (0,1)^T$, whereas $\nabla w(\mathbf{x}_1) = \mathbf{w} = (\sqrt{2}-1,1)^T$. Thus, $\|\nabla w(\mathbf{x}_j) - \nabla v(\mathbf{x}_j)\|_2 = \sqrt{2}-1$, independent of b.*

2. *If we define the fineness of a simplex to be the maximal distance between two vertices, not including the origin, then the fineness may tend to zero without ∇w converging to ∇v as in (3.2).*

   As a counterexample consider the three-dimensional simplex in $[-b,b]^3$ given by $\mathbf{x}_0 = (0,0,0)^T$, $\mathbf{x}_1 = (r,0,b)^T$, $\mathbf{x}_2 = (r\cos\alpha, r\sin\alpha, b)^T$ and $\mathbf{x}_3 = (r\cos\alpha, -r\sin\alpha, b)^T$ where $0 < r < b$ is fixed and $\alpha > 0$. The maximal distance between two vertices excluding the origin is $r|\sin\alpha|\sqrt{2}$ for small α, and tends to 0 if $\alpha \to 0$. For $v(\mathbf{x}) = \|\mathbf{x}\|$ we have $w(\mathbf{x}) = \frac{\sqrt{r^2+b^2}}{b}(0,0,1)^T\mathbf{x}$. Thus at $\mathbf{x}_1$ we have $\nabla v(\mathbf{x}_1) = \frac{1}{\sqrt{r^2+b^2}}(r,0,b)^T$ and $\nabla w(\mathbf{x}_1) = \frac{\sqrt{r^2+b^2}}{b}(0,0,1)^T$, which is independent of α. Note that

   $$\|\nabla v(\mathbf{x}_1) - \nabla w(\mathbf{x}_1)\|_2 = \frac{r}{b}$$

   does not tend to 0 as $\alpha \to 0$. Thus, we have to ensure a certain regularity of the simplices as the fineness tends to zero, which is done by the definition of $\mathcal{T}_{K,b}$ in Section 2.2.

Theorem 3.2 *Consider $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $n \geq 3$ and assume that $\mathbf{x}_0 = \mathbf{0}$ is an exponentially stable equilibrium. Let $v(\mathbf{x}) := \|\mathbf{x}\|_Q$ be the Lyapunov function from Proposition 2.1.*

Then for the standard triangulation $\mathcal{T}_{K,b}$ of $[-b,b]^n$ with fineness $\rho := 2^{-K}b$, the function $w : \mathbb{R}^n \to \mathbb{R}$ defined as in (3.1) is a continuous Lyapunov function for the system, whenever $b > 0$ is small enough and $K \in \mathbb{N}$ is large enough. In particular, we have for every $\mathbf{x} \in]-b,b[^n$ that

$$w(\mathbf{x}) \geq C\|\mathbf{x}\|_2 \text{ and } D^+w(\mathbf{x}) \leq -c\|\mathbf{x}\|_2,$$

where $c := \frac{1}{8\left\|Q^{\frac{1}{2}}\right\|_2}$, $C := \frac{1}{\left\|Q^{-\frac{1}{2}}\right\|_2}$ and the Dini derivative $D^+w(\mathbf{x})$ is defined in (2.2).

Remark 3.3 *Let us make the choice of b (size of the triangulation) and K (fineness of the triangulation) more precise: In the proof we first fix the size b of the cube, which only depends on the linearisation of $f(\mathbf{x}) = J\mathbf{x} + \psi(\mathbf{x})$, in particular J and ψ, cf. Step 2 (3.5). Then we choose K and thus the fineness $\rho = 2^{-K}b$ in Step 4.*

PROOF: We split the proof into several steps.

Step 1: Constants
For the positive definite matrix Q from Proposition 2.1 we set

$$q := \min_{\|\mathbf{x}\|_\infty=1} \|Q\mathbf{x}\|_\infty = \min_{\|\mathbf{x}\|_\infty=1} \max_{i=1,\ldots,n} \left|\mathbf{e}_i^T Q\mathbf{x}\right| > 0. \tag{3.4}$$

Define the following constants, which only depend on q, n and Q:

$$\begin{aligned}
C_1 &:= \frac{(n-1)(2\sqrt{n}+1)\|Q\|_2}{2}\|Q^{-\frac{1}{2}}\|_2 + \|Q^{\frac{1}{2}}\|_2\sqrt{n},\\
C_2 &:= \sqrt{n}\|Q^{\frac{1}{2}}\|_2\Bigg(\frac{1}{4}(n-1)^2(2\sqrt{n}+1)\|Q\|_2^2\|Q^{-\frac{1}{2}}\|_2^2\sqrt{n}\\
&\quad +\frac{1}{2}\|Q^{\frac{1}{2}}\|_2\sqrt{n}\|Q^{-\frac{1}{2}}\|_2(n-1)\|Q\|_2\Bigg),\\
C_3 &:= (n-1)!C_1^{n-2}C_2,\\
C_4 &:= \frac{2C_3}{q\|Q^{-\frac{1}{2}}\|_2^{3-n}},\\
C_c &:= \|Q\|_2\sqrt{n} + (n-1)C_4,\\
C_\gamma &:= \frac{1}{2\|Q^{-\frac{1}{2}}\|_2^2\sqrt{n}\|Q^{\frac{1}{2}}\|_2},\\
\widetilde{C} &:= \frac{C_c}{C_\gamma},
\end{aligned}$$

$$\begin{aligned}
C_0 &:= \max\Bigg[\|Q^{-\frac{1}{2}}\|_2(n-1)C_4\left(\widetilde{C}\|Q^{-\frac{1}{2}}\|_2+1\right), \|Q^{-\frac{1}{2}}\|_2^2(n-1)C_4\widetilde{C}\\
&\quad +\frac{2\sqrt{n}+1}{2}(n-1)\|Q\|_2\|Q^{-\frac{1}{2}}\|_2^3C_c + (n-1)(C_4+\|Q\|_2)\|Q^{-\frac{1}{2}}\|_2\Bigg].
\end{aligned}$$

Step 2: Choice of b
We consider the linearised system $\dot{\mathbf{x}} = J\mathbf{x}$ where $J := Df(\mathbf{0})$. Because f is differentiable we have $f(\mathbf{x}) = J\mathbf{x} + \psi(\mathbf{x})$ with $\lim_{\|\mathbf{x}\|_2\to 0}\frac{\|\psi(\mathbf{x})\|_2}{\|\mathbf{x}\|_2} = 0$. Fix $b > 0$ for the rest of the proof so small that

$$\|\psi(\mathbf{x})\|_2 \leq \frac{c}{\widetilde{C}}\|\mathbf{x}\|_2 \text{ for all } \mathbf{x}\in R, \tag{3.5}$$

where $c := \frac{1}{8\left\|Q^{\frac{1}{2}}\right\|_2}$ and $R := [-b,b]^n$.

Step 3: Description of a simplex
Let $T \in \mathcal{T}_{K,b}$ be an arbitrary simplex. Denote its vertices by $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$, where $\mathbf{x}_0 = \mathbf{0}$. Then $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \partial R$ and there is a $k \in \{1,\ldots,n\}$ such that both $|\mathbf{x}_1\cdot\mathbf{e}_k| = b$ and the vertices $\mathbf{x}_2,\ldots,\mathbf{x}_n$ of the simplex are given by

$$\mathbf{x}_j = \mathbf{x}_1 + \rho\sum_{i=1}^{j-1}\mathbf{u}_i \text{ for } j = 2,\ldots,n,$$

where $\rho = 2^{-K}b$ and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}, \mathbf{u}_n$ is a suitable paraxial orthonormal basis for $\mathbb{R}^n$ as in (2.4), i.e. there is a permutation $\sigma \in S_n$ such that $\mathbf{u}_i = \pm\mathbf{e}_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$ and especially $\mathbf{u}_n = \mathbf{e}_k$, cf. (2.4) and Remark 2.5. The value of $b > 0$ was fixed in Step 2 and a suitable value for $K \in \mathbb{N}$ will be determined later.

We can then write

$$\mathbf{x}_1 = \sum_{i=1}^{n} a_i \mathbf{u}_i$$

with the same orthonormal basis $\mathbf{u}_i$. We have $|a_n| = |\mathbf{x}_1 \cdot \mathbf{u}_n| = b$ and, since $\mathbf{x}_1 \in \partial R$, $|a_i| \leq b$ for all $i = 1, \ldots, n$. Since $\mathbf{x}_j \in \partial R$ for $j = 1, \ldots, n$ we have $b \leq \|\mathbf{x}_j\|_2 \leq b\sqrt{n}$. For the $\|\cdot\|_Q$ norm we thus obtain

$$\|\mathbf{x}_j\|_Q = \|Q^{\frac{1}{2}}\mathbf{x}_j\|_2 \;\leq\; \|Q^{\frac{1}{2}}\|_2 \cdot \|\mathbf{x}_j\|_2 \;\leq\; \|Q^{\frac{1}{2}}\|_2 b\sqrt{n}, \tag{3.6}$$

and

$$\|\mathbf{x}_j\|_2 = \|Q^{-\frac{1}{2}}Q^{\frac{1}{2}}\mathbf{x}_j\|_2 \;\leq\; \|Q^{-\frac{1}{2}}\|_2 \|Q^{\frac{1}{2}}\mathbf{x}_j\|_2 \;=\; \|Q^{-\frac{1}{2}}\|_2 \|\mathbf{x}_j\|_Q,$$

from which

$$\|\mathbf{x}_j\|_Q \geq \frac{1}{\|Q^{-\frac{1}{2}}\|_2}\|\mathbf{x}_j\|_2 \;\geq\; \frac{1}{\|Q^{-\frac{1}{2}}\|_2} b \tag{3.7}$$

follows. We define

$$\begin{aligned} \alpha_i \quad &:= \quad \mathbf{u}_i^T Q \mathbf{x}_1 \text{ for } i = 1, \ldots, n \\ \text{and we have } \; a_i \quad &= \quad \mathbf{u}_i^T \mathbf{x}_1 \text{ for } i = 1, \ldots, n. \end{aligned}$$

Moreover, the following estimates hold

$$|a_i| \;\leq\; b, \tag{3.8}$$

$$|\alpha_i| \;\leq\; \|\mathbf{u}_i\|_2 \|Q\mathbf{x}_1\|_2 \;\leq\; \|Q\|_2 b\sqrt{n}, \tag{3.9}$$

$$\left(\sum_{i=1}^{j}\mathbf{u}_i\right)^T Q \left(\sum_{i=1}^{j}\mathbf{u}_i\right) \;=\; \left\|Q^{\frac{1}{2}}\left(\sum_{i=1}^{j}\mathbf{u}_i\right)\right\|_2^2 \leq \|Q^{\frac{1}{2}}\|_2^2 \left\|\sum_{i=1}^{j}\mathbf{u}_i\right\|_2^2 \;=\; j\|Q\|_2 \tag{3.10}$$

for all $j \in \{1, \ldots, n\}$.

Using the constant $q > 0$ from Step 1, we can conclude that there is an index $i^* = i^*(\mathbf{x}_1) \in \{1, \ldots, n\}$ such that

$$|\alpha_{i^*}| \;\geq\; bq \tag{3.11}$$

holds true. Indeed, define $\mathbf{x} := \frac{1}{b}\mathbf{x}_1$ which satisfies $\|\mathbf{x}\|_\infty = 1$. Note that for q defined in Step 1 there is an index $j^* = j^*(\mathbf{x})$ such that $|\mathbf{e}_{j^*}^T Q\mathbf{x}| = \|Q\mathbf{x}\|_\infty \geq q$. Since the $\mathbf{u}_i$ are a permutation of the $\mathbf{e}_i$, there is an index $i^* = \sigma^{-1}(j^*)$ such that $\mathbf{u}_{i^*} = \pm\mathbf{e}_{j^*}$. Thus, $|\alpha_{i^*}| = \left|\mathbf{u}_{i^*}^T Q\mathbf{x}_1\right| = b\left|\pm\mathbf{e}_{j^*}^T Q\mathbf{x}\right| \geq bq$ holds, which shows (3.11).

Note that we also have

$$\begin{aligned} \|\mathbf{x}_j\|_Q^2 &= \|\mathbf{x}_1\|_Q^2 + 2\rho\left(\sum_{i=1}^{j-1}\mathbf{u}_i\right)^T Q\mathbf{x}_1 + \rho^2\left(\sum_{i=1}^{j-1}\mathbf{u}_i\right)^T Q\left(\sum_{i=1}^{j-1}\mathbf{u}_i\right) \\ &= \|\mathbf{x}_1\|_Q^2 + 2\rho\sum_{i=1}^{j-1}\alpha_i + \rho^2\left(\sum_{i=1}^{j-1}\mathbf{u}_i\right)^T Q\left(\sum_{i=1}^{j-1}\mathbf{u}_i\right). \end{aligned} \tag{3.12}$$

Step 4: Choice of K
We start by showing that:

> For every $\varepsilon > 0$ there is a $K_\varepsilon \in \mathbb{N}$, such that $K \geq K_\varepsilon$ implies, that for any simplex $T \in \mathcal{T}_{K,b}$ and any fixed vertex $\mathbf{x}_1 \neq \mathbf{0}$ of T we have
>
> $$\left|\frac{\|\mathbf{x}_j\|_Q - \|\mathbf{x}_1\|_Q}{\rho} - \frac{1}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{j-1}\alpha_i\right| < \varepsilon, \tag{3.13}$$
>
> where the non-zero vertices $\mathbf{x}_2, \ldots, \mathbf{x}_n$ of T and the numbers $\alpha_1, \ldots, \alpha_n$ are defined as above in Step 3.

Indeed, for any $j = 1, \ldots, n$ and the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ defined as above, we have by (3.12)

$$\begin{aligned} \frac{\|\mathbf{x}_j\|_Q - \|\mathbf{x}_1\|_Q}{\rho} &= \frac{\|\mathbf{x}_j\|_Q^2 - \|\mathbf{x}_1\|_Q^2}{\rho(\|\mathbf{x}_j\|_Q + \|\mathbf{x}_1\|_Q)} \\ &= \frac{2\rho\sum_{i=1}^{j-1}\alpha_i + \rho^2\left(\sum_{i=1}^{j-1}\mathbf{u}_i\right)^T Q\left(\sum_{i=1}^{j-1}\mathbf{u}_i\right)}{\rho(\|\mathbf{x}_j\|_Q + \|\mathbf{x}_1\|_Q)} \\ &= \frac{2\sum_{i=1}^{j-1}\alpha_i + \rho\left(\sum_{i=1}^{j-1}\mathbf{u}_i\right)^T Q\left(\sum_{i=1}^{j-1}\mathbf{u}_i\right)}{\|\mathbf{x}_j\|_Q + \|\mathbf{x}_1\|_Q}. \end{aligned} \tag{3.14}$$

Since $\rho = 2^{-K}b$ and b is fixed, we have

$$|\|\mathbf{x}_j\|_Q - \|\mathbf{x}_1\|_Q| \leq \|\mathbf{x}_j - \mathbf{x}_1\|_Q \leq \|Q^{\frac{1}{2}}\|_2\rho\sqrt{j-1}$$

and the proposition (3.13) follows.

Now, for a $T \in \mathcal{T}_{K,b}$ let i^* be defined as in (3.11) and consider the matrix $A \in \mathbb{R}^{(n-1)\times(n-1)}$ defined for the simplex T by

$$A_{ji} = a_i\frac{\|\mathbf{x}_{j+1}\|_Q - \|\mathbf{x}_1\|_Q}{\rho} - \|\mathbf{x}_1\|_Q\chi_{\{1,\ldots,j\}}(i) \tag{3.15}$$

for $i \in \{1, \ldots, n\} \setminus \{i^*\}$ and $j = 1, \ldots, n-1$, where $\chi_{\{1,\ldots,j\}}(i) = \begin{cases} 1 & \text{if } i \in \{1,\ldots,j\} \\ 0 & \text{otherwise} \end{cases}$. In other words, the matrix A is given by

$$\begin{pmatrix} a_1\frac{\|\mathbf{x}_2\|_Q-\|\mathbf{x}_1\|_Q}{\rho}-\|\mathbf{x}_1\|_Q & a_2\frac{\|\mathbf{x}_2\|_Q-\|\mathbf{x}_1\|_Q}{\rho} & \cdots & a_{n-1}\frac{\|\mathbf{x}_2\|_Q-\|\mathbf{x}_1\|_Q}{\rho} & a_n\frac{\|\mathbf{x}_2\|_Q-\|\mathbf{x}_1\|_Q}{\rho} \\ a_1\frac{\|\mathbf{x}_3\|_Q-\|\mathbf{x}_1\|_Q}{\rho}-\|\mathbf{x}_1\|_Q & a_2\frac{\|\mathbf{x}_3\|_Q-\|\mathbf{x}_1\|_Q}{\rho}-\|\mathbf{x}_1\|_Q & \cdots & a_{n-1}\frac{\|\mathbf{x}_3\|_Q-\|\mathbf{x}_1\|_Q}{\rho} & a_n\frac{\|\mathbf{x}_3\|_Q-\|\mathbf{x}_1\|_Q}{\rho} \\ \vdots & \vdots & \ddots & & \vdots \\ a_1\frac{\|\mathbf{x}_n\|_Q-\|\mathbf{x}_1\|_Q}{\rho}-\|\mathbf{x}_1\|_Q & a_2\frac{\|\mathbf{x}_n\|_Q-\|\mathbf{x}_1\|_Q}{\rho}-\|\mathbf{x}_1\|_Q & \cdots & a_{n-1}\frac{\|\mathbf{x}_n\|_Q-\|\mathbf{x}_1\|_Q}{\rho}-\|\mathbf{x}_1\|_Q & a_n\frac{\|\mathbf{x}_n\|_Q-\|\mathbf{x}_1\|_Q}{\rho} \end{pmatrix}$$

where the i^*-th column is missing.

We now show that

$$|\det A| \geq \frac{1}{2}\|Q^{-\frac{1}{2}}\|_2^{3-n}b^{n-1}q \tag{3.16}$$

for all large enough K independent of the particular $T \in \mathcal{T}_{K,b}$. To see this consider the matrix $A_0 \in \mathbb{R}^{(n-1)\times(n-1)}$, defined by

$$(A_0)_{ji} = a_i\frac{\sum_{l=1}^{j}\alpha_l}{\|\mathbf{x}_1\|_Q} - \|\mathbf{x}_1\|_Q\chi_{\{1,\dots,j\}}(i).$$

In other words, A_0 is given by

$$\begin{pmatrix} \frac{a_1}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{1}\alpha_i-\|\mathbf{x}_1\|_Q & \frac{a_2}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{1}\alpha_i & \cdots & \frac{a_{n-1}}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{1}\alpha_i & \frac{a_n}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{1}\alpha_i \\ \frac{a_1}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{2}\alpha_i-\|\mathbf{x}_1\|_Q & \frac{a_2}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{2}\alpha_i-\|\mathbf{x}_1\|_Q & \cdots & \frac{a_{n-1}}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{2}\alpha_i & \frac{a_n}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{2}\alpha_i \\ \frac{a_1}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{3}\alpha_i-\|\mathbf{x}_1\|_Q & \frac{a_2}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{3}\alpha_i-\|\mathbf{x}_1\|_Q & \cdots & \frac{a_{n-1}}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{3}\alpha_i & \frac{a_n}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{3}\alpha_i \\ \vdots & \vdots & \ddots & & \vdots \\ \frac{a_1}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{n-1}\alpha_i-\|\mathbf{x}_1\|_Q & \frac{a_2}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{n-1}\alpha_i-\|\mathbf{x}_1\|_Q & \cdots & \frac{a_{n-1}}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{n-1}\alpha_i-\|\mathbf{x}_1\|_Q & \frac{a_n}{\|\mathbf{x}_1\|_Q}\sum_{i=1}^{n-1}\alpha_i \end{pmatrix},$$

where the i^*-th column is missing. If $i^* \neq n$ we use Lemma A.2, and if $i^* = n$ we use Lemma A.1, both with $\mathbf{a} = (a_1, \dots, a_{n-1}, a_n)$, where $|a_n| = b$, and $N = \sum_{i=1}^{n} a_i\alpha_i = \left(\sum_{i=1}^{n} a_i\mathbf{u}_i\right)^T Q = \|\mathbf{x}_1\|_Q^2$. The lemmata show that A_0 satisfies in both cases

$$\begin{aligned} |\det A_0| &= \frac{1}{\|\mathbf{x}_1\|_Q^{n-1}}\|\mathbf{x}_1\|_Q^{2(n-2)}b|\alpha_{i^*}| \\ &\geq \|\mathbf{x}_1\|_Q^{n-3}b^2q \text{ by (3.11)} \\ &\geq \|Q^{-\frac{1}{2}}\|_2^{3-n}b^{n-1}q \text{ by (3.7)}. \end{aligned}$$

Because the determinant of a matrix is a continuous function of its entities and by (3.13), it follows that for every $\varepsilon^* > 0$ there is a $K_{\varepsilon^*} \in \mathbb{N}$, such that

$$|\det A - \det A_0| < \varepsilon^* \tag{3.17}$$

for all $K \geq K_{\varepsilon^*}$, and because (3.13) is independent of the particular choice of $T \in \mathcal{T}_{K,b}$ then so is (3.17). Hence, there is a $\widetilde{K} \in \mathbb{N}$ such that whenever $K \geq \widetilde{K}$ we have

$$|\det A| \geq \frac{1}{2}|\det A_0|,$$

i.e. the inequality (3.16) holds true.

We fix $K \in \mathbb{N}$ for the rest of the proof, such that both $K \geq \widetilde{K}$ and

$$K \geq \frac{1}{\ln 2} \max \left[\ln \left(2C_4(n-1)\|Q^{-\frac{1}{2}}\|_2^2 \right), \ln \left(4C_0\|J\|_2\|Q^{\frac{1}{2}}\|_2 \right) \right]. \tag{3.18}$$

With $\rho := 2^{-K}b$ this implies that

$$0 < \rho \leq \min \left(\frac{b}{2C_4(n-1)\|Q^{-\frac{1}{2}}\|_2^2}, \frac{b}{4C_0\|J\|_2\|Q^{\frac{1}{2}}\|_2} \right). \tag{3.19}$$

Step 5: The central equations $A\mathbf{k} = \mathbf{d}$
The function w is linear on the simplex $T \in \mathcal{T}_{K,b}$ and its restriction to T can thus be written as $w(\mathbf{x}) = \mathbf{w}^T\mathbf{x}$, where $\nabla w(\mathbf{x}) = \mathbf{w}$ holds for all $\mathbf{x} \in T$. Moreover, we have

$$\mathbf{w}^T\mathbf{x}_j = w(\mathbf{x}_j) = v(\mathbf{x}_j) = \|\mathbf{x}_j\|_Q$$

for all vertices $\mathbf{x}_j$ of the simplex T. For $j = 2, \ldots, n$ this implies

$$\mathbf{w}^T \left(\mathbf{x}_1 - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q}\mathbf{x}_j \right) = \|\mathbf{x}_1\|_Q - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q}\|\mathbf{x}_j\|_Q = 0.$$

Hence, $\mathbf{w}$ is characterised as the vector which is perpendicular to the $(n-1)$ vectors $\mathbf{x}_1 - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q}\mathbf{x}_j$, $j = 2, \ldots, n$ and satisfies $\mathbf{w}^T\mathbf{x}_1 = \|\mathbf{x}_1\|_Q$.

We now characterise a vector $\mathbf{c}$ through the conditions that it is perpendicular to the $(n-1)$ vectors $\mathbf{x}_1 - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q}\mathbf{x}_j$, $j = 2, \ldots, n$, just as $\mathbf{w}$, and, instead of a norm condition, to satisfy $(\mathbf{c} - Q\mathbf{x}_1)^T\mathbf{u}_{i^*} = 0$, where i^* was defined earlier satisfying $|\alpha_{i^*}| \geq bq$.

We now show that a vector $\mathbf{c}$ satisfying these properties exists and is uniquely determined, and then clearly $\mathbf{c} = \gamma\mathbf{w}$ for some $\gamma \in \mathbb{R}$. In Step 6 we assign a proper value to γ and thus determine $\mathbf{w}$.

Definition of $\mathbf{c}$ and $\mathbf{k}$ as solution of a linear equation
Let $T \in \mathcal{T}_{K,b}$ be arbitrary but fixed, where K and b are as chosen above. We show that there exists one and only one vector $\mathbf{k} = (k_1, \ldots, \hat{k}_{i^*}, \ldots, k_n) \in \mathbb{R}^{n-1}$, where $\hat{k}_{i^*}$ denotes that this entry is missing, such that

$$\mathbf{c} = Q\mathbf{x}_1 + \sum_{i=1, i\neq i^*}^{n} k_i\mathbf{u}_i, \tag{3.20}$$

$$\mathbf{c}^T \left(\mathbf{x}_1 - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q}\mathbf{x}_j \right) = 0 \text{ for all } j = 2, \ldots, n. \tag{3.21}$$

To prove this, we characterise $\mathbf{k}$ as the solution of a system of linear equations. Plugging (3.20) into the $(n-1)$ equations (3.21) for $j = 2, \ldots, n$ gives

$$\begin{aligned} 0 &= \left(Q\mathbf{x}_1 + \sum_{i=1, i \neq i^*}^{n} k_i \mathbf{u}_i \right)^T \left(\mathbf{x}_1 - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q} \mathbf{x}_j \right) \\ &= \left(Q\mathbf{x}_1 + \sum_{i=1, i \neq i^*}^{n} k_i \mathbf{u}_i \right)^T \left(\mathbf{x}_1 \left(1 - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q} \right) - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q} \rho \sum_{i=1}^{j-1} \mathbf{u}_i \right) \\ &= \|\mathbf{x}_1\|_Q^2 \left(1 - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q} \right) - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q} \rho \sum_{i=1}^{j-1} \alpha_i \\ &\quad + \sum_{i=1, i \neq i^*}^{n} k_i \left[a_i \left(1 - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q} \right) - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q} \rho \, \chi_{\{1,\ldots,j-1\}}(i) \right] \end{aligned}$$

where $\chi_{\{1,\ldots,j-1\}}(i) = 1$ if $i \in \{1, \ldots, j-1\}$ and 0 otherwise.

By multiplying the equation by $\frac{\|\mathbf{x}_j\|_Q}{\rho}$ we obtain

$$\begin{aligned} 0 &= \|\mathbf{x}_1\|_Q^2 \frac{\|\mathbf{x}_j\|_Q - \|\mathbf{x}_1\|_Q}{\rho} - \|\mathbf{x}_1\|_Q \sum_{i=1}^{j-1} \alpha_i \\ &\quad + \sum_{i=1, i \neq i^*}^{n} k_i \left[a_i \frac{\|\mathbf{x}_j\|_Q - \|\mathbf{x}_1\|_Q}{\rho} - \|\mathbf{x}_1\|_Q \, \chi_{\{1,\ldots,j-1\}}(i) \right]. \end{aligned}$$

Hence, the vector $\mathbf{k} = (k_1, \ldots, \hat{k}_{i^*}, \ldots, k_n)^T$ is the solution of the linear equation

$$A\mathbf{k} = \mathbf{d}, \tag{3.22}$$

where $A \in \mathbb{R}^{(n-1)\times(n-1)}$ is defined for the simplex T by (3.15) in Step 4 and the vector $\mathbf{d} \in \mathbb{R}^{n-1}$ is given by

$$\mathbf{d} = \|\mathbf{x}_1\|_Q \begin{pmatrix} \sum_{i=1}^{1} \alpha_i - \frac{\|\mathbf{x}_2\|_Q - \|\mathbf{x}_1\|_Q}{\rho} \|\mathbf{x}_1\|_Q \\ \sum_{i=1}^{2} \alpha_i - \frac{\|\mathbf{x}_3\|_Q - \|\mathbf{x}_1\|_Q}{\rho} \|\mathbf{x}_1\|_Q \\ \sum_{i=1}^{3} \alpha_i - \frac{\|\mathbf{x}_4\|_Q - \|\mathbf{x}_1\|_Q}{\rho} \|\mathbf{x}_1\|_Q \\ \vdots \\ \sum_{i=1}^{n-1} \alpha_i - \frac{\|\mathbf{x}_n\|_Q - \|\mathbf{x}_1\|_Q}{\rho} \|\mathbf{x}_1\|_Q \end{pmatrix}.$$

Because of (3.16) A is non-singular and (3.22) has a unique solution $\mathbf{k}$.

By Cramer's rule the solution is given by

$$k_j = \frac{\det A_j}{\det A},$$

where A_j denotes the $(n-1)\times(n-1)$ matrix that is built by taking A and replacing the j-th column by $\mathbf{d}$.

The matrix A
Now we obtain the following estimate for the matrix entities A_{jl} of A, cf. (3.15)

$$\begin{aligned}
|A_{jl}| &\leq |a_l|\left|\frac{\|\mathbf{x}_{j+1}\|_Q - \|\mathbf{x}_1\|_Q}{\rho}\right| + \|\mathbf{x}_1\|_Q \\
&\leq |a_l|\frac{2\sum_{i=1}^{j}|\alpha_i| + \rho(n-1)\|Q\|_2}{\|\mathbf{x}_{j+1}\|_Q + \|\mathbf{x}_1\|_Q} + \|\mathbf{x}_1\|_Q \text{ by (3.14) and (3.10)} \\
&\leq b\left(\frac{2b(n-1)\sqrt{n}\|Q\|_2 + b(n-1)\|Q\|_2}{2b}\right)\|Q^{-\frac{1}{2}}\|_2 + \|Q^{\frac{1}{2}}\|_2 b\sqrt{n} \\
&\qquad \text{by } \rho \leq b, \text{ (3.9), (3.7) and (3.6)} \\
&= b\left(\frac{(n-1)(2\sqrt{n}+1)\|Q\|_2}{2}\|Q^{-\frac{1}{2}}\|_2 + \|Q^{\frac{1}{2}}\|_2\sqrt{n}\right) \\
&= bC_1
\end{aligned} \tag{3.23}$$

due to the definition of C_1.
The vector d
We calculate the j-th component of the vector $\mathbf{d}$.

$$\begin{aligned}
\frac{d_j}{\|\mathbf{x}_1\|_Q} &= \sum_{i=1}^{j}\alpha_i - \frac{\|\mathbf{x}_{j+1}\|_Q - \|\mathbf{x}_1\|_Q}{\rho}\|\mathbf{x}_1\|_Q \\
&= \left(1 - \frac{2\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_{j+1}\|_Q + \|\mathbf{x}_1\|_Q}\right)\sum_{i=1}^{j}\alpha_i \\
&\quad - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_{j+1}\|_Q + \|\mathbf{x}_1\|_Q}\rho\left(\sum_{i=1}^{j}\mathbf{u}_i\right)^T Q\left(\sum_{i=1}^{j}\mathbf{u}_i\right) \text{ by (3.14)} \\
&= \frac{\|\mathbf{x}_{j+1}\|_Q^2 - \|\mathbf{x}_1\|_Q^2}{(\|\mathbf{x}_{j+1}\|_Q + \|\mathbf{x}_1\|_Q)^2}\sum_{i=1}^{j}\alpha_i \\
&\quad - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_{j+1}\|_Q + \|\mathbf{x}_1\|_Q}\rho\left(\sum_{i=1}^{j}\mathbf{u}_i\right)^T Q\left(\sum_{i=1}^{j}\mathbf{u}_i\right)
\end{aligned}$$

$$\begin{aligned}
&= \rho\Bigg[\frac{2\sum_{i=1}^{j}\alpha_i + \rho\left(\sum_{i=1}^{j}\mathbf{u}_i\right)^T Q\left(\sum_{i=1}^{j}\mathbf{u}_i\right)}{(\|\mathbf{x}_{j+1}\|_Q + \|\mathbf{x}_1\|_Q)^2}\sum_{i=1}^{j}\alpha_i \\
&\quad - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_{j+1}\|_Q + \|\mathbf{x}_1\|_Q}\left(\sum_{i=1}^{j}\mathbf{u}_i\right)^T Q\left(\sum_{i=1}^{j}\mathbf{u}_i\right)\Bigg] \text{ by (3.12).}
\end{aligned}$$

Thus, we have by (3.6), (3.7), (3.9) and (3.10)

$$
\begin{aligned}
|d_j| &\le \rho b\sqrt{n}\|Q^{\frac{1}{2}}\|_2\Bigg(\frac{(n-1)\|Q\|_2(2b\sqrt{n}+\rho)}{4b^2}\|Q^{-\frac{1}{2}}\|_2^2(n-1)\|Q\|_2 b\sqrt{n} \\
&\quad +\frac{\|Q^{\frac{1}{2}}\|_2 b\sqrt{n}}{2b}\|Q^{-\frac{1}{2}}\|_2(n-1)\|Q\|_2\Bigg) \\
&\le \rho b C_2. \qquad (3.24)
\end{aligned}
$$

The vector c

By the Leibniz formula for the determinant, namely $\det A = \sum_{\pi\in S_{n-1}} \operatorname{sign}(\pi)\prod_{i=1}^{n-1} A_{i,\pi(i)}$ we obtain with (3.23) and (3.24)

$$
|\det A_j| \;\le\; (n-1)!b^{n-2}C_1^{n-2}\rho b C_2 \;=\; \rho b^{n-1}C_3.
$$

Hence, by (3.16) we have

$$
|k_j| \;=\; \frac{|\det A_j|}{|\det A|} \le \frac{\rho b^{n-1}C_3}{b^{n-1}}\frac{2}{q\|Q^{-\frac{1}{2}}\|_2^{3-n}} = C_4\rho. \qquad (3.25)
$$

Using (3.25) we obtain

$$
\begin{aligned}
\|\mathbf{c}\|_2 &= \left\| Q\mathbf{x}_1 + \sum_{i=1,i\neq i^*}^{n} k_i\mathbf{u}_i \right\|_2 \\
&\le \|Q\|_2 b\sqrt{n} + (n-1)C_4\rho \\
&\le bC_c. \qquad (3.26)
\end{aligned}
$$

Step 6: $\mathbf{c} = \gamma\mathbf{w}$

We showed in the last step that there is a vector $\mathbf{k} = (k_1,\ldots,\hat{k}_{i^*},\ldots,k_n) \in \mathbb{R}^{n-1}$ such that

$$
\mathbf{c} = Q\mathbf{x}_1 + \sum_{i=1,i\neq i^*}^{n} k_i\mathbf{u}_i
$$

fulfills $\mathbf{c}^T\left(\mathbf{x}_1 - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q}\mathbf{x}_j\right) = 0$ for all $j = 2,\ldots,n$. Defining

$$
\gamma \;:=\; \frac{1}{\|\mathbf{x}_1\|_Q}\left(\|\mathbf{x}_1\|_Q^2 + \sum_{i=1,i\neq i^*}^{n} a_i k_i\right)
$$

we obtain with (3.25) that

$$\begin{aligned}\gamma &\geq \frac{1}{\|\mathbf{x}_1\|_Q}\left(\|\mathbf{x}_1\|_Q^2 - C_4(n-1)\rho b\right)\\ &\geq \frac{1}{\|\mathbf{x}_1\|_Q}\left(\frac{b^2}{\|Q^{-\frac{1}{2}}\|_2^2} - \frac{b^2}{2\|Q^{-\frac{1}{2}}\|_2^2}\right) \text{ by (3.7) and (3.19)}\\ &\geq \frac{b^2}{2\|Q^{-\frac{1}{2}}\|_2^2 b\sqrt{n}\|Q^{\frac{1}{2}}\|_2} \text{ by (3.6)}\\ &= bC_\gamma > 0. \end{aligned} \tag{3.27}$$

Moreover, we have with (3.25) and (3.7) that

$$|\gamma - \|\mathbf{x}_1\|_Q| \leq \frac{1}{\|\mathbf{x}_1\|_Q}\sum_{i=1, i\neq i^*}^{n} |a_i k_i| \leq \frac{\|Q^{-\frac{1}{2}}\|_2}{b}(n-1)bC_4\rho. \tag{3.28}$$

Now we prove that $\frac{1}{\gamma}\mathbf{c} = \mathbf{w}$ by showing $\left(\frac{1}{\gamma}\mathbf{c}\right)^T \mathbf{x}_j = \|\mathbf{x}_j\|_Q$ for all $j = 1, \ldots, n$, cf. the beginning of Step 5.

We start with $j = 1$. By definition of γ and (3.20) we have

$$\frac{1}{\gamma}\mathbf{c}^T\mathbf{x}_1 = \frac{1}{\gamma}\left(\|\mathbf{x}_1\|_Q^2 + \sum_{i=1, i\neq i^*}^{n} a_i k_i\right) = \|\mathbf{x}_1\|_Q$$

Now let $j \in \{2, \ldots, n\}$. Since $\mathbf{c}^T\left(\mathbf{x}_1 - \frac{\|\mathbf{x}_1\|_Q}{\|\mathbf{x}_j\|_Q}\mathbf{x}_j\right) = 0$ by (3.21), we have

$$\frac{1}{\gamma}\mathbf{c}^T\mathbf{x}_j = \frac{1}{\gamma}\frac{\|\mathbf{x}_j\|_Q}{\|\mathbf{x}_1\|_Q}\mathbf{c}^T\mathbf{x}_1 = \|\mathbf{x}_j\|_Q.$$

This shows

$$\mathbf{w} = \frac{1}{\gamma}\mathbf{c}.$$

We also derive a bound on $\mathbf{w}$. Using (3.26) and (3.27) we obtain

$$\|\mathbf{w}\|_2 = \frac{1}{\gamma}\|\mathbf{c}\|_2 \leq \frac{bC_c}{bC_\gamma} = \widetilde{C}. \tag{3.29}$$

Step 7: Difference between $\mathbf{w}$ and ∇v
Note that $v(\mathbf{x}) = \|\mathbf{x}\|_Q$ so that $\nabla v(\mathbf{x}) = \frac{Q\mathbf{x}}{\|\mathbf{x}\|_Q}$. We estimate the difference between ∇w and ∇v at $\mathbf{x}_1$, we later consider the other vertices. We have

$$\begin{aligned}
&\|\nabla w(\mathbf{x}_1) - \nabla v(\mathbf{x}_1)\|_2 \\
&= \|\mathbf{w} - \nabla v(\mathbf{x}_1)\|_2 \\
&= \left\| \frac{1}{\gamma}\mathbf{c} - \frac{Q\mathbf{x}_1}{\|\mathbf{x}_1\|_Q} \right\|_2 \\
&\leq \left\| \frac{1}{\gamma}\mathbf{c} - \frac{\mathbf{c}}{\|\mathbf{x}_1\|_Q} \right\|_2 + \left\| \frac{\mathbf{c}}{\|\mathbf{x}_1\|_Q} - \frac{Q\mathbf{x}_1}{\|\mathbf{x}_1\|_Q} \right\|_2 \\
&= |\|\mathbf{x}_1\|_Q - \gamma| \frac{\|\mathbf{c}\|_2}{\gamma\|\mathbf{x}_1\|_Q} + \frac{1}{\|\mathbf{x}_1\|_Q}\|\mathbf{c} - Q\mathbf{x}_1\|_2 \\
&\leq \frac{\|\mathbf{c}\|_2}{\gamma\|\mathbf{x}_1\|_Q}\|Q^{-\frac{1}{2}}\|_2(n-1)C_4\rho + \frac{1}{\|\mathbf{x}_1\|_Q}\sum_{i=1, i\neq i^*}^{n} |k_i| \text{ by (3.28) and (3.20)} \\
&\leq \frac{\|Q^{-\frac{1}{2}}\|_2}{b}\left(\widetilde{C}\|Q^{-\frac{1}{2}}\|_2(n-1)C_4\rho + (n-1)C_4\rho\right) \text{ by (3.29) and (3.25)} \\
&\leq \frac{\rho}{b}C_0, \text{ using the definition of } C_0.
\end{aligned}$$

Now we consider a vertex $\mathbf{x}_j$, $j \geq 2$. We estimate

$$\begin{aligned}
&\|\nabla w(\mathbf{x}_j) - \nabla v(\mathbf{x}_j)\|_2 \\
&= \|\mathbf{w} - \nabla v(\mathbf{x}_j)\|_2 \\
&= \left\| \frac{1}{\gamma}\mathbf{c} - \frac{Q\mathbf{x}_j}{\|\mathbf{x}_j\|_Q} \right\|_2 \\
&\leq \left\| \frac{1}{\gamma}\mathbf{c} - \frac{\mathbf{c}}{\|\mathbf{x}_1\|_Q} \right\|_2 + \left\| \frac{\mathbf{c}}{\|\mathbf{x}_1\|_Q} - \frac{\mathbf{c}}{\|\mathbf{x}_j\|_Q} \right\|_2 + \left\| \frac{\mathbf{c}}{\|\mathbf{x}_j\|_Q} - \frac{Q\mathbf{x}_j}{\|\mathbf{x}_j\|_Q} \right\|_2.
\end{aligned}$$

The first term has already been estimated above by $\frac{\rho}{b}\|Q^{-\frac{1}{2}}\|_2^2(n-1)C_4\widetilde{C}$. For the second term note that we have

$$\begin{aligned}
\left| \frac{1}{\|\mathbf{x}_1\|_Q} - \frac{1}{\|\mathbf{x}_j\|_Q} \right| &= \frac{|\|\mathbf{x}_j\|_Q - \|\mathbf{x}_1\|_Q|}{\|\mathbf{x}_j\|_Q\|\mathbf{x}_1\|_Q} \\
&\leq \frac{2\sum_{i=1}^{j-1}|\alpha_i| + \left(\sum_{i=1}^{j-1}\mathbf{u}_i\right)^T Q \left(\sum_{i=1}^{j-1}\mathbf{u}_i\right)\rho}{\|\mathbf{x}_j\|_Q\|\mathbf{x}_1\|_Q(\|\mathbf{x}_j\|_Q + \|\mathbf{x}_1\|_Q)}\rho \text{ by (3.14)} \\
&\leq \frac{2b\sqrt{n}\|Q\|_2 + \rho\|Q\|_2}{2b^3}(n-1)\rho\|Q^{-\frac{1}{2}}\|_2^3 \text{ by (3.9) and (3.10)} \\
&\leq \frac{2\sqrt{n}+1}{2b^2}(n-1)\|Q\|_2\rho\|Q^{-\frac{1}{2}}\|_2^3
\end{aligned}$$

so that by (3.26) the second term is bounded by $\frac{2\sqrt{n}+1}{2b}(n-1)\|Q\|_2\rho\|Q^{-\frac{1}{2}}\|_2^3 C_c$.

For the third term we obtain

$$\begin{aligned}
\|\mathbf{c} - Q\mathbf{x}_j\|_2 &= \left\| Q\mathbf{x}_1 + \sum_{i=1, i\neq i^*}^{n} k_i \mathbf{u}_i - \left(Q\mathbf{x}_1 + \rho \sum_{i=1}^{j-1} Q\mathbf{u}_i \right) \right\|_2 \\
&\leq (n-1)(C_4\rho + \rho\|Q\|_2), \text{ using (3.25).}
\end{aligned}$$

Hence, by (3.7), the third term is bounded by $(n-1)(C_4 + \|Q\|_2)\frac{\rho}{b}\|Q^{-\frac{1}{2}}\|_2$.

Thus, altogether we obtain by definition of C_0

$$\|\nabla w(\mathbf{x}_j) - \nabla v(\mathbf{x}_j)\|_2 \leq \frac{\rho}{b} C_0 \tag{3.30}$$

for all $j \in \{1, \ldots, n\}$.

Step 8: Negative orbital derivative with respect to linearised system
Recall that $\nabla w(\mathbf{x}) = \mathbf{w} = \frac{1}{\gamma}\mathbf{c}$ for all points $\mathbf{x}$ in the simplex T, since w is linear in the simplex. We now show that $w(\mathbf{x})$ has negative orbital derivative with respect to the linearised system, cf. Step 2, at each vertex. The orbital derivative at the vertex $\mathbf{x}_j$ is, using (3.30), i.e. $\left\|\mathbf{w} - \frac{Q\mathbf{x}_j}{\|\mathbf{x}_j\|_Q}\right\|_2 \leq \frac{\rho}{b}C_0$, given by

$$\begin{aligned}
\mathbf{w}^T J\mathbf{x}_j &= \frac{1}{\|\mathbf{x}_j\|_Q}\mathbf{x}_j^T QJ\mathbf{x}_j + \left(\mathbf{w} - \frac{Q\mathbf{x}_j}{\|\mathbf{x}_j\|_Q}\right)^T J\mathbf{x}_j \\
&\leq \frac{1}{\|\mathbf{x}_j\|_Q}\mathbf{x}_j^T QJ\mathbf{x}_j + \left\|\mathbf{w} - \frac{Q\mathbf{x}_j}{\|\mathbf{x}_j\|_Q}\right\|_2 \|J\|_2\|\mathbf{x}_j\|_2 \\
&\leq \frac{1}{2\|\mathbf{x}_j\|_Q}\mathbf{x}_j^T(J^TQ + QJ)\mathbf{x}_j + \frac{\rho}{b}C_0\|J\|_2\|\mathbf{x}_j\|_2 \\
&\leq -\frac{1}{2\|\mathbf{x}_j\|_Q}\|\mathbf{x}_j\|_2^2 + \frac{1}{4\left\|Q^{\frac{1}{2}}\right\|_2}\|\mathbf{x}_j\|_2 \text{ by (3.19)} \\
&\leq -\frac{1}{4\left\|Q^{\frac{1}{2}}\right\|_2}\|\mathbf{x}_j\|_2,
\end{aligned}$$

where we have used (3.6) and $J^TQ + QJ = -I$.

Step 9: $w(\mathbf{x})$ has negative orbital derivative
In Step 8 we have shown that

$$\mathbf{w}^T J\mathbf{x}_j \quad\leq\quad -2c\|\mathbf{x}_j\|_2 \tag{3.31}$$

where c is defined by $c = \frac{1}{8\left\|Q^{\frac{1}{2}}\right\|_2}$.

Now choose an arbitrary $\mathbf{x} \in R = [-b, b]^n$. We show that there is a simplex $T \in \mathcal{T}_{K,b}$ with vertices $\mathbf{x}_0 = \mathbf{0}, \mathbf{x}_1, \ldots, \mathbf{x}_n$ and $\lambda_0, \lambda_1, \ldots, \lambda_n \in [0,1]$ with $\sum_{j=0}^{n}\lambda_j = 1$ such that

$$\mathbf{x} = \sum_{j=0}^{n}\lambda_j\mathbf{x}_j = \sum_{j=1}^{n}\lambda_j\mathbf{x}_j. \tag{3.32}$$

To show this, we project $\mathbf{x}$ to the boundary of the hypercube: let $\bar{j} \in \{1, \ldots, n\}$ such that $|\mathbf{x}_{\bar{j}}| = \max_{i \in \{1,\ldots,n\}} |\mathbf{x}_i| \leq b$. Then define $\mu := \frac{b}{|\mathbf{x}_{\bar{j}}|} \geq 1$. Obviously, $\mu\mathbf{x} \in \partial R$ so that we can find a $(n-1)$-simplex with vertices $\mathbf{x}_1, \ldots, \mathbf{x}_n$ such that $\mu\mathbf{x} = \sum_{j=1}^n \mu_j \mathbf{x}_j$ with $0 \leq \mu_j \leq 1$ for all $j = 1, \ldots, n$ and $\sum_{j=1}^n \mu_j = 1$. Then

$$\mathbf{x} = \sum_{j=1}^n \frac{\mu_j}{\mu} \mathbf{x}_j + \left(1 - \frac{1}{\mu}\right) \mathbf{x}_0$$

where $\mathbf{x}_0 = \mathbf{0}$, $\lambda_j := \frac{\mu_j}{\mu} \in [0,1]$, $\lambda_0 := 1 - \frac{1}{\mu} \in [0,1]$ and $\sum_{j=0}^n \lambda_j = 1$. This shows (3.32).

We have with $\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{x}_j$

$$\|\mathbf{x}\|_2 = \left\| \sum_{j=1}^n \lambda_j \mathbf{x}_j \right\|_2 \leq \sum_{j=1}^n \lambda_j \|\mathbf{x}_j\|_2 \tag{3.33}$$

$$\text{and also } \|\mathbf{x}\|_Q = \left\| \sum_{j=1}^n \lambda_j \mathbf{x}_j \right\|_Q \leq \sum_{j=1}^n \lambda_j \|\mathbf{x}_j\|_Q$$

$$= \sum_{j=1}^n \lambda_j w(\mathbf{x}_j) = w\left(\sum_{j=1}^n \lambda_j \mathbf{x}_j\right) = w(\mathbf{x}). \tag{3.34}$$

Note that (3.34), together with

$$\|\mathbf{x}\|_2 = \|Q^{-\frac{1}{2}} Q^{\frac{1}{2}} \mathbf{x}\|_2 \leq \|Q^{-\frac{1}{2}}\|_2 \|\mathbf{x}\|_Q$$

shows $w(\mathbf{x}) \geq \frac{1}{\left\|Q^{-\frac{1}{2}}\right\|_2} \|\mathbf{x}\|_2$, which proves the first inequality of the theorem.

Choose an arbitrary simplex $T \in \mathcal{T}_{K,b}$. Within T, the function w is smooth and thus the orbital derivative with respect to the nonlinear system in the simplex T is given by

$$\begin{aligned}
w'(\mathbf{x}) &= \nabla w(\mathbf{x})^T f(\mathbf{x}) \\
&= \mathbf{w}^T (J\mathbf{x} + \psi(\mathbf{x})) \\
&= \sum_{j=1}^n \lambda_j \mathbf{w}^T J \mathbf{x}_j + \mathbf{w}^T \psi(\mathbf{x}) \\
&\leq -2c \sum_{j=1}^n \lambda_j \|\mathbf{x}_j\|_2 + \|\mathbf{w}\|_2 \cdot \|\psi(\mathbf{x})\|_2 \text{ by (3.31)} \\
&\leq -2c\, \|\mathbf{x}\|_2 + \widetilde{C} \cdot \|\psi(\mathbf{x})\|_2
\end{aligned}$$

for $\mathbf{x} \in T$, using (3.33) and (3.29). By (3.5), $\|\psi(\mathbf{x})\|_2 \leq \frac{c}{\widetilde{C}} \|\mathbf{x}\|_2$ holds for all $\mathbf{x} \in R$. Thus, we have

$$w'(\mathbf{x}) \leq -c\|\mathbf{x}\|_2 \text{ for all } \mathbf{x} \in T. \tag{3.35}$$

Now we show that

$$D^+w(\mathbf{x}) \le -c\|\mathbf{x}\|_2$$

holds for all $\mathbf{x} \in]-b,b[^n$. Let $\mathbf{x} \in]-b,b[^n$ be arbitrary. By [16, Theorem 1.17] we have

$$D^+w(\mathbf{x}) = \limsup_{h\to 0+} \frac{w(\mathbf{x}+hf(\mathbf{x})) - w(\mathbf{x})}{h}$$

since for all $h > 0$ small enough there is a $T \in \mathcal{T}_{K,b}$ such that $\text{co}\{\mathbf{x}, \mathbf{x}+hf(\mathbf{x})\} \subset T$, cf. the argumentation at the beginning of Section 6.7 in [10].

Restricting w to this simplex T, w is linear and thus smooth and satisfies $w(\mathbf{x}) = \mathbf{w}^T\mathbf{x}$. Hence, we have

$$\begin{aligned} D^+w(\mathbf{x}) &= \limsup_{h\to 0+} \frac{w(\mathbf{x}+hf(\mathbf{x})) - w(\mathbf{x})}{h} = \limsup_{h\to 0+} \frac{\mathbf{w}\cdot(\mathbf{x}+hf(\mathbf{x})) - \mathbf{w}\cdot\mathbf{x}}{h} \\ &= \limsup_{h\to 0+} \frac{h\mathbf{w}\cdot f(\mathbf{x})}{h} = \nabla w(\mathbf{x})\cdot f(\mathbf{x}) = w'(\mathbf{x}). \end{aligned}$$

Thus, together with (3.35), we have proved the theorem. □

Remark 3.4 *An alternative path to prove* $D^+w(\mathbf{x}) = \limsup_{h\to 0+} \frac{w(\mathbf{x}+hf(\mathbf{x}))-w(\mathbf{x})}{h}$ *is to consider multivalued functions and Clarke's subdifferential, cf. [3].*

4 Conclusions

In this paper we have shown that for any system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $n \ge 3$ with exponentially stable equilibrium, there exists a (local) piecewise linear Lyapunov function $w(\mathbf{x})$. Our result generalises a previous result for dimension $n = 2$, using a different approach.

To construct the Lyapunov function, we first consider a triangulation $\mathcal{T}_{K,b}$ of $[-b,b]^n$ with fineness $\rho = 2^{-K}b$. Then we define w by the values of the function $v(\mathbf{x}) = \sqrt{\mathbf{x}^TQ\mathbf{x}}$ on the vertices of the triangulation, and linear interpolation on each simplex. We have derived a lower bound on $w(\mathbf{x})$ and an upper bound on the Dini derivative $D^+w(\mathbf{x})$.

The significance of this result is that we have given a constructive existence proof for a piecewise linear (local) Lyapunov function. Expanding the triangulation, we can extend this function to a piecewise affine (global) Lyapunov function, which can be constructed by linear programming. We have thus provided the basis for a construction method of a global piecewise Lyapunov function.

A Lemmata on determinants

We prove two lemmata, corresponding to the cases $i^* = n$ and $i^* \neq n$ in the proof of the main theorem. The matrix under consideration in both lemmata is different, the proof, however, is similar.

Lemma A.1 *Let* $\mathbf{a} \in \mathbb{R}^n$ *with* $a_n \neq 0$*, and* $\boldsymbol{\alpha} \in \mathbb{R}^n$ *with* $\alpha_n \neq 0$*. Denote* $N := \sum_{i=1}^n a_i\alpha_i$*. Define the matrix* $A \in \mathbb{R}^{(n-1)\times(n-1)}$ *by*

$$\begin{pmatrix} a_1\sum_{i=1}^1 \alpha_i - N & a_2\sum_{i=1}^1 \alpha_i & \dots & a_{n-2}\sum_{i=1}^1 \alpha_i & a_{n-1}\sum_{i=1}^1 \alpha_i \\ a_1\sum_{i=1}^2 \alpha_i - N & a_2\sum_{i=1}^2 \alpha_i - N & \dots & a_{n-2}\sum_{i=1}^2 \alpha_i & a_{n-1}\sum_{i=1}^2 \alpha_i \\ a_1\sum_{i=1}^3 \alpha_i - N & a_2\sum_{i=1}^3 a_i - N & \ddots & a_{n-2}\sum_{i=1}^3 a_i & a_{n-1}\sum_{i=1}^3 a_i \\ \vdots & \vdots & & \vdots & \vdots \\ a_1\sum_{i=1}^{n-2} \alpha_i - N & a_2\sum_{i=1}^{n-2} \alpha_i - N & \dots & a_{n-2}\sum_{i=1}^{n-2} \alpha_i - N & a_{n-1}\sum_{i=1}^{n-2} \alpha_i \\ a_1\sum_{i=1}^{n-1} \alpha_i - N & a_2\sum_{i=1}^{n-1} \alpha_i - N & \dots & a_{n-2}\sum_{i=1}^{n-1} \alpha_i - N & a_{n-1}\sum_{i=1}^{n-1} \alpha_i - N \end{pmatrix}$$

Then $|\det A| = N^{n-2}|a_n||\alpha_n|$.

Proof: We modify the matrix A by replacing the second row by second row minus first row, then the third row by third row minus second row minus first row, etc. The determinant is the same and the matrix becomes

$$\begin{pmatrix} a_1\alpha_1 - N & a_2\alpha_1 & a_3\alpha_1 & \dots & a_{n-2}\alpha_1 & a_{n-1}\alpha_1 \\ a_1\alpha_2 & a_2\alpha_2 - N & a_3\alpha_2 & \dots & a_{n-2}\alpha_2 & a_{n-1}\alpha_2 \\ \vdots & \vdots & & & \vdots & \vdots \\ a_1\alpha_{n-2} & a_2\alpha_{n-2} & a_3\alpha_{n-2} & \dots & a_{n-2}\alpha_{n-2} - N & a_{n-1}\alpha_{n-2} \\ a_1\alpha_{n-1} & a_2\alpha_{n-1} & a_3\alpha_{n-1} & \dots & a_{n-2}\alpha_{n-1} & a_{n-1}\alpha_{n-1} - N \end{pmatrix}$$

Now assume without loss of generality that $a_1, \dots, a_k \neq 0$ and $a_{k+1}, \dots, a_{n-1} = 0$ with a $k \in \{0, \dots, n-1\}$. Note that we can change the order of $a_1, \dots, a_{n-1}$ by exchanging the i-th and j-th row and then also the i-th and j-th column.

If $k = 0$, then $|\det A| = N^{n-1} = N^{n-2}|a_n| \cdot |\alpha_n|$, since in this case $N = \sum_{i=1}^n a_i\alpha_i = a_n\alpha_n$, which shows the lemma.

Now assume that $k \geq 1$. Using $a_{k+1} = \ldots = a_{n-1} = 0$ the matrix becomes

$$\left(\begin{array}{ccccc|ccc} a_1\alpha_1 - N & a_2\alpha_1 & \ldots & a_{k-1}\alpha_1 & a_k\alpha_1 & 0 & \ldots & 0 \\ a_1\alpha_2 & a_2\alpha_2 - N & \ldots & a_{k-1}\alpha_2 & a_k\alpha_2 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_1\alpha_{k-1} & a_2\alpha_{k-1} & \ldots & a_{k-1}\alpha_{k-1} - N & a_k\alpha_{k-1} & 0 & \ldots & 0 \\ a_1\alpha_k & a_2\alpha_k & \ldots & a_{k-1}\alpha_k & a_k\alpha_k - N & 0 & \ldots & 0 \\ \hline a_1\alpha_{k+1} & a_2\alpha_{k+1} & \ldots & a_{k-1}\alpha_{k+1} & a_k\alpha_{k+1} & -N & & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \ddots & \\ a_1\alpha_{n-1} & a_2\alpha_{n-1} & \ldots & a_{k-1}\alpha_{n-1} & a_k\alpha_{n-1} & 0 & & -N \end{array}\right)$$

The determinant of this matrix satisfies

$$|\det A| \quad = \quad N^{n-k-1}|\det \widetilde{A}_k|, \tag{A.1}$$

where $\widetilde{A}_k$ denotes the left upper $k \times k$ matrix.

If $k = 1$, then we have shown the lemma since in this case we have $N = a_1\alpha_1 + a_n\alpha_n$ and thus $|\det A| = |a_1\alpha_1 - N|N^{n-2} = |a_n\alpha_n|N^{n-2}$.

Now we assume $k \geq 2$ and calculate $|\det \widetilde{A}_k|$, where $a_1, \ldots, a_k \neq 0$. We show by induction that for $2 \leq j \leq k$ we have

$$|\det \widetilde{A}_k| \quad = \quad |\det \widetilde{A}_j|N^{k-j}, \tag{A.2}$$

where $\widetilde{A}_j$, $j = 2, \ldots, k$ denotes the $j \times j$ matrix

$$\widetilde{A}_j := \begin{pmatrix} a_1\alpha_1 - N & a_2\alpha_1 & \ldots & a_{j-1}\alpha_1 & a_j\alpha_1 \\ a_1\alpha_2 & a_2\alpha_2 - N & \ldots & a_{j-1}\alpha_2 & a_j\alpha_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1\alpha_{j-2} & a_2\alpha_{j-2} & \ldots & a_{j-1}\alpha_{j-2} & a_j\alpha_{j-2} \\ a_1\alpha_{j-1} & a_2\alpha_{j-1} & \ldots & a_{j-1}\alpha_{j-1} - N & a_j\alpha_{j-1} \\ \frac{a_1}{a_j}\sum_{i=j}^k a_i\alpha_i & \frac{a_2}{a_j}\sum_{i=j}^k a_i\alpha_i & \ldots & \frac{a_{j-1}}{a_j}\sum_{i=j}^k a_i\alpha_i & \sum_{i=j}^k a_i\alpha_i - N \end{pmatrix}$$

Note that for $j = k$ this definition coincides with the previous definition. We show the formula (A.2) by induction. For $j = k$ the formula holds. We now assume that it holds for $j \in \{3, \ldots, k\}$ and show it for $j - 1$. Indeed, replacing the j-th column by $\frac{a_j}{a_{j-1}} \cdot$ $(j-1)$-st column$-j$-th column gives

$$\det \widetilde{A}_j \quad = \quad \begin{vmatrix} a_1\alpha_1 - N & a_2\alpha_1 & \ldots & a_{j-1}\alpha_1 & 0 \\ a_1\alpha_2 & a_2\alpha_2 - N & \ldots & a_{j-1}\alpha_2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1\alpha_{j-2} & a_2\alpha_{j-2} & \ldots & a_{j-1}\alpha_{j-2} & 0 \\ a_1\alpha_{j-1} & a_2\alpha_{j-1} & \ldots & a_{j-1}\alpha_{j-1} - N & -N\frac{a_j}{a_{j-1}} \\ \frac{a_1}{a_j}\sum_{i=j}^k a_i\alpha_i & \frac{a_2}{a_j}\sum_{i=j}^k a_i\alpha_i & \ldots & \frac{a_{j-1}}{a_j}\sum_{i=j}^k a_i\alpha_i & N \end{vmatrix}$$

We expand along the last column, which gives, using linearity in the last row

$$\begin{aligned}
&\det \widetilde{A}_j \\
&= N \begin{vmatrix} a_1\alpha_1 - N & a_2\alpha_1 & \dots & a_{j-1}\alpha_1 \\ a_1\alpha_2 & a_2\alpha_2 - N & \dots & a_{j-1}\alpha_2 \\ \vdots & \vdots & & \vdots \\ a_1\alpha_{j-2} & a_2\alpha_{j-2} & \dots & a_{j-1}\alpha_{j-2} \\ a_1\alpha_{j-1} & a_2\alpha_{j-1} & \dots & a_{j-1}\alpha_{j-1} - N \end{vmatrix} \\
&\quad + \frac{a_j}{a_{j-1}} N \begin{vmatrix} a_1\alpha_1 - N & a_2\alpha_1 & \dots & a_{j-1}\alpha_1 \\ a_1\alpha_2 & a_2\alpha_2 - N & \dots & a_{j-1}\alpha_2 \\ \vdots & \vdots & & \vdots \\ a_1\alpha_{j-2} & a_2\alpha_{j-2} & \dots & a_{j-1}\alpha_{j-2} \\ \frac{a_1}{a_j}\sum_{i=j}^k a_i\alpha_i & \frac{a_2}{a_j}\sum_{i=j}^k a_i\alpha_i & \dots & \frac{a_{j-1}}{a_j}\sum_{i=j}^k a_i\alpha_i \end{vmatrix} \\
&= \frac{N}{a_{j-1}} \begin{vmatrix} a_1\alpha_1 - N & a_2\alpha_1 & \dots & a_{j-1}\alpha_1 \\ a_1\alpha_2 & a_2\alpha_2 - N & \dots & a_{j-1}\alpha_2 \\ \vdots & \vdots & & \vdots \\ a_1\alpha_{j-2} & a_2\alpha_{j-2} & \dots & a_{j-1}\alpha_{j-2} \\ a_1\sum_{i=j-1}^k a_i\alpha_i & a_2\sum_{i=j-1}^k a_i\alpha_i & \dots & a_{j-1}\left(\sum_{i=j-1}^k a_i\alpha_i - N\right) \end{vmatrix}
\end{aligned}$$

This shows

$$\begin{aligned}
|\det \widetilde{A}_j| &= N \begin{vmatrix} a_1\alpha_1 - N & a_2\alpha_1 & \dots & a_{j-1}\alpha_1 \\ a_1\alpha_2 & a_2\alpha_2 - N & \dots & a_{j-1}\alpha_2 \\ \vdots & \vdots & & \vdots \\ a_1\alpha_{j-2} & a_2\alpha_{j-2} & \dots & a_{j-1}\alpha_{j-2} \\ \frac{a_1}{a_{j-1}}\sum_{i=j-1}^k a_i\alpha_i & \frac{a_2}{a_{j-1}}\sum_{i=j-1}^k a_i\alpha_i & \dots & \sum_{i=j-1}^k a_i\alpha_i - N \end{vmatrix} \\
&= N|\det \widetilde{A}_{j-1}|
\end{aligned}$$

By the induction hypothesis we have $|\det \widetilde{A}_k| = |\det \widetilde{A}_j| N^{k-j} = N|\det \widetilde{A}_{j-1}| N^{k-j} = |\det \widetilde{A}_{j-1}| N^{k-(j-1)}$. This shows (A.2).

Now we use (A.2) for $j = 2$. First, we calculate $\det \widetilde{A}_2$ giving

$$\begin{aligned}
\det \widetilde{A}_2 &= \begin{vmatrix} a_1\alpha_1 - N & a_2\alpha_1 \\ \frac{a_1}{a_2}\sum_{i=2}^k a_i\alpha_i & \sum_{i=2}^k a_i\alpha_i - N \end{vmatrix} \\
&= (a_1\alpha_1 - N)\left(\sum_{i=2}^k a_i\alpha_i - N\right) - a_1\alpha_1 \sum_{i=2}^k a_i\alpha_i \\
&= -Na_1\alpha_1 - N\sum_{i=2}^k a_i\alpha_i + N^2 \\
&= N\left(N - \sum_{i=1}^{n-1} a_i\alpha_i\right) \\
&= Na_n\alpha_n
\end{aligned}$$

since $a_{k+1} = \ldots = a_{n-1} = 0$. Finally, using (A.1) and (A.2) we obtain

$$|\det A| \;=\; N^{n-k-1}N^{k-2}N|a_n| \cdot |\alpha_n| \;=\; N^{n-2}|a_n| \cdot |\alpha_n|.$$

This shows the lemma. □

Lemma A.2 *Let* $\mathbf{a} \in \mathbb{R}^n$ *with* $a_n \neq 0$, *and* $\boldsymbol{\alpha} \in \mathbb{R}^n$ *with* $\alpha_{i^*} \neq 0$ *where* $i^* \in \{1, \ldots, n-1\}$. *Denote* $N := \sum_{i=1}^n a_i\alpha_i$. *Define the matrix* $A \in \mathbb{R}^{(n-1)\times(n-1)}$ *by*

$$\begin{pmatrix}
a_1\sum_{i=1}^1 \alpha_i - N & a_2\sum_{i=1}^1 \alpha_i & \ldots & a_{n-1}\sum_{i=1}^1 \alpha_i & a_n\sum_{i=1}^1 \alpha_i \\
a_1\sum_{i=1}^2 \alpha_i - N & a_2\sum_{i=1}^2 \alpha_i - N & \ldots & a_{n-1}\sum_{i=1}^2 \alpha_i & a_n\sum_{i=1}^2 \alpha_i \\
a_1\sum_{i=1}^3 \alpha_i - N & a_2\sum_{i=1}^3 a_i - N & \ldots & a_{n-1}\sum_{i=1}^3 a_i & a_n\sum_{i=1}^3 a_i \\
\vdots & \vdots & & \vdots & \vdots \\
a_1\sum_{i=1}^{n-2} \alpha_i - N & a_2\sum_{i=1}^{n-2} \alpha_i - N & \ldots & a_{n-1}\sum_{i=1}^{n-2} \alpha_i & a_n\sum_{i=1}^{n-2} \alpha_i \\
a_1\sum_{i=1}^{n-1} \alpha_i - N & a_2\sum_{i=1}^{n-1} \alpha_i - N & \ldots & a_{n-1}\sum_{i=1}^{n-1} \alpha_i - N & a_n\sum_{i=1}^{n-1} \alpha_i
\end{pmatrix}$$

where the i^**-th column is missing.*

Then $|\det A| = N^{n-2}|a_n||\alpha_{i^*}|$.

Proof: We modify the matrix A by replacing the second row by second row minus first row, then the third row by third minus second row minus first row, etc. The determinant is the same and the matrix becomes

$$\begin{pmatrix}
a_1\alpha_1 - N & a_2\alpha_1 & a_3\alpha_1 & \ldots & a_{n-2}\alpha_1 & a_{n-1}\alpha_1 & a_n\alpha_1 \\
a_1\alpha_2 & a_2\alpha_2 - N & a_3\alpha_2 & \ldots & a_{n-2}\alpha_2 & a_{n-1}\alpha_2 & a_n\alpha_2 \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\
a_1\alpha_{n-2} & a_2\alpha_{n-2} & a_3\alpha_{n-2} & \ldots & a_{n-2}\alpha_{n-2} - N & a_{n-1}\alpha_{n-2} & a_n\alpha_{n-2} \\
a_1\alpha_{n-1} & a_2\alpha_{n-1} & a_3\alpha_{n-1} & \ldots & a_{n-2}\alpha_{n-1} & a_{n-1}\alpha_{n-1} - N & a_n\alpha_{n-1}
\end{pmatrix}$$

Now we move the last column to the place of the (missing) i^*-th column, which results possibly in a minus sign. In the next step, we change the order of $a_1, \dots, a_{n-1}$ by exchanging the i-th and j-th row and then also the i-th and j-th column. We can thus move the i^*-th column in front of the first column to obtain the matrix

$$\begin{pmatrix} a_n\alpha_{i^*} & a_1\alpha_{i^*} & a_2\alpha_{i^*} & a_3\alpha_{i^*} & \dots & a_{n-2}\alpha_{i^*} & a_{n-1}\alpha_{i^*} \\ a_n\alpha_1 & a_1\alpha_1 - N & a_2\alpha_1 & a_3\alpha_1 & \dots & a_{n-2}\alpha_1 & a_{n-1}\alpha_1 \\ a_n\alpha_2 & a_1\alpha_2 & a_2\alpha_2 - N & a_3\alpha_2 & \dots & a_{n-2}\alpha_2 & a_{n-1}\alpha_2 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ a_n\alpha_{n-2} & a_1\alpha_{n-2} & a_2\alpha_{n-2} & a_3\alpha_{n-2} & \dots & a_{n-2}\alpha_{n-2} - N & a_{n-1}\alpha_{n-2} \\ a_n\alpha_{n-1} & a_1\alpha_{n-1} & a_2\alpha_{n-1} & a_3\alpha_{n-1} & \dots & a_{n-2}\alpha_{n-1} & a_{n-1}\alpha_{n-1} - N \end{pmatrix}$$

where the i^*-th column and row are missing.

Now assume without loss of generality that $a_1, \dots, a_k \neq 0$ and $a_{k+1} = \dots = a_{n-1} = 0$ with a $k \in \{0, \dots, n-1\}$. Note that we can change the order of $a_1, \dots, a_{n-1}$ by exchanging the i-th and j-th row and then also the i-th and j-th column.

If $k = 0$, then $|\det A| = |a_n| \cdot |\alpha_{i^*}| N^{n-2}$, which shows the lemma.

Now assume that $k \geq 1$. Using $a_{k+1} = \dots = a_{n-1} = 0$ the matrix becomes

$$\left(\begin{array}{cccccc|ccc} a_n\alpha_{i^*} & a_1\alpha_{i^*} & a_2\alpha_{i^*} & \dots & a_{k-1}\alpha_{i^*} & a_k\alpha_{i^*} & 0 & \dots & 0 \\ a_n\alpha_1 & a_1\alpha_1 - N & a_2\alpha_1 & \dots & a_{k-1}\alpha_1 & a_k\alpha_1 & 0 & \dots & 0 \\ a_n\alpha_2 & a_1\alpha_2 & a_2\alpha_2 - N & \dots & a_{k-1}\alpha_2 & a_k\alpha_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ a_n\alpha_{k-1} & a_1\alpha_{k-1} & a_2\alpha_{k-1} & \dots & a_{k-1}\alpha_{k-1} - N & a_k\alpha_{k-1} & 0 & \dots & 0 \\ a_n\alpha_k & a_1\alpha_k & a_2\alpha_k & \dots & a_{k-1}\alpha_k & a_k\alpha_k - N & 0 & \dots & 0 \\ \hline a_n\alpha_{k+1} & a_1\alpha_{k+1} & a_2\alpha_{k+1} & \dots & a_{k-1}\alpha_{k+1} & a_k\alpha_{k+1} & -N & & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots & \\ a_n\alpha_{n-1} & a_1\alpha_{n-1} & a_2\alpha_{n-1} & \dots & a_{k-1}\alpha_{n-1} & a_k\alpha_{n-1} & 0 & & -N \end{array}\right)$$

We assume without loss of generality that $k < i^*$. Then the determinant of this matrix satisfies

$$|\det A| \quad = \quad N^{n-k-2} |\det \widetilde{A}_k|, \tag{A.3}$$

where $\widetilde{A}_k$ denotes the left upper $(k+1) \times (k+1)$ matrix.

Now we calculate $|\det \widetilde{A}_k|$, where $a_1, \dots, a_k \neq 0$. We show by induction that for $1 \leq j \leq k$ we have

$$|\det \widetilde{A}_k| \quad = \quad |\det \widetilde{A}_j| N^{k-j}, \tag{A.4}$$

where $\widetilde{A}_j$, $j = 1, \ldots, k$ denotes the $(j+1) \times (j+1)$ matrix

$$\begin{pmatrix} a_n\alpha_{i^*} & a_1\alpha_{i^*} & a_2\alpha_{i^*} & \ldots & a_{j-1}\alpha_{i^*} & a_j\alpha_{i^*} \\ a_n\alpha_1 & a_1\alpha_1 - N & a_2\alpha_1 & \ldots & a_{j-1}\alpha_1 & a_j\alpha_1 \\ a_n\alpha_2 & a_1\alpha_2 & a_2\alpha_2 - N & \ldots & a_{j-1}\alpha_2 & a_j\alpha_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_n\alpha_{j-2} & a_1\alpha_{j-2} & a_2\alpha_{j-2} & \ldots & a_{j-1}\alpha_{j-2} & a_j\alpha_{j-2} \\ a_n\alpha_{j-1} & a_1\alpha_{j-1} & a_2\alpha_{j-1} & \ldots & a_{j-1}\alpha_{j-1} - N & a_j\alpha_{j-1} \\ \frac{a_n}{a_j}\sum_{i=j}^k a_i\alpha_i & \frac{a_1}{a_j}\sum_{i=j}^k a_i\alpha_i & \frac{a_2}{a_j}\sum_{i=j}^k a_i\alpha_i & \ldots & \frac{a_{j-1}}{a_j}\sum_{i=j}^k a_i\alpha_i & \sum_{i=j}^k a_i\alpha_i - N \end{pmatrix}$$

Note that for $j = k$ this definition coincides with the previous definition. We show the formula (A.4) by induction. For $j = k$ the formula holds. We now assume that it holds for $j \in \{2, \ldots, k\}$ and show it for $j - 1$. Indeed, replacing the j-th column by $\frac{a_j}{a_{j-1}} \cdot$ $(j-1)$-st column minus the j-th column gives the following expression for $\det \widetilde{A}_j$

$$\begin{vmatrix} a_n\alpha_{i^*} & a_1\alpha_{i^*} & a_2\alpha_{i^*} & \ldots & a_{j-1}\alpha_{i^*} & 0 \\ a_n\alpha_1 & a_1\alpha_1 - N & a_2\alpha_1 & \ldots & a_{j-1}\alpha_1 & 0 \\ a_n\alpha_2 & a_1\alpha_2 & a_2\alpha_2 - N & \ldots & a_{j-1}\alpha_2 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_n\alpha_{j-2} & a_1\alpha_{j-2} & a_2\alpha_{j-2} & \ldots & a_{j-1}\alpha_{j-2} & 0 \\ a_n\alpha_{j-1} & a_1\alpha_{j-1} & a_2\alpha_{j-1} & \ldots & a_{j-1}\alpha_{j-1} - N & -N\frac{a_j}{a_{j-1}} \\ \frac{a_n}{a_j}\sum_{i=j}^k a_i\alpha_i & \frac{a_1}{a_j}\sum_{i=j}^k a_i\alpha_i & \frac{a_2}{a_j}\sum_{i=j}^k a_i\alpha_i & \ldots & \frac{a_{j-1}}{a_j}\sum_{i=j}^k a_i\alpha_i & N \end{vmatrix}.$$

We expand along the last column, which gives, using linearity in the last row

$$\begin{aligned} &\det \widetilde{A}_j \\ = \;& N \begin{vmatrix} a_n\alpha_{i^*} & a_1\alpha_{i^*} & a_2\alpha_{i^*} & \ldots & a_{j-1}\alpha_{i^*} \\ a_n\alpha_1 & a_1\alpha_1 - N & a_2\alpha_1 & \ldots & a_{j-1}\alpha_1 \\ a_n\alpha_2 & a_1\alpha_2 & a_2\alpha_2 - N & \ldots & a_{j-1}\alpha_2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_n\alpha_{j-2} & a_1\alpha_{j-2} & a_2\alpha_{j-2} & \ldots & a_{j-1}\alpha_{j-2} \\ a_n\alpha_{j-1} & a_1\alpha_{j-1} & a_2\alpha_{j-1} & \ldots & a_{j-1}\alpha_{j-1} - N \end{vmatrix} \\ & + \frac{a_j}{a_{j-1}} N \begin{vmatrix} a_n\alpha_{i^*} & a_1\alpha_{i^*} & a_2\alpha_{i^*} & \ldots & a_{j-1}\alpha_{i^*} \\ a_n\alpha_1 & a_1\alpha_1 - N & a_2\alpha_1 & \ldots & a_{j-1}\alpha_1 \\ a_n\alpha_2 & a_1\alpha_2 & a_2\alpha_2 - N & \ldots & a_{j-1}\alpha_2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_n\alpha_{j-2} & a_1\alpha_{j-2} & a_2\alpha_{j-2} & \ldots & a_{j-1}\alpha_{j-2} \\ \frac{a_n}{a_j}\sum_{i=j}^k a_i\alpha_i & \frac{a_1}{a_j}\sum_{i=j}^k a_i\alpha_i & \frac{a_2}{a_j}\sum_{i=j}^k a_i\alpha_i & \ldots & \frac{a_{j-1}}{a_j}\sum_{i=j}^k a_i\alpha_i \end{vmatrix} \end{aligned}$$

$$
= \frac{N}{a_{j-1}} \begin{vmatrix} a_n\alpha_{i^*} & a_1\alpha_{i^*} & \dots & a_{j-1}\alpha_{i^*} \\ a_n\alpha_1 & a_1\alpha_1 - N & \dots & a_{j-1}\alpha_1 \\ a_n\alpha_2 & a_1\alpha_2 & \dots & a_{j-1}\alpha_2 \\ \vdots & \vdots & & \vdots \\ a_n\alpha_{j-2} & a_1\alpha_{j-2} & \dots & a_{j-1}\alpha_{j-2} \\ a_n\sum_{i=j-1}^k a_i\alpha_i & a_1\sum_{i=j-1}^k a_i\alpha_i & \dots & a_{j-1}\left(\sum_{i=j-1}^k a_i\alpha_i - N\right) \end{vmatrix}
$$

This shows

$$
|\det \widetilde{A}_j| \quad = \quad N|\det \widetilde{A}_{j-1}|.
$$

By the induction hypothesis we have $|\det \widetilde{A}_k| = |\det \widetilde{A}_j|N^{k-j} = N|\det \widetilde{A}_{j-1}|N^{k-j} = |\det \widetilde{A}_{j-1}|N^{k-(j-1)}$. This shows (A.4).

Now we use (A.4) for $j = 1$. First, we calculate $\det \widetilde{A}_1$ giving

$$
\begin{aligned}
\det \widetilde{A}_1 &= \begin{vmatrix} a_n\alpha_{i^*} & a_1\alpha_{i^*} \\ \frac{a_n}{a_1}\sum_{i=1}^k a_i\alpha_i & \sum_{i=1}^k a_i\alpha_i - N \end{vmatrix} \\
&= a_n\alpha_{i^*}\sum_{i=1}^k a_i\alpha_i - Na_n\alpha_{i^*} - a_1\alpha_{i^*}\frac{a_n}{a_1}\sum_{i=1}^k a_i\alpha_i \\
&= -Na_n\alpha_{i^*}.
\end{aligned}
$$

Finally, using (A.3) and (A.4) we obtain

$$
|\det A| \quad = \quad N^{n-k-2}N^{k-1}N|a_n| \cdot |\alpha_{i^*}| \; = \; N^{n-2}|a_n| \cdot |\alpha_{i^*}|.
$$

This shows the lemma. □

References

[1] R. Baier, L. Grüne and S. Hafstein. Linear programming based Lyapunov function computation for differential inclusions. *Discrete Contin. Dyn. Syst. Ser. B*, **17-1**, pp. 33–56, 2012.

[2] R. Bartels and G. Stewart. Algorithm 432: Solution of the matrix equation $AX + XB = C$. *Comm. ACM*, **15-9**, pp. 820–826, 1972.

[3] F. Clarke. *Optimization and nonsmooth Analysis*, SIAM, Philadephia, PA, 1990.

[4] A. Garcia and O. Agamennoni. Attraction and stability of nonlinear ODE's using continuous piecewise linear approximations. (submitted)

[5] P. Giesl. *Construction of global Lyapunov functions using radial basis functions.* Volume 1904 of *Lecture Notes in Mathematics*, Springer, Berlin, 2007.

[6] P. Giesl and S. Hafstein. Existence of piecewise affine Lyapunov functions in two dimensions. *J. Math. Anal. Appl.*, **371**, pp. 233–248, 2010.

[7] P. Giesl and S. Hafstein. Construction of Lyapunov functions for nonlinear planar systems by linear programming. (submitted)

[8] S. Hafstein. A constructive converse Lyapunov theorem on exponential stability. *Discrete Contin. Dyn. Syst. - Series A*, **10-3**, pp. 657–678, 2004.

[9] S. Hafstein. A constructive converse Lyapunov theorem on asymptotic stability for nonlinear autonomous ordinary differential equations. *Dynamical Systems: An International Journal*, **20**, pp. 281–299, 2005.

[10] S. Hafstein. *An algorithm for constructing Lyapunov functions.* Volume 8 of *Electron. J. Differential Equ. Monogr.* Texas State Univ., Dep. of Mathematics, San Marcos, TX, 2007. Available electronically at `http://ejde.math.txstate.edu/`.

[11] T. Johansen. Computation of Lyapunov Functions for Smooth Nonlinear Systems using Convex Optimization. *Automatica*, **36**, pp. 1617–1626, 2000.

[12] M. Johansson and A. Rantzer. On the computation of piecewise quadratic Lyapunov functions. In: *Proceedings of the 36th IEEE Conference on Decision and Control*, 1997.

[13] P. Julian. *A high level canonical piecewise linear representation: Theory and applications.* Ph.D. Thesis: Universidad Nacional del Sur, Bahia Blanca, Argentina, 1999.

[14] P. Julian, J. Guivant and A. Desages. A parametrization of piecewise linear Lyapunov function via linear programming. *Int. Journal of Control*, **72**, pp. 702–715, 1999.

[15] H. K. Khalil. *Nonlinear Systems*, 3rd edition, Prentice Hall, New Jersey, 2002.

[16] S. Marinosson. *Stability analysis of nonlinear systems with linear programming: A Lyapunov functions based approach.* Ph.D. Thesis: Gerhard-Mercator-University, Duisburg, Germany, 2002.

[17] S. Marinosson. Lyapunov function construction for ordinary differential equations with linear programming. *Dynamical Systems: An International Journal*, **17**, pp. 137–150, 2002.

[18] A. Papachristodoulou and S. Prajna. The construction of Lyapunov functions using the sum of squares decomposition. In: *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 3482–3487, 2002.

[19] P. Parrilo. *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization.* Ph.D. Thesis: Caltech, Pasadena, USA, 2000.

[20] M. Peet. Exponentially stable nonlinear systems have polynomial Lyapunov functions on bounded regions. *IEEE Trans. Automatic Control*, **54-5**, pp. 979–987, 2009.

[21] V. Zubov. *Methods of A. M. Lyapunov and their application.* P. Noordhoff, Groningen, Netherlands, 1964.